Preconditioning Methods for Multidimensional Aerodynamics

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1 Introduction

Over the past years several preconditioning methods have appeared with the aim to solve nearly incompressible flow problems with numerical algorithms designed for the compressible flows. The developments are motivated by several observations. First, there exist flow problems of mixed compressible/incompressible type. That is, part of the flow region is incompressible with locally low Mach numbers, whereas significant compressibility effects occur in other regions of the flow. A typical example in aerodynamics is the flow over a wing at a high angle of attack. Other examples include duct flow where the flow in most of the region is highly subsonic but sharp changes in the geometry of the duct cause the flow to become supersonic in a poertion of the region. Moreover, low speed flows may be compressible, because of surface heat transfer or volumetric heat addition. Finally, engineers prefer to use existing compressible flow codes over the broadest range of flow conditions possible in order to avoid to deal with multiple flow codes.

The difficulty in solving the compressible equations for low Mach numbers are associated with the large disparity of the acoustic wave speed, $u + a$, and the waves convected at the fluid speed, u. The application of preconditioning changes the eigenvalues of the system of compressible flow equations in order to remove this large disparity of wave speeds. The time derivatives are premultiplied by a matrix which slows the speed of the acoustic waves down towards the fluid speed. A review of preconditioning methods for convection dominated flow problems has been published by Turkel [35]. Successful applications to low Reynolds number viscous flows are reported by Choi and Merkle [6].

There are two basic views of preconditioning methods. The first, which arose in applications to elliptic equations , views preconditioning merely as a way to accelerate the convergence of the iterative method. This is usually done on the matrix level. Thus, once convergence is achieved the solution should be identical to the original solution. The second approach, which we advocate for low speed flow is to view preconditioning as changing the differential equations to be solved. Thus, as a consequence of preconditioning the basic scheme can be changed, e.g. the artificial viscosity or upwinding. Similarly, the boundary conditions satisfied are based on the preconditioned equations rather than the physical equations. The justification for this is presented in [37] is that the "standard" numerical schemes for the compressible equations do not converge to a scheme for the incompressible equations (using a pseudo-compressibility approach) as the Mach number goes to zero. However, a proper preconditioning leads to a numerical scheme that does behave appropriately for low Mach numbers.

We present a generalization of the preconditioners given by Turkel [35, 36], Choi-Merkle [6] and van Leer et. al [39]. The characteristic variables needed to compute flow conditions at inflow or outflow boundaries are also derived. The accuracy of numerical solutions of the preconditioned flow equations is also investigated. We discuss both nonconservative and conservative artificial dissipation models. Applications to two and three dimensional inviscid and viscous flows are presented.

We demonstrate that preconditioning can be combined with well-known convergence acceleration techniques such as residual smoothing and multigrid. Indeed, it is no surprise that the clustering of eigenvalues with preconditioning improves the damping of transient high-frequency modes to an extent, that efficient multigrid computations of flows with very low Mach numbers are now possible.

1.1 General requirements

For both the incompressible and compressible equations we consider systems of the form

$$w_t + f_x + g_y = 0.$$

This system is written in conservation form though for some applications this is not necessary. Our analysis will be based on the linearized equations so that the conservation form does not appear in the analysis though it does appear in the final numerical approximation. This system is now replaced by

$$\mathbf{P}^{-1} w_t + f_x + g_y = 0,$$

or in linearized form

$$\mathbf{P}^{-1} w_t + A w_x + B w_y = 0, \tag{1.1.1}$$

with A and B constant matrices.

In order for this system to be equivalent to the original system, in the steady state, we demand that $\mathbf{P}^{-1}$ have an inverse. This only need be true in the flow regime under consideration. We shall see later that frequently $\mathbf{P}$ is singular at stagnation points and also along sonic lines. Thus, we will consider strictly subsonic flow without a stagnation point. For transonic flow it is necessary to smooth out the singularity in a neighborhood of the sonic line. We also assume that the Jacobian matrices $A = \frac{\partial f}{\partial w}$ and $B = \frac{\partial g}{\partial w}$ are simultaneously symmetrizable . In terms of the 'symmetrizing' variables we also demand that $\mathbf{P}$ be positive definite.

Assuming the steady state has a unique solution, it does not matter which system we march to a steady state. We shall later see that for the finite difference approximations the steady state solutions are not necessarily the same and usually the preconditioned system leads to a better behaved steady state.

2 Incompressible Equations

We first consider the incompressible inviscid equations in primitive variables.

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + u u_x + v u_y + p_x &= 0 \\ v_t + u v_x + v v_y + p_y &= 0 \end{aligned}$$

We consider generalizations of Chorin's pseudo-compressibility method [7]. Using the preconditioning suggested in [35] (with $\alpha = 1$) we have

$$\begin{aligned} \frac{1}{\beta^2} p_t + u_x + v_y &= 0 \\ \frac{u}{\beta^2} p_t + u_t + u u_x + v u_y + p_x &= 0 \\ \frac{v}{\beta^2} p_t + v_t + u v_x + v v_y + p_y &= 0 \end{aligned} \tag{2.1}$$

We can also write this in matrix form using

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} \beta^2 & 0 & 0 \\ -u & 1 & 0 \\ -v & 0 & 1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & v & 0 \\ 1 & 0 & v \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \tag{2.2}$$

Multiplying by **P** we rewrite this as

$$w_t + \mathbf{P} A w_x + \mathbf{P} B w_y = 0.$$

We also define

$$D = \omega_1 A + \omega_2 B \qquad -1 \leq \omega_1, \omega_2 \leq 1 \tag{2.3}$$

where ω_1, ω_2 are the Fourier transform variables in the x and y directions respectively. The speeds of the waves are now given by the roots of $det(\lambda I - \mathbf{P} A \omega_1 - \mathbf{P} B \omega_2) = 0$. Let $q = u\omega_1 + v\omega_2$. Then the eigenvalues of **P**D are

$$d_0 = q \quad , \quad d_\pm = \pm\beta \tag{2.4}$$

and so the 'acoustic' speed is isotropic.

The spatial derivatives involve symmetric matrices, i.e. D is a symmetric matrix but **P** is not symmetric. Thus, while the original system was symmetric hyperbolic the preconditioned system is no longer symmetric. In [35] it is shown that as long as

$$\beta^2 > (u^2 + v^2)$$

then the equations can be symmetrized. On the other hand the eigenvalues are most equalized if $\beta^2 = (u^2 + v^2)$ [35]. Hence, we wish to choose β^2 slightly larger than $u^2 + v^2$. However, numerous calculations verify, that in general, a constant β is the best for the convergence rate.

We wish to stress that β has the dimensions of a speed. Therefore, β cannot be a universal constant. There are papers that claim that $\beta = 1$ or $\beta = 2.5$ are optimal. Such claims cannot be true in general. It is simple to see that if one nondimensionalizes the equation then β gets divided by a reference velocity. Hence, the optimal ‘constant’ β depends on the dimensionalization of the problem and in particular depends on the inflow conditions. In many calculations the inflow mass flux is equal to 1 or else $p + (u^2 + v^2)/2 = 1$. Such conditions will give an optimal β close to one. However, if one chooses the incoming mass flux as ten then the optimal β will be larger.

We next define the Bernoulli function

$$H = p + (u^2 + v^2)/2.$$

Bernoulli’s theorem states that for steady inviscid flow H is constant along streamlines. We now multiply the second equation of (2.1) by u and the third equation of (2.1) by v and add these two equations. If $\beta^2 = u^2 + v^2$, the result is

$$H_t + uH_x + vH_y = 0. \tag{2.5}$$

Thus, by altering the time dependence of the equations we have constructed a new equation in which H is convected along streamlines. Furthermore, if H is a uniform constant both initially and at inflow then H will remain constant for all time. On the numerical level this will usually not be true because of the introduction of an artificial viscosity or because of upwinding. For viscous flow, (2.5) is replaced by

$$H_t + uH_x + vH_y = \frac{1}{Re}(u\Delta u + v\Delta v)$$

We note that these relationships for H follow from the momentum equations and do not depend on the form of the continuity equation. Hence, we consider the following generalization of (2.1)

$$\begin{aligned} \frac{1}{\beta^2}p_t + aH_t + u_x + v_y &= 0 \\ \frac{\alpha u}{\beta^2}p_t + u_t + uu_x + vu_y + p_x &= 0 \\ \frac{\alpha v}{\beta^2}p_t + v_t + uv_x + vv_y + p_y &= 0 \end{aligned}$$

where, a is a free parameter. The eigenvalues of $\mathbf{P}$D are independent of the parameter a and are given by (2.4). For $a = 0, \alpha = 1$ we recover our original scheme. For $a = -1$ the time derivative of the pressure no longer appears in the continuity equation. For general β we have

$$\mathbf{P}^{-1} = \frac{1}{\beta^2}\begin{pmatrix} (a+1) & au & av \\ \alpha u & \beta^2 & 0 \\ \alpha v & 0 & \beta^2 \end{pmatrix},$$

$$\mathbf{P} = \frac{1}{d}\begin{pmatrix} \beta^2 & -au & -av \\ -\alpha u & 1 + a - \frac{a\alpha v^2}{\beta^2} & \frac{a\alpha uv}{\beta^2} \\ -\alpha v & \frac{a\alpha uv}{\beta^2} & 1 + a - \frac{a\alpha u^2}{\beta^2} \end{pmatrix}$$

where $d = 1 + a - a\alpha\frac{u^2+v^2}{\beta^2}$ and we require that $d \geq 0$. If $\beta^2 = u^2 + v^2$ and $\alpha = 1$ then

$$\mathbf{P} = \begin{pmatrix} u^2+v^2 & -au & -av \\ -u & 1+\frac{au^2}{u^2+v^2} & \frac{auv}{u^2+v^2} \\ -v & \frac{auv}{u^2+v^2} & 1+\frac{av^2}{u^2+v^2} \end{pmatrix}$$

In [36] an analogy to the symmetric preconditioning of van Leer, Lee and Roe was constructed for the incompressible equations. If we choose $a = 1, \alpha = 1$ we get this preconditioning of van Leer et. al., i.e. $\mathbf{P}$ is symmetric .

These examples show that the preconditioning is not unique. If fact, since the determinant of the transpose of a matrix is equal to the determinant of the original matrix it follows that the transpose of $\mathbf{P}$ is also a preconditioner with the same eigenvalues for the preconditioned system. In general, these various systems will have similar eigenvalues but different eigenvectors for the preconditioned system. Calculations show that the system given by $\mathbf{P}$ in (2.1) is more robust and converges faster than that with the transpose preconditioner. This shows that it is not sufficient to consider just the eigenvalues but that the eigenvectors are also of importance. However, even when $\mathbf{P}$ is symmetric $\mathbf{P}$D is not symmetric and so the eigenvectors of the preconditioned system do not form an orthogonal basis.

We next examine some general form that the preconditioner can have. For this analysis it is easier to use streamwise coordinates as suggested in [39] and so $v = 0$. Let u_* be some normalization of the velocity components, then

$$A = \begin{pmatrix} 0 & u_* & 0 \\ u_* & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & u_* \\ 0 & 0 & 0 \\ u_* & 0 & 0 \end{pmatrix}$$

Then the "convective" eigenvector for the non-preconditioned system is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{-u\omega_1+\sqrt{(u\omega_1)^2+4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{-u\omega_1-\sqrt{(u\omega_1)^2+4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}.$$

We now consider preconditioners of the form

$$\mathbf{P} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

Given D (2.3), we want the eigenvalues of $\mathbf{P}D$ to be $\omega_1 u$, $\pm u$. This gives us three relations for the four unknowns:

$$a = \frac{u^2}{u_*^2}$$

$$(b+c)u_* + du = 0$$

$$u^2 d - bcu_*^2 = u^2$$

The values suggested in [35] are $b = 0, c = -\frac{u}{u_*}, d = 1$ while the values suggested in [39] are $b = c = -\frac{u}{u_*}, d = 2$. We next present the eigenvectors of $\mathbf{P}D$ in terms of the elements of $\mathbf{P}$. We exclude the case $\omega_2 = 0, \omega_1 = \pm 1$ as in this case $\mathbf{P}$D has a double eigenvalue u and the eigensystem completely changes. Then the "convective" eigenvector is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -(1 + b\frac{u_*}{u})\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 - \frac{u^2\omega_1}{u_*} + bu\omega_1 \\ u_*(a + b\omega_1^2) - u_*(b+c)\omega_1 \\ (u_* b\omega_1 + u)\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 + \frac{u^2\omega_1}{u_*} - bu\omega_1 \\ u_*(a + b\omega_1^2) + u_*(b+c)\omega_1 \\ (u_* b\omega_1 - u)\omega_2 \end{pmatrix}.$$

We note that the convective eigenvector is the same as before the preconditioning for the choice $b = 0$. The two acoustic eigenvectors are orthogonal to each other if we choose $b = 0$ and $c^2 = \frac{u^2(1-\frac{u^2}{u_*^2})}{u_*^2}$. This is similar, but not identical, to the choice suggested in [35]. There is no way to make the convective eigenvector normal to both acoustic eigenvectors for preconditioners of the form (2.6).

3 Algorithm I

We consider the preconditioned Euler equation written as

$$P^{-1}w_t + Aw_x + Bw_y = 0 \tag{3.1}$$

The form of the matrices A, B depend on the choice of the variables w . We shall first consider the variables $w = w_0 = (p, u, v, S)$ with $dS = dp - a^2 d\rho$. We then have

$$A_0 = \begin{pmatrix} u & \rho a^2 & 0 & 0 \\ \frac{1}{\rho} & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \quad B_0 = \begin{pmatrix} v & 0 & \rho a^2 & 0 \\ 0 & v & 0 & 0 \\ \frac{1}{\rho} & 0 & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

The preconditioners we shall consider contain a free parameter β. Unfortunately, the papers that have dealt with this subject all use different definitions of this free variable. Thus, for Choi and Merkle $\beta \sim a^2$ while for Turkel $\beta^2 \sim u^2 + v^2$ (based on Chorin's work on pseudo-compressibility) while for van Leer $\beta = \sqrt{1 - M^2}$ (based on fluid dynamic notation). In this paper we shall concentrate on the Choi-Merkle and Turkel preconditioners. To avoid confusion we shall use the Choi-Merkle choice for β within this paper. In terms of these variables we consider the following preconditioner which is a generalization of previous preconditioners. For $\delta = 0$ we recover the preconditioner suggested by Turkel [35, 36] while for $\delta = 1, \alpha = 0$ we recover the preconditioner suggested by Choi and Merkle [6]. Note that δ is a new free parameter to generalize the Choi-Merkle and Turkel preconditioners. It does not correspond to the δ in the Choi-Merkle paper.

$$P_0^{-1} = \begin{pmatrix} \frac{a^2}{\beta M_r^2} & 0 & 0 & \delta \\ \frac{\alpha u}{\rho \beta M_r^2} & 1 & 0 & 0 \\ \frac{\alpha v}{\rho \beta M_r^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_0 = \begin{pmatrix} \frac{\beta M_r^2}{a^2} & 0 & 0 & -\frac{\beta M_r^2}{a^2}\delta \\ -\frac{\alpha u}{\rho a^2} & 1 & 0 & \frac{\alpha u}{\rho a^2}\delta \\ -\frac{\alpha v}{\rho a^2} & 0 & 1 & \frac{\alpha v}{\rho a^2}\delta \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

Choi and Merkle chose

$$M_r^2 = \begin{cases} \epsilon, & M < \epsilon, \\ M^2, & \epsilon \leq M < 1 \\ 1 & M \geq 1; \end{cases} \qquad \beta = a^2 \quad \text{(for inviscid flows)}$$

Here we shall define βM_r^2 directly. The guiding principle is that near stagnation points this should be cut off and not allowed to go to zero. We also desire that for supersonic flow that preconditioner be turned off, i.e. for high speed flow we want that $\beta M_r^2 = a^2$, $\alpha = 0$, $\delta = 0$. We choose

$$\begin{aligned} \beta M_r^2 &= \min\left[\max\left(K_1(u^2+v^2)(1+\frac{(1-K_1M_0^2)}{K_1M_0^4}M^2), K_2(u_\infty^2+v_\infty^2)\right), a^2\right] \\ &= \min\left[\max\left((u^2+v^2)(K_1(1-\frac{M^2}{M_0^2})+\frac{M^4a^2}{M_0^4}), K_2(u_\infty^2+v_\infty^2)\right), a^2\right] \end{aligned} \tag{3.3}$$

This returns βM_r^2 to its non-preconditioned value at M_0. K_1 typically varies between 1.0 and 1.1. K_2 seems to depend on the numbers of mesh nodes near the stagnation points. For viscous flow K_2 needs to be chosen larger ($\sim .15$) because of the large number of points in the boundary layer (and also to account for the viscous time step). For duct flow one expects to again choose a very small K even for viscous flows since there usually are no stagnation points. K also depends on α [35] and may depend on the local cell Reynolds number [6]. We note that by using the variables $(\frac{dp}{\rho a}, du, dv, dS)$ the matrices A and B would be symmetric. However, since we do not presently consider symmetric preconditioners this is not of any consequence. Thus,

$$P_0A_0 = \begin{pmatrix} \frac{\beta M_r^2 u}{a^2} & \rho\beta M_r^2 & 0 & \frac{-\beta M_r^2 u\delta}{a^2} \\ \frac{1}{\rho}\left(1-\frac{\alpha u^2}{a^2}\right) & (1-\alpha)u & 0 & \frac{\alpha u^2\delta}{\rho a^2} \\ \frac{-\alpha uv}{\rho a^2} & -\alpha v & u & \frac{\alpha uv\delta}{\rho a^2} \\ 0 & 0 & 0 & u \end{pmatrix} \tag{3.4}$$

The eigenvalues of P_0A_0 are

$$\lambda_0 = u$$

and

$$\lambda_\pm = 0.5\left(zu \pm \sqrt{z^2u^2 + 4\beta M_r^2(1-\frac{u^2}{a^2})}\right), \quad z = 1-\alpha+\frac{\beta M_r^2}{a^2} \tag{3.5}$$

In particular if $\alpha = 1+\frac{\beta M_r^2}{a^2}$ then $z=0$ and $\lambda_\pm = \pm\sqrt{\beta M_r^2(1-M^2)}$. The right eigenvectors are given by the columns of

$$R^{-1} = \begin{pmatrix} \rho\beta M_r^2 & \rho\beta M_r^2 & 0 & 0 \\ \lambda_+ - \frac{\beta M_r^2}{a^2}u & \lambda_- - \frac{\beta M_r^2}{a^2}u & 0 & \frac{\delta u}{\rho a^2} \\ \frac{\alpha v\lambda_+}{u-\lambda_+} & \frac{\alpha v\lambda_-}{u-\lambda_-} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

We note that for $\delta = 0$ and $\alpha = 0$ the two eigenvectors for the convective speeds are orthogonal to each other and to the acoustic eigenvectors. Of course, orthogonality of eigenvectors depends on the basis chosen, i.e. on the variables used in constructing P_0 and A_0. Furthermore, as the Mach number goes to zero the determiant of R^{-1} depends only on βM_r^2. Thus, the condition number of the transformation is well behaved as long as βM_r^2 is bounded away from zero as the Mach number approaches zero. See [8] for the importance of orthogonal eigenvectors.

For generalized coordinates we replace this by

$$U = us_x + vs_y \qquad |S_\xi|^2 = s_x^2 + s_y^2$$

$$\lambda_\pm = 0.5\left(zU \pm \sqrt{z^2U^2 + 4\beta M_r^2(|S_I|^2 - \frac{U^2}{a^2})}\right) \tag{3.7}$$

where $S_\xi = (s_x, s_y)$ is the cell face vector in ξ-direction. The largest eigenvalue is then used to determine the (inviscid) time step. We note that these eigenvalues are independent of δ. For M ~ 0, $\lambda_\pm \sim 0.5u\left((1-\alpha) \pm \sqrt{(1-\alpha)^2+4}\right)$. So for $\alpha = 0$ the condition number is $\frac{1+\sqrt{5}}{1-\sqrt{5}} \sim 2.6$ while for $\alpha = 1$ the condition number is $1+O(M)$. If we choose $\alpha = 1+\frac{\beta M_r^2}{a^2}$ and $\beta M_r^2 = \frac{M^2c^2}{1-M^2}$ then we get the same eigenvalues as those of the symmetric optimal preconditioner to be presented in the next section (see 5.1). For small Mach numbers this is similar to what we obtained with $\alpha = 1$ while at the sonic line α and β become infinite which reduces the condition number of the preconditioned system near the sonic line. We shall also express these matrices in terms of the variables $w_4 = (p,u,v,T)$. Let

$$g = 1+(\gamma-1)\delta \quad = \quad \begin{cases} 1, & \text{if} \quad \delta = 0 \\ \gamma, & \text{if} \quad \delta = 1 \end{cases}$$

Then,

$$A_4 = \begin{pmatrix} u & \rho a^2 & 0 & 0 \\ \frac{1}{\rho} & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & (\gamma-1)T & 0 & u \end{pmatrix} \quad B_4 = \begin{pmatrix} v & 0 & \rho a^2 & 0 \\ 0 & v & 0 & 0 \\ \frac{1}{\rho} & 0 & v & 0 \\ 0 & 0 & (\gamma-1)T & v \end{pmatrix}$$

Let $m^2 = \frac{\beta M^2}{a^2}$, then

$$P_4^{-1} = \begin{pmatrix} \frac{1}{m^2} - (\gamma-1)\delta & 0 & 0 & \frac{\gamma p}{T}\delta \\ \frac{\alpha u}{\rho\beta M^2} & 1 & 0 & 0 \\ \frac{\alpha v}{\rho\beta M^2} & 0 & 1 & 0 \\ \frac{1}{\rho c_p}\left(\frac{1}{m^2} - g\right) & 0 & 0 & g \end{pmatrix}, P_4 = \begin{pmatrix} m^2 g & 0 & 0 & -m^2\frac{\gamma p}{T}\delta \\ -\frac{\alpha u g}{\rho a^2} & 1 & 0 & \frac{\alpha u}{\rho a^2}\frac{\gamma p}{T}\delta \\ -\frac{\alpha v g}{\rho a^2} & 0 & 1 & \frac{\alpha v}{\rho a^2}\frac{\gamma p}{T}\delta \\ \frac{1}{\rho c_p}(m^2 g - 1) & 0 & 0 & 1 - (\gamma-1)m^2\delta \end{pmatrix} \tag{3.8}$$

The matrix presented in (3.2) while improving the condition in the neighborhood of $M = 0$ does not improve the condition number in the neighborhood of the sonic line. Indeed in (3.3) we explicitly removed the preconditioning as $M \to 1$. An alternative is to generalize (3.2) so that it is basically the same for low Mach numbers but reduces the condition number near the sonic line.

4 Algorithm II

In [9, 37] an alternate way of preconditioning the fluid dynamics system was introduced. In this approach we do not update the conservation variables, rather we update the variables

$$W = \begin{pmatrix} p' \\ \rho u \\ \rho v \\ E' \end{pmatrix}$$

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p' \\ \rho u v \\ \rho H' u \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p' \\ \rho H' v \end{pmatrix}, \quad Q = \begin{pmatrix} p' \\ \rho u \\ \rho v \\ H' \end{pmatrix}$$

$$\frac{\partial W}{\partial t} = \mathbf{P}\left[\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right]$$

where $p' = p - p_\infty$, $E' = c_p\rho(T - T_\infty) - (p - p_\infty) + \frac{\rho(u^2+v^2)}{2} = E + p_\infty - \frac{\gamma p_\infty}{(\gamma-1)\rho_\infty}\rho$ and $\rho H' = E' + p'$. Then

$$\mathbf{P} = \mathbf{I} \quad + \quad \Delta \cdot \begin{pmatrix} 1 - \frac{1}{\Delta} & -\frac{u}{G+h_\infty} & -\frac{v}{G+h_\infty} & \frac{1}{G+h_\infty} \\ -B_2 & \frac{uB_2}{G+h_\infty} & \frac{vB_2}{G+h_\infty} & -\frac{B_2}{G+h_\infty} \\ -B_3 & \frac{uB_3}{G+h_\infty} & \frac{vB_3}{G+h_\infty} & -\frac{B_3}{G+h_\infty} \\ -B_4 & \frac{uB_4}{G+h_\infty} & \frac{vB_4}{G+h_\infty} & -\frac{B_4}{G+h_\infty} \\ & & & \end{pmatrix}$$

$$\begin{aligned} B_1 &= \frac{1}{\beta M_r^2} - \frac{1}{(\gamma-1)h} \\ B_2 &= (B_1 + \frac{\alpha}{\beta M_r^2})u \end{aligned}$$

$$\begin{aligned} B_3 &= (B_1 + \frac{\alpha}{\beta M_r^2})v \\ B_4 &= B_1 H' + \frac{\alpha(u^2+v^2)}{\beta M_r^2} \end{aligned}$$

where βM_r^2 was previously defined and $h = c_p T$, $G = \frac{u^2+v^2}{2}$, $\Delta = \frac{(G+h_\infty)\beta M_r^2}{h}$ and $\alpha = 1$,

With this approach all operations (e.g. residual smoothing, multigrid etc.) are done on the new variables and not the conservative variables. The rationale is that since the pressure is more appropriate than density at lower speeds we should interpolate the pressure between grids rather than the density. Similarly, since the density is almost constant it is more beneficial to smooth the residual of the pressure rather than the residual of the density. The eigenvalues of this system, and hence the time steps needed for stability, are the same as for Algorithm I, (3.7).

5 Symmetric Preconditioner

Until now we have stressed the importance of preconditioning for low Mach number flows. Indeed we have constructed the preconditioners by starting from the incompressible equations and analyzing what changes are needed to the compressible equations in order to recover the incompressible equations in the limit. Van Leer, Lee and Roe [39] began at the other end. They first noticed that the two dimensional supersonic equations can be diagonalized if an appropriate preconditioner is added to the system. They then constructed a preconditioner for the subsonic case that it some sense was the continuation of the supersonic case. This continuation can never be exact since there must be a singularity at the sonic line.

Furthermore, in contrast to the variables that we have used until now van Leer et al. expressed the inviscid equations in $(\frac{dp}{\rho a}, du, dv, dS)$ variables. This differs from what we have done above in the normalization of the pressure variable. With this new set of variables the Jacobian matrices for inviscid flow are symmetric. Adding a symmetric positive definite preconditioner yields a symmetric hyperbolic system:

$$P^{-1}\frac{\partial w}{\partial t} + A\frac{\partial w}{\partial x} + B\frac{\partial w}{\partial y} = 0$$

where P, A, B are symmetric and P is positive definite. Taking the inner product with we have by integration by parts (ignoring boundary contributions) $(w, A\frac{\partial w}{\partial x}) = (w, B\frac{\partial w}{\partial x}) = 0$ Hence, we conclude that $\frac{\partial}{\partial t}(P^{-1}w, w) = 0$ and so $||w^2||_{P^{-1}}$ is constant in time. Since P^{-1} is positive definite, this norm is equivalent to the standard L^2 norm if there exists positive constants K_1 and K_2 such that

$$0 < K_1 \leq ||P|| \leq K_2$$

For the preconditioners that we shall consider the norm of P depends on the Mach number. Hence, we have well-posedness only as long as the Mach number is bounded away from zero. In the limit of the Mach number approaching zero, P becomes singular and the above analysis is no longer valid.

We now present the van Leer-Lee-Roe preconditioning for streamwise coordinates where the x axis is aligned with the flow and so $v = 0$. We express the preconditioner for the equations using $(\frac{dp}{\rho a}, du, dv, dS)$ variables. The coefficient matrices are given by

$$A = c\begin{pmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{pmatrix} \quad B = c\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define $D = A\omega_1 + B\omega_2$; then

$$\mathbf{P}_V = \begin{pmatrix} \frac{\tau}{\hat{\beta}^2}M^2 & -\frac{\tau}{\hat{\beta}^2}M & 0 & 0 \\ -\frac{\tau}{\hat{\beta}^2}M & 1+\frac{\tau}{\hat{\beta}^2} & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\hat{\beta} = \begin{cases} \sqrt{1-M^2}, & M < 1, \\ \sqrt{M^2-1}, & M \geq 1; \end{cases} \quad , \tau = \begin{cases} \sqrt{1-M^2} = \hat{\beta}, & M < 1, \\ \sqrt{1-M^{-2}} = \frac{\hat{\beta}}{M}, & M \geq 1. \end{cases}$$

β was introduced to change the envelope of the wave form while τ was introduced to change the amplitude of the waves. More details of the development of this preconditioner are given in [44]. Hence, we include both in $\mathbf{P}_V$. The eigenvalues of $\mathbf{P}_V D$, for subsonic flow, are

$$\begin{aligned} \lambda_0 &= u\omega_1 \quad \text{twice} \\ \lambda_\pm &= \pm\tau M c\sqrt{\omega_1^2 + \frac{\omega_2^2}{1-M^2}} \end{aligned} \tag{5.1}$$

In order to compare this formula with the formulae of previous sections we wish to reformulate this preconditioner for the case where the flow is not aligned in the x direction. We denote the matrices in the streamwise and perpendicular directions as $A_{||}$ and $A_\perp$ respectively. We next define the rotation matrices as

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To get the streamwise direction we shall choose

$$cos\theta = \frac{u}{\sqrt{u^2+v^2}}$$

$$sin\theta = \frac{v}{\sqrt{u^2+v^2}}$$

One can then verify that given the original matrices A, B.

$$\begin{aligned} A_{||} &= U(Acos\theta + Bsin\theta)U^{-1} \\ A_{\perp} &= U(-Asin\theta + Bcos\theta)U^{-1} \end{aligned} \tag{5.2}$$

Given numbers $\hat{\omega}_1, \hat{\omega}_2$ for $A_{||}, A_{\perp}$ we define

$$\begin{aligned} \omega_1 &= \hat{\omega}_1 cos\theta - \hat{\omega}_2 sin\theta \\ \omega_2 &= \hat{\omega}_1 sin\theta + \hat{\omega}_2 cos\theta \end{aligned}$$

note

$$\hat{\omega_1^2} + \hat{\omega_1^2} = \omega_1^2 + \omega_2^2.$$

Also define

$$\mathbf{P} = U^{-1}\hat{\mathbf{P}}U.$$

Then it is easy to verify that

$$\hat{\mathbf{P}}(A_{||}\hat{\omega}_1 + A_{\perp}\hat{\omega}_2) = U\left[\mathbf{P}(A\omega_1 + B\omega_2)\right]U^{-1}$$

The van Leer-Lee-Roe preconditioning for general nonaligned flow in $(\frac{dp}{\rho a}, du, dv, dS)$ variables is

$$\mathbf{P}_V = \begin{pmatrix} \frac{\tau}{\hat{\beta}^2}M^2 & -\frac{\tau}{\hat{\beta}^2}u/a & -\frac{\tau}{\hat{\hat{\beta}}^2}v/a & 0 \\ -\frac{\tau}{\hat{\beta}^2}u/a & (\frac{\tau}{\hat{\beta}^2}+1)\frac{u^2}{u^2+v^2} + \tau\frac{v^2}{u^2+v^2} & (\frac{\tau}{\hat{\beta}^2}+1)\frac{uv}{u^2+v^2} & 0 \\ -\frac{\tau}{\hat{\beta}^2}v/a & (\frac{\tau}{\hat{\beta}^2}+1)\frac{uv}{u^2+v^2} & (\frac{\tau}{\hat{\beta}^2}+1)\frac{v^2}{u^2+v^2} + \tau\frac{u^2}{u^2+v^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.3}$$

The symmetric preconditioner has the nice property that it always leads to a well posed problem, at least for Mach numbers bounded away from zero. It also is useful in constructing canonical forms that separate the euler equations into its elliptic and hyperbolic constitutive parts (see section 12). However, this preconditioner seems to have difficulties near stagnation points. One possible reason is that the general preconditioner (5.3) depends on the flow angle even near stagnation points. However, for very small Mach numbers the angle the flow makes with the x axis is not well defined. This leads to a numerical indeterminacy near stagnation points. Possible cures for this are discussed in [40]. These cures depend on modifications to (5.3) that remove this dependency on the flow angle usually at the cost of a loss of optimality.

In general it is not clear that a condition number of one at $M = 0$ is a desirable trait. One purpose of preconditioning is to transform the infinite condition number at $M = 0$ to a finite and small condition number. However, in practice, choosing the optimal condition number of one results in a less robust scheme. Thus, for example, it is found that for Algorithm I and II (3, 4) that $\alpha = 0$ leads to a more robust scheme even though $\alpha = 1$ is optimal at $M = 0$. One possible reason for this is that the preconditioning affects not only the time step but also the artificial viscosity. An optimal preconditioning lowers the artificial viscosity which improves the accuracy but deteriorates the robustness of the scheme. Further comparisons between Algorithm I and the symmetric preconditioner are presented in section 13.

6 Convergence to the Incompressible Equations

The time dependent Euler equations can be written as

$$\begin{aligned} p_t + up_x + vp_y + \rho a^2(u_x + v_y) &= 0 \\ u_t + uu_x + vu_y + \frac{p_x}{\rho} &= 0 \\ v_t + uv_x + vv_y + \frac{p_y}{\rho} &= 0 \\ S_t + uS_x + vS_y &= 0 \end{aligned} \tag{6.1}$$

where a is the speed of sound given by $a^2 = \frac{\gamma p}{\rho}$.

The form of this system is unchanged if we nondimensionalize the equations. From now on we shall assume that u, v, p, ρ are nondimensional quantities where the dimensional variables are nondimensionalized by u_*, p_*, ρ_* with $p_* = \rho_* u_*^2$. Following [15] we define $\epsilon = \frac{u_*}{a_*} = M_*$. If the fluid is isentropic then

$$p = \frac{\rho^\gamma}{\gamma \epsilon^2} \quad , \quad a = \frac{\rho^{\frac{\gamma-1}{2}}}{\epsilon} \quad . \tag{6.2}$$

Hence, as ϵ goes to zero the speed of sound, a, goes to infinity and so the first equation in (6.1) reduces to $u_x + v_y = 0$.

It was pointed out in [35, 36] that these equations can be symmetrized by using $\frac{dp}{\rho a}$ as the independent variable rather than dp . Hence, we define ϕ by $d\phi = \frac{dp}{\rho a}$. For isentropic flow both p and a are functions only of the density and so using (6.2) this can be integrated explicitly. This gives $\phi = \frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}\epsilon}$. As the Mach number goes to zero ϕ tends to infinity and therefore, Gustafsson and Stoor [15] subtract a constant and define

$$\phi = \frac{\rho^{\frac{\gamma-1}{2}} - 1}{\frac{\gamma-1}{2}\epsilon} \quad . \tag{6.3}$$

This amounts to specifying the constant in the integration of $d\phi$ from dp. They then prove, using energy methods, that for the linearized equations

$$a\phi_x \rightarrow \frac{\partial p_{incompressible}}{\partial x} \quad .$$

Hence, ϕ and all its derivatives behave as $O(M)$ as $M \rightarrow 0$. Since $\rho \rightarrow 1$ and using the definition of $d\phi$ this is equivalent to

$$dp_{compressible} \rightarrow dp_{incompressible}. \tag{6.4}$$

Several papers [15, 19, 21], have proven the convergence of the solution to the compressible equations to the solution of the incompressible equations, for isentropic flow, as the Mach number goes to zero. For nonisentropic flow there are no formal proofs. However, it is clear that for viscous flows that the boundary condition on the temperature, adiabatic or isothermal is very important.

All these results refer to the time dependent physical equations. Once preconditioning is introduced time accuracy is lost and one can only discuss convergence of the steady state solutions. In this case one would hope that the time dependent preconditioned compressible equations converge to some time dependent preconditioning of the incompressible equations. In addition, one would also want this to be true on the numerical level. Thus, one would want to solve the preconditioned compressible equations by some numerical technique, on a fixed mesh and compare that with the solution of the preconditioned incompressible equations on the same mesh. Mathematically, we have two limit processes occurring: the Mach number going to zero and the mesh size going to zero. These two limits need not commute. If one first converges the mesh size and then the Mach number it is equivalent to the convergence proofs for the analytic case. The more interesting problem is to converge the Mach number and then converge the mesh, i.e. we use a fix mesh as the Mach number is reduced.

In particular this requires a careful study of the viscosities introduced by the scheme. We first consider an upwinding scheme. For the compressible case the Riemann solver should depend on the preconditioned problem. One would then need to show that this Riemann problem converges to a Riemann problem for some preconditioning of the incompressible equations. We next consider a central difference scheme with a scalar viscosity. In this case a high order even difference of some quantity is added separately to each equation, e.g. for the incompressible equations: pressure for the continuity equation, u and v for the momentum equations. For the compressible equations one normally adds a density difference to the continuity equation. In such a case it is obvious that the numerical scheme for the compressible equations cannot converge to the numerical scheme for the incompressible equations. Furthermore, for low Mach number flows the density is almost constant and so the higher order difference of the density does not add much of a viscosity to the continuity equation. As such, we conclude that the artificial viscosity for the compressible continuity equation should be based on pressure and not density (at least for low Mach numbers).

By convergence of the compressible equations to the incompressible equations we are merely verifying what happens to the difference equations as the Mach number goes to zero. This gives neccessary but not sufficient conditions for the convergence of the solution of the numerical approximation to the preconditioned compressible equations to the numerical solution of the incompressible equations. We first consider a central difference approximation together with a scalar artificial viscosity for the nondimensionalized preconditioned inviscid equations. For the incompressible equations in nonconservative form we consider the preconditioned system

$$\begin{aligned}
p_t + \beta^2(u_x + v_y) &= h\left[(K_1 p_x)_x + (K_2 p_y)_y\right] \\
\frac{u}{\beta^2} p_t + u_t + u u_x + v u_y + p_x &= h\left[(K_1 u_x)_x + (K_2 u_y)_y\right] \qquad (6.5) \\
\frac{v}{\beta^2} p_t + v_t + u v_x + v v_y + p_y &= h\left[(K_1 v_x)_x + (K_2 v_y)_y\right]
\end{aligned}$$

where each space derivative is approximated by a central difference with spacing h in each direction. We can also consider higher order accurate artificial viscosities but this doesn't change any of the properties that are of interest for the convergence to the incompressible limit. The time derivatives are replaced by a multistage scheme. K_1, K_2 are proportional to the largest eigenvalues of the coefficient matrix in the respective direction.

We next consider the same scheme for the preconditioned compressible inviscid equations, under the assumption that $p = p(\rho)$. It easier to analyze the convergence for the nonsymmetrical form since the pressure, p, converges and not $\frac{dp}{\rho a}$, see (6.4). Then for the compressible equations we have

$$\begin{aligned}
p_t + \beta^2 \left[(\rho u)_x + (\rho v)_y)\right] &= h\left[(K_1 p_x)_x + (K_2 p_y)_y\right] \\
\frac{u}{\beta^2} p_t + u_t + u u_x + v u_y + \frac{p_x}{\rho} &= h\left[(K_1 u_x)_x + (K_2 u_y)_y\right] \\
\frac{v}{\beta^2} p_t + v_t + u v_x + v v_y + \frac{p_y}{\rho} &= h\left[(K_1 v_x)_x + (K_2 v_y)_y\right]
\end{aligned} \tag{6.6}$$

Comparing (6.5) with (6.6) it is obvious that if $\rho \to 1$ as $M \to 0$ then (6.5) converges to (6.6). It is crucial for both the time derivative and the artificial viscosity in the compressible continuity equation to be pressure based rather than density based. The preconditioning of the momentum equations is not important for this convergence.

For the incompressible equations in conservative form we multiply the first equation in (6.5) by u and add it to the second and third equations. However, we do not change the artificial viscosity. Then

$$\begin{aligned}
p_t + \beta^2 (u_x + v_y) &= h\left[(K_1 p_x)_x + (K_2 p_y)_y\right] \\
\frac{2u}{\beta^2} p_t + u_t + (u^2 + p)_x + (uv)_y &= h\left[(K_1 u_x)_x + (K_2 u_y)_y\right] \\
\frac{2v}{\beta^2} p_t + v_t + (uv)_x + (v^2 + p)_y &= h\left[(K_1 v_x)_x + (K_2 v_y)_y\right]
\end{aligned} \tag{6.7}$$

For the compressible equations in conservative form we have two choices. One choice is to multiply the first equation in (6.6) by u and the second by ρ and add the two. The spatial derivatives are then in conservation form. However, the time derivative is of the form ρu_t rather than $(\rho u)_t$ and the artificial viscosity terms are not in conservation form. Hence, we instead choose to apply the preconditioning directly to the conservative form. The resultant preconditioned compressible equations in conservative form is

$$\begin{aligned}
p_t + \beta^2 \left[(\rho u)_x + (\rho v)_y)\right] &= h\left[(K_1 p_x)_x + (K_2 p_y)_y\right] \\
\frac{2u}{\beta^2} p_t + (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y &= h\left[(K_1 (\rho u)_x)_x + (K_2 (\rho u)_y)_y\right] \\
\frac{v}{\beta^2} p_t + (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y &= h\left[(K_1 (\rho v)_x)_x + (K_2 (\rho v)_y)_y\right]
\end{aligned} \tag{6.8}$$

Note that (6.8) is not equivalent to (6.6).

We now consider how to construct a general artificial viscosity that will enable us to reach the incompressible limit without the difficulties previous presented. Consider

$$\mathbf{P}^{-1} w_t + f_x + g_y = h\left[(Q_1 w_x)_x + (Q_2 w_y)_y\right] \quad . \tag{6.9}$$

We wish to find the dependence of $\mathbf{P}$ and Q_i on the Mach number as $M \to 0$ so that we get the proper convergence. We therefore consider the isentropic equations based on $w = (\phi, u, v)$ (see (6.3)). This has the symmetric form

$$\mathbf{P}^{-1} w_t + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{11} & a_{23} & a_{33} \end{pmatrix} w_x + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{11} & b_{23} & b_{33} \end{pmatrix} w_y = h\left[(Q_1 w_x)_x + (Q_2 w_y)_y\right] \quad .$$

As $M \to 0, a_{12}$ and $b_{13} = O(1/M)$ while $d\phi = O(M)$ while all other quantities are bounded. Hence, the leading terms in the first equation are all $O(1/M)$ while they are $O(1)$ for the second and third equations. Multiplying the first equation by M and taking the limit we get $u_x + v_y$ for the space derivatives on the left hand side. Using $d\phi = O(M), du = O(1), dv = O(1)$ we see that a necessary condition for convergence as $M \to 0$ is that P^{-1}, Q_1, Q_2 all have the form

$$\mathbf{P}^{-1}, \mathbf{Q_1}, \mathbf{Q_2} \sim \begin{pmatrix} O(1/M^2) & O(1/M) & O(1/M) \\ O(1/M) & O(1) & O(1) \\ O(1/M) & O(1) & O(1) \end{pmatrix} \quad . \tag{6.10}$$

If we would have considered the variables (p, u, v) instead of (ϕ, u, v) then all the terms would have been of order 1. However, the matrices A, B would not be symmetric. Hence, each set has its advantages and we shall consider both sets of variables.

As an example, consider the equations in $(\frac{dp}{\rho c}, du, dv)$ variables. Then

$$\mathbf{A_{comp}} = \begin{pmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{pmatrix}, \qquad \mathbf{B_{comp}} = \begin{pmatrix} v & 0 & c \\ 0 & v & 0 \\ c & 0 & v \end{pmatrix} \quad .$$

Since these equations are nondimensionalized $c = O(1/M)$. Let

$$\mathbf{D} = A\omega_1 + B\omega_2 = \begin{pmatrix} Rq\omega_1 & \omega_1 c & \omega_2 c \\ \omega_1 c & Rq\omega_1 & 0 \\ \omega_2 c & 0 & Rq\omega_1 \end{pmatrix},$$

where $q^2 = u^2 + v^2$, $R = \frac{u\omega_1 + v\omega 2}{q}$ and we define $\mathbf{P_{comp}^{-1}}$ and $\mathbf{P_{comp}}$ by

$$\mathbf{P^{-1}} = \begin{pmatrix} \frac{2}{M^2} - 1 & \frac{u}{qM} & \frac{v}{qM} \\ \frac{u}{qM} & 1 & 0 \\ \frac{v}{qM} & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \frac{1}{1 - M^2} \begin{pmatrix} M^2 & -\frac{Mu}{q} & -\frac{Mv}{q} \\ -\frac{Mu}{q} & 1 - M^2 + \frac{u^2}{q^2} & \frac{uv}{q} \\ -\frac{Mv}{q} & \frac{uv}{q} & 1 - M^2 + \frac{v^2}{q^2} \end{pmatrix}.$$

By inspection $\mathbf{P_{comp}^{-1}}$ has the form desired of (6.10). To find the form of the artificial viscosity we return to the numerical approximation for

$$\mathbf{P}^{-1} w_t + f_x + g_y = h\left[(Q_1 w_x)_x + (Q_2 w_y)_y\right] \quad .$$

For constant $\mathbf{P}$ we want $\mathbf{P}Q_1 = |\mathbf{P}A|$. Hence, $Q_1 = \mathbf{P}^{-\mathbf{1}}|\mathbf{P}A|, Q_2 = \mathbf{P}^{-\mathbf{1}}|\mathbf{P}B|$. Then

$$(\mathbf{P}^{-\mathbf{1}}|\mathbf{PD}|)_{comp} = q \begin{pmatrix} \frac{S+R}{M^2} & \frac{|u|R}{qM} & \frac{|v|R}{qM} \\ \frac{|u|R}{qM} & \frac{u^2|R|}{q^2} + \frac{v^2 S}{q^2(1-M^2)} & \frac{|uv|R}{q^2} - \frac{|uv|S}{q^2(1-M^2)} \\ \frac{|v|R}{qM} & \frac{|uv|R}{q^2} - \frac{|uv|S}{q^2(1-M^2)} & \frac{v^2|R|}{q^2} + \frac{u^2 S}{q^2(1-M^2)} \end{pmatrix},$$

where $q^2 = u^2 + v^2$, $R = \frac{u\omega_1 + v\omega_2}{q}$, $S = \sqrt{(1-M^2)(1-M^2R^2)}$.

Letting $M \to 0$ we get $S \to 1$. To get the right limit of $(\mathbf{P}^{-\mathbf{1}}|\mathbf{PD}|)_{comp}$ we must first multiply each element by the appropriate factor of M (or equivalently go to (p,u,v) variables). We then get

$$\left(\frac{dp}{\rho c}, du, dv\right)_{comp} \to \left(\frac{dp}{q}, du, dv\right)_{incomp}$$

and

$$(\mathbf{P}^{-\mathbf{1}}|\mathbf{PD}|)_{comp} \to (\mathbf{P}^{-\mathbf{1}}|\mathbf{PD}|)_{incomp} = q \begin{pmatrix} |R|+1 & \frac{|uR|}{q} & \frac{|vR|}{q} \\ \frac{|uR|}{q} & \frac{u^2|R|+v^2}{q^2} & \frac{|uv|(|R|-1)}{q^2} \\ \frac{|vR|}{q} & \frac{|uv|(|R|-1)}{q^2} & \frac{v^2|R|+u^2}{q^2} \end{pmatrix}.$$

Changing from the variables $(\frac{dp}{q}, du, dv)_{incomp}$ to $(dp, du, dv)_{incomp}$ we get the same viscosity matrix given by Arnone and Turkel [3].

We next point out that the conditions on the matrix $(\mathbf{P}^{-\mathbf{1}}|\mathbf{PD}|)_{\mathbf{comp}}$ are not satisfied by the non-preconditioned Roe matrices. Furthermore, even reasonable preconditioners need not satisfy these conditions. Consider, for example, the one dimensional system

$$\mathbf{P}^{-\mathbf{1}} w_t + A w_x = h(Q w_x)_x.$$

A reasonable choice is $\mathbf{P}^{-\mathbf{1}} = |A|$ i.e. $\mathbf{P} = |A^{-1}|$. In this case all the wave speeds of $\mathbf{P}A$ are ±1. Now

$$Q = \mathbf{P}^{-1}|\mathbf{P}A| = |A| \left| |A|^{-1} A \right| = |A| \sim \begin{pmatrix} O(\frac{1}{M}) & O(1) & O(1) \\ O(1) & O(\frac{1}{M}) & O(1) \\ O(1) & O(1) & O(1) \end{pmatrix},$$

i.e. Q is the nonpreconditioned Roe matrix which does not have the desired property. We therefore conclude that for an upwind difference scheme the Riemann solver should be based on the preconditioned system and not the original scheme. In [11] plots are shown to illustrate the greatly improved accuracy for low Mach number flows when the Riemann solver is based on the preconditioning. For central difference schemes with a scalar artificial viscosity the viscosity in the continuity equation should include pressure terms. For a matrix viscosity $Q_1 = \mathbf{P}^{-\mathbf{1}}|\mathbf{P}A|, Q_2 = \mathbf{P}^{-\mathbf{1}}|\mathbf{P}B|$. Characteristics in the boundary conditions these should be based on the characteristics of the modified system and not the physical system. When using multigrid it is better to transfer the residuals based on the preconditioned system to the next grid since these residuals are more balanced than the physical residuals. However, conservation of the variables is now lost. Preconditioning is even more important when using multigrid than with an explicit scheme. With the original system the disparity of the eigenvalues greatly affects the smoothing rates of the slow components and so slows down the multigrid method, [23]. As indicated above there are accuracy difficulties at low Mach numbers [43]. Some of these can be alleviated by preconditioning the dissipation terms. For very small Mach numbers there is also a difficulty with round off errors as $\frac{p}{u^2+v^2} \to \infty$. Several people have suggested subtracting out a constant pressure from the dynamic pressure. A more detailed analysis [14] suggests replacing the pressure p by $\tilde{p}$ where $p = \frac{p_0+\epsilon\tilde{p}}{\epsilon^2}$ and ϵ is a representative Mach number.

We conclude from the above remarks that the steady state solution of the preconditioned system may be different from that of the physical system. Thus, on the finite difference level the preconditioning can improve the accuracy as well as the convergence rate.

7 Artificial Viscosity

We have seen that the details of the artificial viscosity (or equivalently the Roe matrix) determines whether the approximation to the compressible equation converges to that of the incompressible equations as the Mach number approaches zero. The preconditionings discussed i.e. Algorithm I, Algorithm II and Van Leer et. al. all satisfy the necessary condition (6.10). We have previously shown that the "natural" one dimensional preconditioning $P^{-1} = |A|$ does not satisfy (6.10). Similarly the preconditioner proposed by Allmaras though it has nice high frequency properties provides no help for the low Mach limit.

We now discuss the implementation of the artificial viscosity in more detail.

7.1 Basic Dissipation Scheme

For a central difference scheme it is necessary to add artificial dissipative terms to the central difference approximation of the spatial derivatives. One usually adds a nonlinear second difference to control oscillations near shocks and a linear fourth difference to damp high frequency oscillations [16]. We are interested in the functional form of these differences. Hence, our examples will be with a second-difference artificial dissipation. Extensions to fourth differences and nonlinearities are straightforward. Similarly, for one sided schemes the central difference plus artificial dissipation is replaced by a Roe matrix formulation. Extension of our formulation to these schemes is also straightforward.

A typical preconditioned finite difference schemes can be expressed in differntial form as

$$\Delta w_c = \Delta t P \left[\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right] = \Delta t P \left[A \frac{\partial w_c}{\partial x} + B \frac{\partial w_c}{\partial y} \right] \tag{7.1.1}$$

where w_c represents the conservative variables and the partial spatial derivatives are replaced by central difference approximations, either cell centered or at nodal points. We shall consider both conservative and non-conservative ways of adding artificial dissipative terms. The dissipation need not be expressed in terms of the conservative variables w_c. Using another set of variables w_V we have $\Delta w_c = \frac{\Delta w_c}{\Delta w_V} \Delta w_V$. Using different variables we can introduce the appropriate preconditioner in two ways. One choice is to completely change variables in all places and the the matrices A and B in (7.1.1) change as well as the preconditioner P. The new precondioner P_i is then a function of P_0, see (3.2), given by $P_i = \frac{\partial w_i}{\partial w_0} P_0 \frac{\partial w_0}{\partial w_i}$. An alternative is to leave the spatial derivatives in conservation form and to transform only the time derivatives. Essentially following Choi and Merkle's notation we denote this preconditioner by Γ (our Γ is the inverse of that used by Choi and Merkle). Thus, in conservative variables we have $\Gamma_c = P_c$. We choose as our "basic" form that given in (p, u, v, S) coordinates (i.e. P_0). Let w_c be the conservative variables then for any other set of variables we have $\Gamma_i = \frac{\partial w_i}{\partial w_c} P_c = P_i \frac{\partial w_i}{\partial w_c} = \frac{\partial w_i}{\partial w_0} P_0 \frac{\partial w_0}{\partial w_c}$.

We now include an artificial dissipation and replace (7.1.1) by differences yielding:

$$\frac{\Delta (w_V)_{i,j}}{\Delta t} = (\Gamma_V)_{i,j} \left[\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} \right] - \left[\frac{d_{i+1/2,j} - d_{i-1/2,j}}{\Delta x} + \frac{d_{i,j+1/2} - d_{i,j-1/2}}{\Delta y} \right]. \tag{7.1.2}$$

where

$$d_{i+1/2,j} = \left(|\sigma(P_V A_V)| \right)_{i+1/2,j} \left((w_V)_{i+1,j} - (w_V)_{i,j} \right).$$

$$d_{i,j+1/2} = \left(|\sigma(P_V B_V)| \right)_{i,j+1/2} \left((w_V)_{i,j+1} - (w_V)_{i,j} \right).$$

$A_V = \frac{\partial w_V}{\partial w_c} A \frac{\partial w_c}{\partial w_V}$, B is defined similarly, and σ is matrix function. For example, if σ is proportional to the spectral radius then we have a scalar artificial viscosity, in w_V variables, while if $\sigma(A) \sim A$ we have a matrix-valued artificial viscosity. Once we have found Δw_V we can either update this to find w_V or else we can convert back to find Δw_c. Thus, for example, residual smoothing and multigrid can be done on either the w_c variables or the w_V variables.

The formula (7.1.2) is easy to implement. For a scalar viscosity (i.e. σ is a scalar function) the viscosity remains scalar after preconditioning. The main difference is that now differences of the w_V variables are added to each of the conservation variables rather than differences of the conserved variables. We note that in the original paper of Jameson, Schmidt and Turkel [16] differences of ρH rather than E were added to the energy equation. However, the formulation (7.1.2) is nonconservative. Since we shall also be interested in transonic flows with shocks we present the addition of the artificial dissipation in a conservative formulation. Furthermore, for internal flow the conservative form is important even for subsonic flows without shocks in order to preserve global quantities, e.g. total mass flow. We introduce a conservative formulation:

$$\frac{\Delta (w_V)_{i,j}}{\Delta t} = (\Gamma_V)_{i,j} \left\{ \frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} - \left[\frac{d_{i+1/2,j} - d_{i-1/2,j}}{\Delta x} + \frac{d_{i,j+1/2} - d_{i,j-1/2}}{\Delta x} \right] \right\}. \tag{7.1.3}$$

where

$$d_{i+1/2,j} = \Gamma^{-1}_{i+1/2,j} \left(|\sigma(P_V A_V)| \right)_{i+1/2,j} \left[(w_V)_{i+1,j} - (w_V)_{i,j} \right].$$

$$d_{i,j+1/2} = \Gamma^{-1}_{i,j+1/2} \left(|\sigma(P_V B_V)| \right)_{i,j+1/2} \left[(w_V)_{i,j+1} - (w_V)_{i,j} \right].$$

Even when σ is a scalar function we now have to evaluate matrix-vector product and so the numerical effort corresponds to a matrix-valued viscosity. We should mention that

$A_V = \frac{\partial w_V}{\partial w_c} A \frac{\partial w_c}{\partial w_V}$, and similar expressions can be written for the other coordinate directions. Let $\hat{A_V} = \frac{\partial F}{\partial w_V}$ be the Jacobian matrix; then, $\hat{A_V} = \frac{\partial w_c}{\partial w_V} A_V$. Hence,

$$P_V A_V = \frac{\partial w_V}{\partial w_c} P_c A_c \frac{\partial w_c}{\partial w_V} = \Gamma_V A_c \frac{\partial w_c}{\partial w_V}$$

Because the spectral radius is invariant under a similarity transformation for a scalar viscosity, we can use either $P_V A_V$ or $P_c A_c$.

In the present work we present numerical results which are obtained for $\sigma(PA)$ and $\sigma(PB)$ defined as scalar functions. Following previous work on multigrid schemes for the Navier-Stokes equations these functions take into account the ratio of the spectral radii in the different coordinate directions, that is

$$\sigma(PA) = k^{(4)} \tilde{\sigma}(PA) \left(1 + \left(\frac{\tilde{\sigma}(PB)}{\tilde{\sigma}(PA)}\right)^{\zeta}\right) \tag{7.1.4}$$

where $\tilde{\sigma}$ denotes the spectral radius and $k^{(4)}$ is the artificial viscosity coefficient corresponding to fourth-difference terms. Similar formulas occur in the other directions. These scaling functions yield sufficient artificial dissipation for general curvilinear grids with cells of high aspect ratio.

In previous sections we introduced a preconditioner that depended on parameters α and δ. δ does not affect the eigenvalues of PA and so does not affect a scalar artificial viscosity. $\alpha = 1$ reduces the largest eigenvalue compared with $\alpha = 0$. This improves the condition number but also decreases the artificial viscosity. We thus expect $\alpha = 1$ to converge slower than $\alpha = 0$ in spite of the fact that we have reduced the condition number. However, we expect $\alpha = 1$ to be more accurate. For a matrix viscosity (or a Roe matrix) we expect similar but a less pronounced dependence on α.

For a matrix artificial viscosity coupled with preconditioning (either conservative or non-conservative) we need to evaluate $|P_V A_v|$ and similarly for the other directions. We never need the matrices by themselves only a matrix vector product. We now show how to calculate this efficiently.

To do all the cases together and in three dimensions we define $D = A\omega_1 + B\omega_2 + C\omega_3$ and let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_3)$, where λ_1, λ_2, and λ_3 represent λ_+, λ_-, and λ_0, (i.e. the eigenvalues of PD), modified by cutoffs near the stagnation points. Define $|\Lambda| = \text{diag}(|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_3|, |\lambda_3|)$. Then, $|PD| = R^{-1}|\Lambda|R$ and for an arbitrary vector $x = (x_1, x_2, x_3, x_4, x_5)^t$, we have $|PD|x = (R^{-1}|\Lambda|)z$, where $z = Rx$. Define $Y = \omega_1 x_2 + \omega_2 x_3 + \omega_3 x_4$. Then

$$z = \begin{pmatrix} \frac{1}{\lambda_+ - \lambda_-}\left[\frac{\lambda_+ - (1-\alpha)q}{\rho\beta^2} x_1 + Y\right] \\ -z_1 + \frac{x_1}{\rho\beta^2} \\ z_3 \\ z_4 \\ (1-\gamma)x_1 + \frac{\gamma p}{T} x_5 \end{pmatrix}$$

where z_1 is the first element of the vector z. We shall shortly see that z_3 and z_4 do not appear by themselves in subsequent formulas but rather in conjunction with other variables in the form $\omega_2 z_3 = -x_2 + \frac{1}{(\lambda_+ - q)(\lambda_- - q)}\left[\frac{\alpha(u|\omega|^2 - q\omega_1)}{\rho}x_1 - (\beta^2\omega_1 - \alpha u q)Y\right]$ and $\omega_1 z_3 + \omega_3 z_4 = x_3 - \frac{1}{(\lambda_+ - q)(\lambda_- - q)}\left[\frac{\alpha(v|\omega|^2 - q\omega_2)}{\rho}x_1 - (\beta^2\omega_2 - \alpha v q)Y\right]$. So

$$|PD|x = R^{-1}|\Lambda|z = \left(\begin{array}{c} \rho\beta^2(|\lambda_1|z_1 + |\lambda_2|z_2) \\ \frac{\beta^2\omega_1 - \alpha u\lambda_+}{\lambda_+ - q}|\lambda_1|z_1 + \frac{\beta^2\omega_1 - \alpha u\lambda_-}{\lambda_- - q}|\lambda_1|z_2 - (\omega_2 z_3)|\lambda_3| \\ \frac{\beta^2\omega_2 - \alpha v\lambda_+}{\lambda_+ - q}|\lambda_1|z_1 + \frac{\beta^2\omega_2 - \alpha v\lambda_-}{\lambda_- - q}|\lambda_1|z_2 - (\omega_1 z_3 + \omega_3 z_4)|\lambda_3| \\ \frac{\beta^2\omega_3 - \alpha w\lambda_+}{\lambda_+ - q}|\lambda_1|z_1 + \frac{\beta^2\omega_3 - \alpha w\lambda_-}{\lambda_- - q}|\lambda_1|z_2 - (\omega_2 z_3)|\lambda_3| \\ \frac{T}{\gamma p}\left[(\gamma - 1)V_1 + |\lambda_3|z_5\right] \end{array}\right)$$

where V_1 is the first element of this vector.

7.2 Preconditioned CUSP Scheme

In this section the use of specific dissipation schemes designed for shock capturing with a single-point shock structured is presented. More specifically, we describe the use of the *CUSP* scheme in combination with preconditioning.

We briefly review our previous description of *CUSP* for a Cartesian mesh. Extensions to a general mesh are straightforward.

$$\Delta w_c = \Delta t\left[\mathcal{L}_C + \mathcal{L}_{AD}\right], \tag{7.2.1}$$

where w_c represents the conservative variables. $\mathcal{L}_C$ is the central difference approximation to the inviscid flux balance (possibly also including the physical viscous Navier-Stokes terms) while $\mathcal{L}_{AD}$ represents the artificial dissipation as a linear combination of second differences of the state vector w and F where F is the flux through a cell face.

$$\mathcal{L}_{AD} = -\left[\frac{d_{i+1/2,j} - d_{i-1/2,j}}{\Delta x} + \frac{d_{i,j+1/2} - d_{i,j-1/2}}{\Delta y}\right],$$

$$d_{i+1/2,j} = \left[\frac{\nu c}{2}(w_{i+1,j} - w_{i,j}) + \frac{\beta}{2}(F_{i+1/2,j} - F_{i,j})\right]|\vec{S}|$$

The dissipation coefficients, ν and β, can be defined such that a steady-state shock structure with a single interior point is obtained for shocks aligned with the cell face $(i + 1/2, j)$. The particular form chosen for these parameters supports a clean shock capturing provided that the convective cell face Mach number is larger than 0.5. Moreover, it was shown that the eigenvalues of the artificial viscosity matrix are all equal to the convective cell face velocity for Mach numbers less than 0.5.

These observations yield sufficient guidance for the selection of appropriate preconditioning parameters for *CUSP*: First, the preconditioning matrix, P_c should be turned off within the shock structure so that the shock capturing properties of the *CUSP* scheme are not perturbed. This is accomplished with the choice $\alpha = 0, M_0 = 0.5$ in (3.3). The nonconservative preconditioned *CUSP* can then be written as

$$\Delta w_c = \Delta t \left[P_c \mathcal{L}_C + \mathcal{L}_{AD} \right], \tag{7.2.2}$$

Note, that we use conservative w_c variables and not (p, u, v, T) variables. The nonconservative form should give clean shocks if the post-shock Mach number is larger than 0.5. This is true, in particular, for transonic flows which create only moderate strength shocks. However, low post-shock Mach numbers can be observed for hypersonic flows. This is a partial motivation for considering a conservative form for the dissipation. This conservative is given by

$$\Delta w_c = \Delta t P_c \left[\mathcal{L}_C - ((P_c w)_{i+1/2,j} - (P_c w)_{i-1/2,j}) - ((P_c w)_{i,j+1/2} - (P_c w)_{i,j-1/2}) \right], \tag{7.2.3}$$

This form is conservative for all Mach numbers. The shock capturing analysis holds for all cell faces where the local Mach number is larger than 0.5 and so $P_c = I$. Hence, this preconditioned scheme is consistent with the basic *CUSP* scheme formulation. The disadvantage of the conservative formulation is that it is more expensive than the nonconservative form since P_c must be calculated at each of the neighbors $(i + 1/2, j)$ etc. and a matrix-vector multiply is needed at each neighbor rather than just at the single point (i, j) for the nonconservative formulation. The computational efficiency of the the dissipation coding is improved by using the transformations given in the appendix rather than computing P_c directly.

8 Boundary Conditions

In many codes the boundary conditions in the far field are based on characteristic variables, even for viscous flow. Thus at inflow the incoming variables, corresponding to positive eigenvalues, are specified while the outgoing variables, corresponding to negative eigenvalues, are extrapolated. Once we change the time dependent equations we also change the characteristics of the system (though not the signs of the eigenvalues). Hence, it is necessary to also modify the boundary conditions for the preconditioned system. In addition, the original characteristic variables do not scale correctly as the Mach goes to zero and so we do not obtain well-posed boundary conditions in the limit.

Since the eigenvalues and eigenvectors are invariant under a similarity transform we shall use the (p, u, v, S) set of variables to determine the new eigenvalues and characteristic variables. We shall only consider the flow normal to the boundary which we take to be a line $x = const$. Thus, we need to use the characteristic variables we found above corresponding to the left eigenvectors of $P_0 A_0$ see (3.4). Flow variables at a boundary are then obtained by taking the values for R_1, R_2 and R_+ from upstream flow whereas downstream values are used for R_-. When we diagonalize $P_0 A_0 = R D R^{-1}$ the rows of R are the left eigenvectors of $P_0 A_0$ while the columns of R^{-1} are the right eigenvectors of $P_0 A_0$.

Let variables with a subscript "b" denote the variables to be computed at the boundary. Variables with a subscript "a" denote the flow state approaching the boundary, i.e. specified at inflow, whereas variables with a subscript "l" are used for the flow leaving the boundary, e.g. extrapolated at inflow. Note, that this notation can be consistently used for inflow and outflow boundaries. The characteristic boundary conditions are expressed in (p, u, v, T) variables as:

$$p_b = \frac{\lambda_+ p_a - \lambda_- p_l}{\lambda_+ - \lambda_-} - \frac{u(1 - \alpha - \delta\frac{(\gamma-1)\beta M_r^2}{a^2})(p_a - p_l) - \rho\beta M_r^2(u_a - u_l) + \frac{\delta\rho\beta M_r^2 u}{T}(T_a - T_l)}{\lambda_+ - \lambda_-}$$

or equivalently

$$p_b = p_l + \frac{\lambda_+(p_a - p_l)}{\lambda_+ - \lambda_-} - \frac{u(1 - \alpha - \delta\frac{(\gamma-1)\beta M_r^2}{a^2})(p_a - p_l) - \rho\beta M_r^2(u_a - u_l) + \frac{\delta\rho\beta M_r^2 u}{T}(T_a - T_l)}{\lambda_+ - \lambda_-}$$

$$u_b = u_a - \frac{\lambda_+ - u(1-\alpha)}{\rho\beta M^2}(p_b - p_a) \tag{8.1}$$

$$T_b = T_a + \frac{(\gamma - 1)T}{\gamma p}(p_b - p_a)$$

$$v_b = v_a - \frac{\alpha v}{\beta M^2 - \alpha u^2}\left[\frac{p_b - p_a}{\rho} + u(u_b - u_a)\right]$$

In generalized coordinates u and v are replaced by the normal and tangential components of the velocity relative to the boundary.

Computationally we observed that a simplified set of non-characteristic conditions yielded almost the same results as those of the characteristic boundary conditions. These are given by

$$v_b = v_a, \quad p_b = p_l, \quad u_b = u_a, \qquad \text{either} \quad T_b = T_a \quad \text{or} \quad \rho_b = \rho_a + \frac{p_b - p_a}{a^2} \tag{8.2}$$

The simplified boundary condition has been used in all of the computations of section 14.

These boundary conditions are reasonable for exterior problems. However, for flow in interior regions one customarily specifies the total temperature, T_t, and total pressure, p_t, profiles together with the angle of the flow at inflow. At the exit a back pressure is usually given. Let

$$\begin{aligned} \frac{T_t}{T} &= 1 + \frac{\gamma - 1}{2}M^2 \\ \frac{p_t}{p} &= \left\{1 + \frac{\gamma - 1}{2}M^2\right\}^{\frac{\gamma-1}{\gamma}} \\ \tan(\theta) &= \frac{v}{u} \end{aligned}$$

A reasonable boundary treatment for the inflow boundary is

$$\begin{aligned}
p_b &= p_l \\
T_b &= T_t \left(\frac{p_b}{p_t}\right)^{\frac{\gamma-1}{\gamma}} \\
u_b^2 &= \frac{\frac{2\gamma}{\gamma-1}(T_t - T_b)}{1+\tan^2(\theta)} \\
v_b &= u_b \tan(\theta)
\end{aligned}$$

At the exit we specify the back pressure and extrapolate u, v, T.

9 Implementation

We present the steps used to introduce preconditioning into a previously existing code based on Algorithm I. For simplicity we shall assume an explicit iteration scheme (e.g. Runge-Kutta) possibly augmented by implicit residual smoothing and multigrid.

1. Precondition the residual
    - Select a reasonable set of variables for the problem e.g. (p, u, v, T). We stress that other sets of variables are possible. This set is used for illustration only.
    - For nonconservative scalar artificial viscosity
        - multiply physical residual (inviscid and Navier-Stokes portions and physical forcing functions (if applicable)) by $\Gamma = P_T \frac{\partial w_4}{\partial w_1}$.
        - add scalar artificial viscosity in these new variables to the residual.
    - For conservative scalar viscosity
        - viscosity includes Γ^{-1} (see (7.1.3)). Hence, even a scalar viscosity requires a matrix-vector multiply.
        - add inviscid, viscous and artificial viscosity fluxes to obtain the total residual
        - multiply this total residual by Γ
2. Perform residual smoothing. At the end of the Runge-Kutta cycle restrict the residual to the next coarser grid. Thus both residual smoothing and restrictions are performed on the residuals in (p, u, v, T) variables.
3. Having calculated the total preconditioned residual in (p, u, v, T) multiply these by Δt. There are then two ways of updating the solution to the next Runge-Kutta stage
    - One: multiply the above total residual by $\frac{\partial w_1}{\partial w_4}$ to get the residual in conservative variables. Add the new residual to the old conservative variables to get new conservative variables.

- Two: Add the residuals in (p, u, v, T) variables to the old (p,u,v,T) variables to get (p, u, v, T) at the next Runge-Kutta stage. Calculate the conservative variables as nonlinear functions of (p, u, v, T).

4. Choose a new time step based on the preconditioning (3.5).
5. Modify the far field boundaries based on the new characteristics. If characteristic data is also used at physical boundaries (e.g. inflow to a nozzle) these should also be modified. Alternatively, specify or extrapolate (p, u, v, T) at the boundaries.

For Algorithm II (section 3) all operations including residual smoothing and interpolations between grids are done in terms of the $(p', \rho u, \rho v, E')$ variables. This scheme has the advantage that, in theory, setting $M_\infty = 0$ would yield identical results to an incompressible code based on pseudo-compressibility. Algorithm I as presently implemented performs the residual smoothing and restriction in terms of (p, u, v, T) variables. However, restriction of the variables rather than residuals for the FAS algorithm is based on conservative variables. Similarly the prolongation from coarse to fine grids is based on conservative variables.

10 Time Dependent Problems

We consider two kinds of time dependent problems. For one, which is typified by acoustic problems, the requirements of accuracy and stability are approximately the same. Thus, the time restrictions required by stability of the scheme do not impose an unnecessary burden on the scheme as the same time steps are needed to resolve the changes in the time behavior of the scheme. For such problems explicit methods are the best. It is not clear how to combine this with preconditioning since the preconditioning that we have discussed until now destroys the time accuracy of the method.

A second time dependent problem is one where the time steps required by accuracy are much larger than those required by stability. One application is to forced motion where the time scale of the outside forces are much larger than the stability time step of an implicit scheme. For such problems it is useful to introduce an implicit method. However, to avoid time errors associated with large time steps it is advisable to use a fully implicit method and not an ADI type algorithm. However, this leads to the neccesity of inverting large non-sparse matrices for a linear problem plus the linearization difficulties of a non-linear proble. One alternative is to consider the implicit method as a steady state problem for the next time step. Thus, any steady solver can be used to go from time t to time $t + \delta t$. It is only required that the steady state solver converge within a short enough time that this approach is more efficient than a straightforward explicit method. Thus, for example, if accuracy requirements lead to a CFL of 100 relative to an explicit method then this approach is efficient if we can solve for the steady state in 50 iterations or less. In order for this to be feasible it is necessary to start with a good initial guess which can be obtained by extrapolation from previous time data. In addition we only want the steady state solver to converge to the discretization level of the time dependent implicit scheme. Any attempt to solve the steady state problem to lower residuals is a waste of time.

As an example consider the simple time dependent problem

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} = 0$$

We discretize this equation by the backward Euler method which is only first order in time.

$$w^{n+1} + \delta t(\frac{\partial f}{\partial x})^{n+1} = w^n$$

To solve this problem we introduce a pseudo-time variable τ and solve

$$w_\tau + \frac{w}{\delta t} + \frac{\partial f}{\partial x} = \frac{w^n}{\delta t}$$

until a steady state in τ is reached. At this level we can now introduce preconditioning into the system and replace w_τ by $P^{-1}w_\tau$ as before.

See [45, 29, 18, 24, 42] for further details. For a different approach see [13] and [15].

11 Differential Preconditioners

In the previous sections the preconditioner P was a matrix. For the nonlinear fluid dynamic equations the elements of P involved the dependent variables. There are several limitations with this approach.

We first consider a scalar equation

$$w_t + aw_x + bw_y = 0, \tag{11.1}$$

We consider a uniform cartesian mesh with constant $\Delta x, \Delta y$. We define the aspect ratio for this problem as

$$\mathbf{ar} = \text{aspect ratio} = \frac{a/\Delta x}{b/\Delta y}.$$

This can be interpreted as the ratio of time for a wave to traverse a mesh in the x direction relative to the time in the y direction. We note that the ratio $\frac{\Delta y}{\Delta x}$ is meaningless since this can be changed by a trivial change of variables.

If this aspect ratio differs greatly from one then the standard schemes will converge slowly since a time step appropriate for one direction is inappropriate for the other direction. For a scalar equation, this is an artificial problem since, in practice, the mesh would be chosen so that the aspect ratio is close to one. However, for a system of equations there are many waves. If the aspect ratio is close to one for one wave it will not be close to one for other waves. In the boundary layer for the acoustic wave $\mathbf{ar} = \frac{(u+c)/\Delta x}{(v+c)/\Delta y} \sim \frac{\Delta y}{\Delta x}$. However, for the shear wave $\mathbf{ar} \sim \frac{u}{v}\frac{\Delta y}{\Delta x}$ and away from the wall but in the boundary layer u is much larger than v. Hence, any mesh that is appropriate for the acoustic wave is not appropriate for the shear and entropy waves and vice versa. For complex geometries it is also difficult to construct a mesh that conforms with the solid surfaces and also has aspect ratios that are reasonable. In addition there are viscous effects that we are ignoring, so that in practice the mesh is constructed based on viscous effects and ignores both the acoustic and entropy waves. For the scalar equation we are considering algebraic preconditioning cannot help. For a system the preconditionings we have considered can partially rectify the difference of speeds between the various waves but does not alleviate the aspect ratio difficulty.

The matrix preconditioners we have considered until now have a second difficulty. For one dimensional flow one can choose the preconditioner as the absolute value of the matrix A. Then all the resultant waves have identical speeds with only differences in the direction, positive or negative. However, in two space dimension when the matrices A and B do not commute it is not possible, in general , to equalize all the speeds. Equivalently, we cannot diagonalize the system and reduce it to a sequence of scalar equations even for the frozen coefficient problem. To alleviate these two problems we shall allow the preconditioner **P** to contain derivatives. However, similar to section 5 we demand that for the symmetric equations that **P** be invertible and positive definite.

For the scalar equation (11.1) we considered in Part I a preconditioner based on residual smoothing [17]. This is given by

$$(1-\beta_x\partial_{xx})(1-\beta_y\partial_{yy})Res_{new} = Res_{old} \tag{11.2}$$

where Res refers to the residual before and after smoothing. This residual smoothing is usually introduced to improve the time step and smoothing properties of an explicit scheme as Runge-Kutta or Lax-Wendroff. Here, we analyze the scheme from a different perspective, that of wave speeds. We assume that the aspect ration for the problem is very large (i.e. b is large compared to a or Δy is small compared to Δx). The question we wish to address is whether β_x and β_y can be chosen so as to reduce this aspect ratio.

We first consider residual smoothing in one space dimension. In this case there is no aspect ratio. Instead we will show how the concept of wave speeds explains the phenomena that one should not use residual smoothing with a very large time step even though it can be stabilized by choosing an appropriately large β..

$$(1-\beta\partial_{xx})w_t + aw_x = 0$$

We analyze this for a semi-discrete equation with time continuous, the first x derivative approximated by a central difference and the second space derivative by a three point central difference. In order to find the phase and group velocities we consider solutions of the form $w = e^{i(kx-\omega t)}$. Here k is given and we find ω from the dispersion relation. For the one dimensional residual smoothing we have

$$k = asin\theta, \quad \omega = \frac{asin\theta/\Delta x}{1+2\beta(1-cos\theta)}$$

$\theta = k\Delta x$

To find a stability condition for a Runge-Kutta scheme in time we maximize ω and find that the worst case is $cos\theta = \frac{2\beta}{1+2\beta}$. We then find that the scheme is stable if

$$\beta \geq \frac{1}{4}(r^2-1)$$

where $r = \frac{\Delta t_{new}}{\Delta t_{original}}$ .Thus, from the viewpoint of stability we can choose any time step we wish by choosing β sufficiently large. Nevertheless, one finds computationally that convergence to a steady state is slowed down by choosing Δt, and hence β, too large. Optimal values are $r \sim 2$. We shall now show from the viewpoint of wave propagation that it is not good to choose a very large time step.

Residual smoothing adds a term w_{xxt} to the original differential equation. Such a term is a dispersive term i.e. the energy is not reduced but now the speed of a plane wave is no longer constant but instead depends on the wave number. The main purpose of this term is to increase the time stability limit. However, as in defining the aspect ratio, increasing the time step is meaningful only if we normalize the solution in some way, otherwise we are merely rescaling the time dimension. Hence, the appropriate quantity is not the time step but rather the time it takes a wave to transverse one cell (assuming Δx is constant). The phase speed of a plane wave is given by

$$v_p = \frac{\omega}{k} = \frac{a}{1 + 4\beta sin^2\theta/2}$$

Let $\beta = \frac{1}{4}(r^2 - 1)$ and multiply v_p by r to get the distance transversed in time Δt. Then

$$s_p = relative\ phase\ distance = \frac{2r}{(r^2+1) - (r^2-1)cos\theta} \tag{11.3}$$

For the long wave lengths $cos\theta \sim 1$ and so $s_p \sim r$, i.e. the long wave lenths move r times further in one time step. If we look at $\theta = \pi/2$, we have $s_p = \frac{2r}{r^2+1} \leq 1$. Thus this frequency moves slower than without residual smoothing. For the highest frequency on the mesh we have $\theta = \pi$ and $s_p = \frac{1}{r}$. We therefore, see that the high frequencied are actually slowed down by the residual smoothing and so take longer to exit from the domain, furthermore the larger Δt is chosen the slower these waves go. Even more important the larger Δt is chosen the more frequencies that are slowed down even though the lowest frequencies travel faster. The breakeven frequency is given by $cos\theta = \frac{r-1}{r+1}$.

We can also consider the group velocity. For the optimal β this is given by

$$v_g = \frac{d\omega}{dk} = 2\frac{(r^2+1)cos\theta - (r^2-1)cos2\theta}{[(r^2+1) - (r^2-1)cos\theta]^2}$$

and

$$s_g = s_p\frac{(r^2+1)cos\theta - (r^2-1)cos2\theta}{(r^2+1) - (r^2-1)cos\theta} \tag{11.4}$$

The situation now is even less favorable than before. Again, the lowest frequencies are sped up by a factor r. The frequency $\theta = \pi/2$ is slowed down by an additional factor of $\frac{r^2-1}{r^2+1}$ and the highest frequency $\theta = \pi$ now reverses direction and goes upstream .

Because we are considering the semi-discrete equation and residual smoothing is purely dispersive there is no damping of the waves. For a Runge-Kutta scheme one finds that as r is increased that the damping of high frequencies decreases. Thus, for large r the high frequencies do not propagate very fast and are not damped either. This explains one in practice one chooses an r of about two for the greatest increase in the convergence rate to a steady state.

We next consider the two dimensional equation. To ease the derivations we shall consider the partial differential equation (11.2) rather than the finite difference approximation. We rewrite (11.2) as

$$u_t - \beta_x u_{xxt} - \beta_y u_{yyt} + \beta_x \beta_y u_{xxyyt} = a u_x + b u_y \tag{11.5}$$

We are interested in the effect of high aspect ratios. So we consider $\Delta y << \Delta x$. By rescaling we instead consider a uniform mesh but $a << b$. In particular we shall choose $a = \epsilon, \quad b = 1$.

Consider solutions of the form $u = e^{i(k_x x + k_y y - \omega t)}$ or equivalently $u = e^{i(\vec{k}\vec{x} - \omega t)}$ where $\vec{k} = (k_x, k_y)$ and $\vec{x} = (x, y)$. Substituting this into (11.5) we get

$$\omega(k_x, k_y) = \frac{\epsilon k_x + k_y}{(1 + \beta_x k_x^2)(1 + \beta_y k_y^2)}$$

Hence, $\omega(1,0) = \frac{\epsilon}{1+\beta_x}$ and $\omega(0,1) = \frac{\epsilon}{1+\beta_y}$. If we want these to be equal then we need $\beta_x = O(1), \beta_y = O(\frac{1}{\epsilon})$. This is different than what is normally chosen for in residual smoothing [31].

We now consider differential preconditioners for the Euler equations. We shall only consider the linearized equations with constant coefficients. We first rewrite (6.1) in differential form. The Euler equations can be written as

$$w_t + \mathbf{L} w = 0 \tag{11.6}$$

with $w = (p, u, v, S)^t$. Define

$$Q = u\partial_x + v\partial_y$$

Since, all coefficients are assumed constant Q commutes with ∂_x and ∂_y and

$$\mathbf{L} = \begin{pmatrix} Q & \rho c^2 \partial_x & \rho c^2 \partial_y & 0 \\ \frac{1}{\rho}\partial_x & Q & 0 & 0 \\ \frac{1}{\rho}\partial y & 0 & Q & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \tag{11.7}$$

Let

$$D = Q^2 - c^2(\partial_x^2 + \partial_y^2). \tag{11.8}$$

We now replace (11.6) by the preconditioned system

$$w_t + \mathbf{P_D} \mathbf{L} w = 0 \tag{11.9}$$

with

$$\mathbf{P_D} = \begin{pmatrix} Q^2 & -\rho c^2 \partial_x Q & -\rho c^2 \partial_y Q & 0 \\ -\frac{1}{\rho}\partial_x & Q^2 - c^2\partial_y^2 & c^2\partial_x\partial_y & 0 \\ -\frac{1}{\rho}\partial_y & c^2\partial_x\partial_y & Q^2 - c^2\partial_x^2 & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \tag{11.10}$$

One can then verify that

$$\mathbf{P_D L} = QD\mathbf{I}, \qquad \mathbf{P}_D^{-1} = D^{-1}Q^{-1}\mathbf{L}$$

One can replace the D in the lower right corner of $\mathbf{P}_D$ by the identity matrix. Then $\mathbf{P_D L}$ is not the identity matrix but is still a diagonal matrix. We can use simpler matrices than $\mathbf{P}_D$ by considering congruent transformations. Consider the symmetrizing variables $(\frac{dp}{\rho c}, u, v, S)$, then

$$\mathbf{L} = \begin{pmatrix} Q & c\partial_x & c\partial_y & 0 \\ c\partial_x & Q & 0 & 0 \\ c\partial y & 0 & Q & 0 \\ 0 & 0 & 0 & Q \end{pmatrix}$$

Let,

$$\mathbf{P_E} = \begin{pmatrix} Q & -c\partial_x & -c\partial_y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P_E^t} = \begin{pmatrix} Q & 0 & 0 & 0 \\ -c\partial_x & 1 & 0 & 0 \\ -c\partial_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.11}$$

then

$$\mathbf{P_E L P_E^t} = \begin{pmatrix} DQ & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & Q \end{pmatrix}$$

so we have diagonalized L by a congruent transformation. But,

$$\mathbf{P_E L P_E^t} = \mathbf{P_E^{t^{-1}}}(\mathbf{P_E^t P_E L})\mathbf{P_E},$$

so the congruent transform is similar to a preconditioning with a positive definite matrix $\mathbf{P_E^t P_E}$. Alternatively, $(\mathbf{P_E^t P_E})\mathbf{L}$ is similar to a diagonal matrix.

$$\mathbf{P_E^t P_E} = \begin{pmatrix} Q & -c\partial_x & -c\partial_y & 0 \\ -c\partial_x & 1 + c^2\partial_x^2 & c^2\partial_x\partial_y & 0 \\ -c\partial_y & c^2\partial_x\partial_y & 1 + c^2\partial_y^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $\mathbf{P_E^t P_E}$ looks similar to $\mathbf{P_D}$ but is not identical. $\mathbf{P_E^t P_E}$ has fewer derivatives along the identical but $\mathbf{P_E^t P_E L}$ is only similar to a diagonal matrix while $\mathbf{P_D L}$ is diagonal and even a scalar differential operator multiplying the identity matrix. These transformations are independent of the flow regime as long as the preconditioner is nonsingular.

These preconditioners are connected with the techniques used in distributive Gauss-Seidel smoothers for multigrid methods [5].

It remains to show that **P** is nonsingular. We have four eigenvalues and corresponding eigenfunctions. As usual the entropy wave decouples. For this wave **P** has an eigenvalue D and an eigenfunction (0,0,0,1) . For the shear wave **P** has an eigenvalue D and the eigenvector is $(v_1, v_2, v_3, 0)$ where

$$\begin{aligned} Dv_1 &= 0 \\ D(\frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial y}) &= 0 \end{aligned}$$

The other two 'acoustic' eigenvalues of **P** are $Q^2 \pm cQ\sqrt{\partial_x^2 + \partial_y^2}$ and the eigenvectors satisfy the pseudo-differential equation

$$\begin{aligned} \left[c(\partial_x^2 + \partial_y^2) \pm Q\sqrt{\partial_x^2 + \partial_y^2} \quad \right] \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_2}{\partial y}\right) &= 0 \\ \mp\sqrt{\partial_x^2 + \partial_y^2} = \rho c \left(\frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial y}\right) \end{aligned}$$

We therefore have to show that the eigenvalues are all nonzero so that **P** is nonsingular. The operator D is just the potential operator i.e. for any variable w

$$Dw = (u^2 - c^2)w_{xx} + 2uvw_{xy} + (v^2 - c^2)w_{yy}$$

For subsonic flow this is an elliptic operator and so invertible. For supersonic flow D is a hyperbolic operator . Similarly, Q is a hyperbolic operator denoting convection along a streamline. Thus, given appropriate boundary conditions this too should be invertible. At a stagnation point Q is singular and so it is necessary to limit the values of u and v in the definition of Q so that they do not become too small in a neighborhood of the stagnation point. A similar smoothing is needed near the sonic line.

With residual smoothing and $\mathbf{P_D}$ we have increased the order of the system and so changed the number of boundary conditions needed for the equation to be well posed. To avoid this difficulty we do not solve the equation (11.9). Instead these preconditioners are used as a post processor for the usual Euler or Navier-Stokes equations. Thus, at each time step we calculate a residual based on one's favorite scheme. This gives a predicted value of the change in time, $\Delta w_{predicted}$. We also update the boundary conditions for the standard fluid dynamic equations. We then operate on Δw with **P** with the boundary condition that $\Delta w_{corrected} = 0$, i.e. we don't change the boundary values calculated by the predictor. When we reach a steady state for the fluids equations we are solving $\mathbf{P}\Delta w_{corrected} = 0$ with zero boundary conditions. Since **P** is invertible $\Delta w = 0$, i.e. we preserve the steady state. Thus, in essence we are imposing the fluid dynamic boundary conditions between the **P** operator and the **L** operator.

We note that if the partial derivarives are all replaced by central differences then everything goes over since it is based only a formal multiplication and uses no properties of the derivative. Hence, beginning with a central difference approximation to the constant coefficient multidimensional Euler equations we can precondition it so that we have a set of scalar equations. This applied equally well for any aspect ratio. Hence, we need consider high aspect ratio problems only for the scalar equation. This use of the differential preconditioner is similar to the explicit/implicit residual averaging previously considered except that now the explicit portion is a matrix involving second differences rather than the scalar second differences considered before.

12 Canonical Decomposition

This is joint unpublished work with Phil Roe.

12.1 Introduction

We shall consider first-order systems of partial differential equations. Within this class we seek to identify canonical forms to which the systems may be reduced. Our analysis is restricted to the linear case with constant coefficients.

In one space dimension, any strongly hyperbolic $(m \times m)$ system of the form

$$\partial_t \mathbf{u} + A \partial_x \mathbf{u} = 0 \tag{12.1}$$

is reducible to m scalar convection equations of the form

$$\partial_t (\ell_{\mathbf{k}} \cdot \mathbf{u}) + \lambda_k \partial_x (\ell_{\mathbf{k}} \cdot \mathbf{u}) = 0 \tag{12.2}$$

where $\ell_{\mathbf{k}}$ is a left eigenvector of A, and λ_k is the corresponding eigenvalue. This follows since for a strongly hyperbolic system A is diagonalizable. Thus, the canonical form is simply the familiar linear advection equation

$$\partial_t u + a \partial_x u = 0$$

In two dimensions we consider first the steady-state equations

$$A \partial_x \mathbf{u} + B \partial_y \mathbf{u} = 0 \tag{12.3}$$

When A and B are symmetric it is known [4] that this equation can be diagonalised by a congruence transformation if and only if some real linear combination of A and B is positive definite. In that case the canonical form is again linear advection

$$a \partial_x u + b \partial_y u = 0. \tag{12.4}$$

If the above condition is not met, then we shall show that some of the eigenvalues of $A^{-1}B$ appear in conjugate pairs, and a second canonical form appears which is a (2×2) system, equivalent under scaling and axis transformations to the Cauchy-Riemann equations

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{12.5}$$

The general two-dimensional system can then be written in a form that is block diagonal, with a mixture of scalar and (2×2) blocks. When the steady solution is sought by marching in a time-like variable, we show that a preconditioning matrix P can always be found, such that this block structure is preserved in the transient solution of

$$P^{-1}\partial_t \mathbf{u} + A\partial_x \mathbf{u} + B\partial_y \mathbf{u} = 0$$

The scalar blocks have then the canonical form

$$\partial_t u + a\partial_x u + b\partial_y u = 0. \tag{12.6}$$

The (2×2) blocks, although they behave elliptically in the steady state, behave hyperbolically in the transient phase, and the associated variables follow a wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{12.7}$$

Although both of these equations are hyperbolic, considered as time-dependent problems, only (12.6) is associated with any distinct direction (namely $dy/dx = b/a$). By contrast, (12.7), even during the transient phase, is associated with characteristic surfaces, $(x-x_0)^2 + (y-y_0)^2 - (t-t_0)^2 = const$, rather than with individual directions. In the steady state, the solution is clearly elliptic, satisfying Laplace's equation, and has no uniquely distinguished directions.

In three dimensions, we have found three canonical forms, which are

$$\partial_t u + a\partial_x u + b\partial_y u + c\partial_x u = 0, \tag{12.8}$$

and (3×3) systems equivalent to either

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{12.9}$$

or

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^2 \psi = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{12.10}$$

In this paper we will only present the decomposition and not discuss any applications (see however, ([25])). However, to provide motivation, one possible application would be to numerical approximations of these equations. Then one could consider applying quite different numerical techniques to the different subproblems that appear (see e.g. [34]).

We will begin by presenting the complete analysis for the general two-dimensional problems, and then apply this to the special case of the Euler equations. In the following section we shall treat, less rigorously, the general three-dimensional case, but our results for the Euler equations are precise.

12.2 The Two-dimensional Case

12.2.1 General Steady-state Problems

We consider

$$A\partial_x \mathbf{u} + B\partial_y \mathbf{u} = 0 \tag{12.11}$$

with A and B symmetric.

According to a result of Au-Yeung [4], this equation can be diagonalized by a congruence transformation if and only if some real linear combination of A and B is positive definite. In the present context, this condition has a physical interpretation. Suppose that we add a time-dependent term $\partial_t \mathbf{u}$ to (12.11). Then the speeds with which waves propagate in a direction θ are the eigenvalues of $A\cos\theta + B\sin\theta$. If this matrix is positive definite for $\theta = \theta^*$, then the domain of influence of a disturbance at the origin lies in the half-plane $x\cos\theta^* + y\sin\theta^* > 0$. Therefore the domain of influence does not enclose the origin, and tangents to it from the origin may be drawn. These are the characteristic directions for the steady solution. This situation arises whenever the flow speed is greater than the speed of any signal relative to the medium (supersonic case).

To deal with the subsonic case, we begin by attempting a formal diagonalization of (12.11). We premultiply the equation by a matrix S, and effect a change of variable $Q\mathbf{v} = \mathbf{u}$, so that

$$(SAQ)\partial_x \mathbf{v} + (SBQ)\partial_y \mathbf{v} = 0. \tag{12.12}$$

We require that SAQ, SBQ be representable as

$$\begin{aligned} SAQ &= \Lambda_A^{\#}, \\ SBQ &= \Lambda_B^{\#}, \end{aligned}$$

where $\Lambda_A^{\#}, \Lambda_B^{\#}$ denote matrices having the same block diagonal structure. (Note that matrices sharing such a structure form a closed group under multiplication and inversion.) Then

$$(\Lambda_A^{\#})^{-1}\Lambda_B^{\#} = (Q^{-1}A^{-1}S^{-1})(SBQ) = Q^{-1}A^{-1}BQ$$

so that Q must represent a similarity transformation that block diagonalises $A^{-1}B$. (If A happens to be singular we may diagonalize $B^{-1}A$ instead). At the present stage, the matrix S is still arbitrary. We may first attempt a complete diagonalisation

$$A^{-1}B = R\Lambda L \qquad \text{where} \qquad R = L^{-1}$$

If this succeeds, we can take $S = LA^{-1}$ and $Q = R$, to derive

$$\partial_x(L\mathbf{u}) + \Lambda\partial_y(L\mathbf{u}) = 0.$$

representing a set of characteristic equations for the characteristic variables $v_k = \ell_k \cdot u$ propagating along directions $dy/dx = \lambda_k$.

Suppose this fails in that

$$\det(A^{-1}B - \lambda I) = 0$$

has a pair of conjugate roots

$$\lambda_{1,2} = \lambda_R \pm i\lambda_I.$$

Then we will also find conjugate eigenvectors

$$\begin{aligned} \mathbf{r}_{1,2} &= \mathbf{r}_R \pm i\mathbf{r}_I, \\ \ell_{1,2} &= \ell_R \pm i\ell_I. \end{aligned}$$

That is,

$$A^{-1}B = \begin{bmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \mathbf{r}_1 & \mathbf{r}_2, & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \ell_1 & \cdot & \cdot \\ \cdot & \cdot & \ell_2 & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \end{bmatrix} \tag{12.13}$$

Now introduce the complex matrices

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & & \\ i & -i & & & \\ & & \sqrt{2} & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}, \quad J^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & & & \\ 1 & -i & & & \\ & & \sqrt{2} & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{12.14}$$

and rewrite the factorization

$$A^{-1}B = (RJ^T)((J^T)^{-1}\Lambda J^{-1})(JL) = R^{\#}\Lambda^{\#}L^{\#}$$

Then it can be verified that $R^{\#}, L^{\#}$ are real matrices, and that $\Lambda^{\#}$ is the real symmetric matrix

$$\Lambda^{\#} = (J^T)^{-1}\Lambda J^{-1} = \begin{bmatrix} \lambda_R & \lambda_I & & & \\ \lambda_I & -\lambda_R & & & \\ & & \lambda_3 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{12.15}$$

Now we write the governing equations in the form (12.12) with $S = (R^{\#})^{-1}A^{-1}$ and $Q = (L^{\#})^{-1}$; the result, after partial simplification, is

$$(R^{\#})^{-1}(L^{\#})^{-1}\partial_x \mathbf{v} + \Lambda^{\#}\partial_y \mathbf{v} = 0$$

Here,

$$\begin{aligned} (R^{\#})^{-1}(L^{\#})^{-1} &= (L^{\#}R^{\#})^{-1} \\ &= (JLRJ^T)^{-1} \\ &= (JJ^T)^{-1} \\ &= \begin{bmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}. \end{aligned}$$

Thus, when expressed in terms of variables $\mathbf{v} = L^{\#}\mathbf{u}$, the equations decouple into $(m-2)$ advection equations, and the 2×2 symmetric elliptic system

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \lambda_R & \lambda_I \\ \lambda_I & -\lambda_R \end{pmatrix} \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{12.16}$$

If there is more than one conjugate pair of eigenvalues, the treatment above can easily be extended by adding additional (2×2) blocks to the matrix J, so we have

Theorem 12.1 *Any $(m \times m)$ first order constant coefficient hyperbolic system in two dimensions decomposes into $(m-2k)$ convection equations and k independent (2×2) symmetric elliptic systems.*

Remark 12.2 *The $(m \times 2)$ blocks that appear in $R^{\#}$ provide basis vectors for the subspaces inhabited by the elliptic subproblems; the $(2 \times m)$ blocks that appear in $L^{\#}$ are operators that project an arbitrary vector into these subspaces.*

12.2.2 The Steady Euler Equations

Take the vector of unknowns to be $\mathbf{u} = (dp/(\rho a), du, dv, dp - a^2 d\rho)^T$, and adopt coordinates aligned with the flow direction. Then A, B are the symmetric matrices

$$A = \begin{bmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We then find that the equation $A^{-1}B = R\Lambda L$ becomes

$$A^{-1}B = \begin{bmatrix} 0 & 0 & \frac{M}{M^2-1} & 0 \\ 0 & 0 & \frac{-1}{M^2-1} & 0 \\ \frac{1}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} M & M & 0 & 0 \\ -1 & -1 & 1 & 0 \\ \sqrt{M^2-1} & -\sqrt{M^2-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{M^2-1}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{M^2-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2M} & 0 & \frac{1}{2\sqrt{M^2-1}} & 0 \\ \frac{1}{2M} & 0 & \frac{-1}{2\sqrt{M^2-1}} & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For supersonic flow, the transformation $\mathbf{v} = L\mathbf{u}$ produces four real characteristic equations, which are, in primitive variables,

$$\begin{aligned} \partial_x S &= 0 \qquad dS = dp - a^2 d\rho \\ \partial_x H &= 0 \qquad dH = dp + \frac{\rho}{2}(u^2) = dp + \rho q\, dq \\ (\beta\partial_x + \partial_y)(\beta dp + \rho u dv) &= 0 \\ (\beta\partial_x - \partial_y)(\beta dp - \rho u dv) &= 0 \end{aligned}$$

where $\beta^2 = M^2 - 1$.

If $M^2 < 1$, two of these characteristic equations break down. We may then have recourse to the alternative factorization $R^{\#}\Lambda^{\#}L^{\#}$, which is

$$A^{-1}B = \begin{bmatrix} 0 & 0 & \frac{M}{M^2-1} & 0 \\ 0 & 0 & -\frac{1}{M^2-1} & 0 \\ \frac{1}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2}M & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 1 & 0 \\ 0 & -\sqrt{2(1-M^2)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{\sqrt{1-M^2}} & 0 & 0 \\ \frac{-1}{\sqrt{1-M^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}M} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2(1-M^2)}} & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this factorization we recover the same streamwise characteristic equations as in the supersonic case. It is interesting to note that the entropy characteristic corresponds to a shared eigenvector of the matrices A, B, whereas the enthalpy characteristic does not. The two acoustic characteristics collapse into the elliptic system deriving from (12.16), which is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} \frac{\beta_*}{\rho u} dp \\ dv \end{pmatrix} + \begin{pmatrix} 0 & \frac{-1}{\beta_*} \\ \frac{-1}{\beta_*} & 0 \end{pmatrix} \partial_y \begin{pmatrix} \frac{\beta_*}{\rho u} dp \\ dv \end{pmatrix} = 0. \qquad (12.17)$$

where $\beta_*^2 = 1 - M^2$. These are the Cauchy-Riemann equations written in Prandtl-Glauert coordinates $(x, y/\beta_*)$.

This decomposition differs from that found by Ta'asan [33, 34]. He employed, as canonical variables, (S, H, u, v) and his decomposition was not completely block diagonal, containing some additional coupling terms. The variables which emerge from the present treatment are (S, H, p, v) and there is no coupling between blocks.

12.2.3 Preconditioning

Suppose that the time-dependent two-dimensional problem obeys

$$\partial_t \mathbf{u} + A\partial_x \mathbf{u} + B\partial_y \mathbf{u} = 0 \qquad (12.18)$$

where A and B are symmetric. To find a decomposition of this equation as it stands would require the three-dimensional treatment given in the next section. However, there are occasions, when only the steady state is of interest, but it is convenient to compute it by following some time-like evolution. In such cases (12.18) may be replaced (sometimes advantageously [35, 36, 39]) by

$$P^{-1}\partial_t \mathbf{u} + A\partial_x \mathbf{u} + B\partial_y \mathbf{u} = 0 \qquad (12.19)$$

where P is a 'preconditioning matrix'. To retain well-posedness, P must be positive definite. In this section we point out that P can always be chosen to preserve the canonical block-diagonal form of the steady state equations.

As in the steady case, premultiply (12.19) by $(R^{\#})^{-1}A^{-1}$ and make the change of variable $\mathbf{v} = L^{\#}\mathbf{u}$; this gives

$$(R^{\#})^{-1}A^{-1}P^{-1}(L^{\#})^{-1}\partial_t\mathbf{v} + (JJ^T)^{-1}\partial_x\mathbf{v} + \Lambda^{\#}\partial_y\mathbf{v} = 0.$$

For the decomposition to be preserved, it is neccessary and sufficient that

$$(R^{\#})^{-1}A^{-1}P^{-1}(L^{\#})^{-1} = D^{\#}$$

where $D^{\#}$ is block diagonal of the same form as the matrices that multiply the space derivatives. Solving for P gives

$$P = (R^{\#}ADL^{\#})^{-1}. \tag{12.20}$$

where the only constraint on $D^{\#}$ is that it shall be positive definite to ensure well-posedness.

To apply this analysis to the Euler equations we take

$$D^{\#} = \begin{bmatrix} P & Q & 0 & 0 \\ R & S & 0 & 0 \\ 0 & 0 & \frac{1}{M} & 0 \\ 0 & 0 & 0 & \frac{1}{M} \end{bmatrix}. \tag{12.21}$$

This assigns the fluid velocity u to each of the scalar waves. We require $\Delta = PS - QR > 0$. After carrying through the required operations we find that

$$P^{-1} = \begin{bmatrix} \frac{1}{M^2} - P\frac{\beta_*^2}{M} & \frac{1}{M} & Q\beta_* & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ R\beta_* & 0 & -SM & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12.22}$$

and also

$$P = \begin{bmatrix} -\frac{S}{\Delta}\frac{M}{\beta_*^2} & \frac{S}{\Delta}\frac{1}{\beta_*^2} & -\frac{Q}{\Delta}\frac{1}{\beta_*} & 0 \\ \frac{S}{\Delta}\frac{1}{\beta_*^2} & 1 - \frac{S}{\Delta}\frac{1}{M\beta_*^2} & \frac{Q}{\Delta}\frac{1}{M\beta_*} & 0 \\ -\frac{R}{\Delta}\frac{1}{\beta_*} & \frac{R}{\Delta}\frac{1}{M\beta_*} & -\frac{P}{\Delta}\frac{1}{M} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12.23}$$

This is the most general preconditioner that preserves the canonical decomposition. It may be noted that $\det P = (\beta_*^2\Delta)^{-1}$. The preconditioner of van Leer *et. al.* [39] corresponds to the special case $P = S = -1/(M\beta_*)$, $Q = R = 0$, $\Delta = 1/(M^2\beta_*^2)$.

When the preconditioned equations are written in Cartesian coordinates not aligned with the flow, it can be shown that terms may appear which depend on the flow direction even though the flow velocity vanishes. It can also be shown that a necessary and sufficient condition to avoid this phenomenon is that $P_{22} = P_{33}$, which implies

$$\beta_*^2(M\Delta + A) = D. \tag{12.24}$$

The wavespeeds in the direction θ are given by

$$\det[A\cos\theta + B\sin\theta - P^{-1}\lambda]$$

which gives a twice repeated root $\lambda = M\cos\theta$, together with the quadratic term

$$\Delta\beta^2\lambda^2 + [(P-S)\beta^2\cos\theta + (Q+R)\beta\sin\theta]\lambda = \beta^2\cos^2\theta + \sin^2\theta. \tag{12.25}$$

To keep this symmetrical with respect to the streamline, we must take $Q+R=0$, and we consider first the case $Q=R=0 \Rightarrow \Delta = AD$. The wavespeeds in the x-direction are then $1/P$ and $-1/S$, and in the y-direction $\pm 1/(\beta_*\sqrt{PS})$. It can be shown that the best condition number is obtained from the preconditioner of van Leer *et. al.*, but unfortunately this does not satisfy the condition (12.24) at low Mach numbers. We require, in fact, that as $M \to 0$, then

$$(S + 1/M)(P - 1/M) = -1/M^2.$$

This is the graph of a hyperbola in the SP plane. We are only concerned with the branch that passes through the origin (the other branch leads to imaginary wavespeeds), and only with the half of that branch for which P, S are both negative (the other half does not make P positive definite). The condition number based solely on the 'acoustic' waves is

$$K = \max\left(S/P, P/S\right)$$

which is unity at the origin and degrades away from it. Taking into account that the flow velocity is M, and that $|P| > |S|$,we actually have

$$K = \max\left(\frac{S}{P}, \frac{P}{S}, \frac{1}{MS}, MP\right).$$

The point on the hyperbola that minimises this is

$$P = \frac{-1}{M} \qquad S = \frac{-1}{2M}$$

leading to K=2.

Of course, the preconditioner does not have to satisfy (12.24) away from $M=0$, and can be blended in with the van Leer *et al.* preconditioner, by some such choice as

$$P = \frac{-1}{M\beta_*}, \qquad S = \frac{-1}{M * \beta_*(1+\beta_*)}$$

which leads to simple preconditioner

$$P = \begin{bmatrix} \frac{M^2}{\beta_*} & \frac{-M}{\beta_*} & 0 & 0 \\ \frac{-M}{\beta_*} & \frac{1+\beta_*}{\beta_*} & 0 & 0 \\ 0 & 0 & \beta_*(1+\beta_*) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} \frac{1+\beta_*}{M^2} & \frac{1}{M} & 0 & 0 \\ \frac{1}{M} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta_*(1+\beta_*)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12.26}$$

Rather strikingly, these are identical with those given by van Leer *et al.*, except for the factors $(1+\beta_*)$ in the $()_{33}$ elements.

12.3 The Three-dimensional Case

12.3.1 General Steady-state Problems

The most common block diagonal forms in three-dimensional problems appear to have blocks that are either scalar or else (3 × 3). The possibility of (2 × 2) blocks is ruled out, apart from exceptional cases, by an anlysis of Gilquin, Laurens and Rosier [10], who considered the general (2×2) symmetric system in multi-dimensions. They only require that the system be strongly hyperbolic (i.e. combinations of the coefficient matrices can always be diagonalized). They then prove that such systems can always be decomposed into a convective part (possibly with speed zero) and a *two dimensional* elliptic part. Therefore, (2×2) blocks are incapable of representing three-dimensional elliptic systems.

To see better what happens in three space dimension we consider the specific case of the Euler equations. In addition we add a preconditioner of the time derivative as described previously.

12.3.2 Subsonic - three dimensional

In three space dimensions the preconditioned subsonic equations are

$$\begin{aligned}
\frac{\beta+1}{M^2}p_t + \frac{\rho u}{M^2}u_t + up_x + \rho a^2(u_x + v_y + w_z) &= 0 \\
\frac{1}{\rho u}p_t + u_t + uu_x + \frac{p_x}{\rho} &= 0 \\
\frac{1}{\beta}v_t + uv_x + \frac{p_y}{\rho} &= 0 \\
\frac{1}{\beta}w_t + uw_x + \frac{p_z}{\rho} &= 0 \\
S_t + uS_x &= 0
\end{aligned}$$

As always S and H are convected along the stream line. We now get three equations for the unknowns (p,v,w). Let $(dq, dr, ds) = (\frac{\beta}{\rho u}dp, du, dv)$.

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_t + \begin{pmatrix} -\beta u & 0 & 0 \\ 0 & \beta u & 0 \\ 0 & 0 & \beta u \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_x + \begin{pmatrix} 0 & u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_z = 0$$

We next rescale x by defining $x = \beta x'$. We rewrite this equation in terms of x' and then drop the $'$. The effect is to eliminate the β term from the x derivative. Then in this modified coordinate system we get

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_t + \begin{pmatrix} -u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_x + \begin{pmatrix} 0 & u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}_z = 0$$

We now consider the three dimensional Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

Let $u = \psi_x$, $v = -\psi_y$ and $w = -\psi_z$. Then writing the second order scalar equation as a first order system we get

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_z = 0$$

However, by the existence of the potential ψ we also have the equation $v_z - w_y = 0$. Hence, to go from the system back to the second order Laplace equation for a potential ψ we have a compatibility requirement on the solution. Converting this into a time dependent system we get

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_y + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_z = 0$$

Thus the "acoustic" portion of the three dimensional Euler equation corresponds to the reduction of the Laplace equation (actually wave equation) to a first order system. However, in distinction to two dimensions, this system is no longer "elliptic". To see this we start with the system form for three space dimensions and convert back to a second order equation (assuming u is constant).

Then q satisfies the wave equation $\frac{\partial^2 q}{\partial t^2} = u^2 \Delta q$. However r and s do not obey a wave equation. Instead we get

$$\begin{aligned} \frac{\partial^2 r}{\partial t^2} &= u^2 (\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial y \partial z}) \\ &= u^2 (\Delta r + \frac{\partial}{\partial z} (\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})) \\ \frac{\partial^2 s}{\partial t^2} &= u^2 (\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial z^2} + \frac{\partial^2 r}{\partial y \partial z}) \\ &= u^2 (\Delta s - \frac{\partial}{\partial y} (\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})) \end{aligned}$$

Differentiate the equation for r by y and the equation for s by z and add we find that $\frac{\partial^2 (r_y + s_z)}{\partial t^2} = u^2 \Delta (r_y + s_z)$, i.e.$(r_y + s_z)$ also satisfies the wave equation. Since q satisfies a wave equation so does q_x and hence so does the divergence $q_x + r_y + s_z$. Define $\omega = r_z - s_y$ i.e. the z component of the vorticity. Differentiating the equation for r by z and the equation for s by y and subtracting we find that $\omega_t + u \omega_x =$, Thus, we have two "elliptic" variables, q and div(q,r,s) and one "hyperbolic" variable ω in addition to the previous two hyperbolic variables S and H. Hence, if we solve this three dimensional system with an initial condition that the vorticity is zero then the vorticity remains zero for all time. In this case we can go from the first order system back to the second order wave equation. An equivalent way to view this is to remember that the components of the first order system are the gradients of a potential ψ. In order for this potential to exist there are compatibilty conditions. In particular we need that the vorticity component ω be zero. In two dimensions the system is purely elliptic since ω is always identically zero. Hence, on a differential level the first order equivalent to the Laplace equation has a hyperbolic part. It is not clear how to do this split between elliptic and hyperbolic portions within an algebraic context.

12.3.3 Supersonic - three dimensional

In three dimensions the preconditioned equations are

$$\begin{aligned}
\frac{\beta M+1}{M^2}p_t + \frac{\rho a}{M}u_t + up_x + \rho a^2(u_x+v_y+w_z) &= 0 \\
\frac{1}{\rho u}p_t + u_t + uu_x + \frac{p_x}{\rho} &= 0 \\
\frac{M}{\beta}v_t + uv_x + \frac{p_y}{\rho} &= 0 \\
\frac{M}{\beta}w_t + uw_x + \frac{p_z}{\rho} &= 0 \\
S_t + uS_x &= 0
\end{aligned}$$

Solving for (p,u,v,w,S) we get

$$\begin{aligned}
p_t + \frac{\beta u}{M}p_x + \frac{\rho a^2 M}{\beta}(v_y+w_z) &= 0 \\
u_t + uu_x - \frac{1}{\rho}\left(\frac{\beta}{M}-1\right)p_x - \frac{a}{\beta}(v_y+w_z) &= 0 \\
v_t + \frac{\beta}{M}(uv_x + \frac{p_y}{\rho}) &= 0 \\
w_t + \frac{\beta}{M}(uw_x + \frac{p_z}{\rho}) &= 0 \\
S_t + uS_x &= 0
\end{aligned}$$

As before the system for (p,v,w) decouples from the rest. Written in vector form we get

$$\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_t + \frac{\beta u}{M}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_x + \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_y \qquad (12.27)$$

$$+ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\beta}{\rho u}dp \\ dv \\ dw \end{pmatrix}_z = 0 (12.28)$$

Attempting the same variables as before we find

$$
\begin{aligned}
S_t + uS_x &= 0 \qquad dS = dp - a^2 d\rho \\
H_t + uH_x &= 0 \qquad dH = dp + \frac{\rho}{2}(u^2) \\
(\partial_t + \frac{\beta u}{M}\partial_x + \frac{u}{M}\partial_y + \frac{u}{M}\partial_z)(\beta dp + \rho u(dv + dw)) &= \frac{\rho u^2}{M}(w_y + v_z) \\
(\partial_t + \frac{\beta u}{M}\partial_x - \frac{u}{M}\partial_y - \frac{u}{M}\partial_z)(\beta dp - \rho u(dv + dw)) &= \frac{\rho u^2}{M}(w_y + v_z)
\end{aligned}
$$

We see that these equations are no longer uncoupled due to the right hand side of $\frac{\rho u^2}{M}(w_y+v_z)$. To get the additional convective quantity define the vorticity (component) as $\omega = v_z - w_y$. Then we find from the equations for v and w, that $\omega_t + u\omega_x = 0$. So ω is convected. We are left with three equations in the form

$$w_t + w_x + Bw_y + Cw_z = 0$$

with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Since the x direction reduces to the identity matrix we can introduce new variables $\tau = t$ and $\xi = x - t$. In terms of the new variables we have

$$W_\tau + BW_y + CW_z = 0 \quad , \qquad W = (q, r, s)^t$$

This is just the subsonic case considered above. Hence, q satisfies a wave equation and r, s satisfy

$$
\begin{aligned}
\frac{\partial^2 r}{\partial t^2} &= \Delta r + \frac{\partial}{\partial z}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z}) \\
\frac{\partial^2 s}{\partial t^2} &= \Delta s - \frac{\partial}{\partial y}(\frac{\partial s}{\partial y} - \frac{\partial r}{\partial z})
\end{aligned}
$$

So, $r_y + s_z$ satisfies a two dimensional wave equation while $(s_y - r_x)_\tau = 0$ in the moving coordinate system. Hence, the vorticity component $s_y - r_x$ satisfies a one dimensional convection, in the steamwise direction, as we already have found.

Thus, the supersonic system consists of three convective equations for the entropy, enthalpy and vorticity that decouple. In addition we have a system of two equations that is a combined hyperbolic-elliptic system in the sense that it is an elliptic system in the y and z directions in a system moving in the x direction.

13 Comparison of Preconditioners

We now compare properties of the symmetric preconditioner with that of Algorithm I.

Symmetric	Algorithm I
gives canonical splitting into hyperbolic and elliptic parts	
symmetric hyperbolic - well posed	symmetrizable if $\beta^2 > \alpha(u^2 + v^2)$
for supersonic flow all equations are scalar and optimal	
reduces condition number at sonic point by order of magnitude	reduces condition number near M=1 only if α, β become singular
condition no. = 1 at M=0	condition no. =1 at M=1 if $\beta^2 = \alpha(u^2 + v^2)$
near M= 0 depends on flow angle or is suboptimal	independent of flow angle

One possibility is to use the Van Leer preconditioner for high (transonic) Mach numbers and switch to the Turkel preconditioner near stagnation points.

There are no computational comparisons of the two preconditioners on the identical problem.

14 Computational Results

The preconditioning approach described above is studied numerically for inviscid and viscous flows over airfoils and wings with freestream conditions corresponding to incompressible and compressible flow conditions.

14.1 Flows over 2D Airfoils

14.1.1 Flows over two Dimensional Foils with Basic Dissipation Model

Two codes were used to simulate the airfoil flows in the present work. Both are central-difference finite volume codes with an addition of artificial dissipative terms. A multi-stage Runge-Kutta scheme is used to advance the solution in time. Local time steps, residual smoothing and multigrid are used to accelerate the convergence to a steady state. In one code the unknown conservative variables are stored at the nodes of the stencil [27]. In this code a scalar artificial dissipation is used based on fourth differences of (p,u,v,T) in the four conservation equations: continuity, momentum and energy respectively. In the second code the variables are located at the cell centers. This is the basic code used in [31]. For the preconditioning the variables are changed from the conservative variables to $W_p = (p', \rho u, \rho v, E')^t$. These are used for all operations including implicit residual smoothing and multigrid transfers between grids is discussed in [6].

Both codes use an artificial viscosity based on a blend of second and fourth differences. Since we are not interested in shocks we set the coefficient of the second difference, $k^{(2)}$, to zero. The fourth difference is still needed to prevent the occurrence of odd-even oscillations. Its coefficient, $k^{(4)}$, is given for each case. We denote the first code with variables at the nodes and updating (p, u, v, T) as *Code1* and the second code with variables at the center of the cells and using $(p', \rho u, \rho v, E')$ variables as *Code2.* Both codes used the same five stage Runge-Kutta formula coupled with an implicit residual smoothing. Code1 used CFL=7 (based on the preconditioned time step). The residual smoothing needs the CFL of the unsmoothed scheme and this was taken as 3.5. These are the same values typically chosen for the non-preconditioned scheme. *Code2* required lower CFL numbers and a significantly reduced value of CFL of the unsmoothed scheme. It is unclear why there is a need for additional residual smoothing with *Code2.* Both codes use a multigrid acceleration with a fixed sawtooth and W cycles with one smoothing on the finest mesh and two smoothings on all coarser meshes.

The first numerical test case is inviscid flow over NACA0012 airfoil. The c-type grid has 224x40 cells clustered near the leading and trailing edges in order to allow accurate drag computations. The Mach number contours for $M_\infty = 0.01, \alpha = 2°$ are displayed in Figure 1a. Grid convergence of the total force coefficients is displayed in Figure 1b for *Code1.* The results are plotted versus $1/N$ where N is the total number of grid cells. As a reference we have included the results for subsonic compressible flow at $M_\infty = 0.63$ which was computed without preconditioning. Grid convergence for the drag, which should be zero for inviscid flow without shocks, is quite sensitive to the details of the implementation of the wall slip boundary condition, and also to the amount of artificial viscosity used. For example, the pressure drag on the coarse 56x10 grid may even change sign when changing the input value of $k^{(4)}$ from 1/32 to 1/64.

The convergence behavior of the inviscid flow computations towards the steady state is displayed in Figure 1c. A 4-level multigrid was used. Convergence is monitored by plotting the L_2-norm of the residual of the continuity equation computed on the finest mesh.

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Figure 1a: Numerical solution for incompressible inviscid flow

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Figure 1b: Grid convergence of Code1 for inviscid flow, $k^{(4)} = 1/32, \zeta = 2/3$

-bb-error = =

Figure 1c: Residual convergence of Code1 for inviscid flow, $\delta = 1, k^{(4)} = 1/32, \zeta = 2/3$

We next consider viscous flow about a RAE 2822 airfoil. The flow conditions chosen are $M_\infty = 0.01, \alpha = 1.89°$, Re = 5.7 million. We use a Baldwin-Lomax turbulence model with transition at 11% of the chord. In Figure 2a we present the grid used for the calculations. This grid was constructed for subsonic problems and clusters the cells near the leading and trailing edges. The farfield boundary is located about 20 chord lengths away from the airfoil. The numerical flow solution is presented in Figure 2b for *Code2*. The agreement between both codes in the representation of the flow details is within plotting accuracy. The pressure distribution is typical for subsonic flow over a supercritical airfoil. The skin friction displays the effect of fixed transition at $x/c = 0.11$. Note, that pressure oscillations near the trailing edge, for which compressible flow solvers are notorious, are absent in the numerical result due to the correct scaling of artificial viscosity.

The residual convergence is presented in Figure 2c. A 5-level multigrid was used because of the finer mesh. This displays a comparison for the two flow solvers and both codes converge at about the same rate. Recall that the algorithmic differences between *Code1* and *Code2* concern the flow variables for which residuals are computed and the artificial dissipation model. Numerical implementations of boundary conditions and multigrid transfer operators differ because *Code1* stores the flow variables at the grid nodes whereas *Code2* is cell centered. *Code2* transfers the primitive variables between the grids rather than the conservative variables used by *Code1*.

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Figure 2a: Mesh lines near RAE 2822 airfoil.

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Figure 2b: Solution of Code2 for incompressible viscous flow, $\alpha = 1, \delta = 0, k^{(4)} = \frac{1}{32}, \zeta = 2/3$.

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Figure 2c: Grid convergence of Code1 and Code2 for $k^{(4)} = 1/32, \zeta = 2/3$.

14.1.2 Flows over Two Dimensional Foils with Preconditioned *CUSP*

Code1 is extended to include the preconditioned *CUSP* scheme as described in 7.2. The *CUSP* scheme is implemented by using $w = w_h$, i.e. the *CUSP* variant which yields a constant enthalpy in the steady-state, *H-CUSP*. Higher-order solutions are obtained via the *SLIP* switch. (see Part I for more details). The preconditioner parameters chosen were $\alpha = 0, \delta = 0, M_0 = 0.5, K_2 = 0.25, K_1 = 1.0$.

Numerical computations of inviscid flow over a NACA 0012 airfoil are used to assess the behavior of *HCUSP* over a broad speed range. Figure 3a displays the typical flow features obtained for incompressible, transonic and hypersonic flow. It is seen that shocks are captured with a single interior point for both the transonic and hypersonic flow regimes. The numerical behavior of *HCUSP* for low subsonic conditions is further examined in Figure 3b. The preconditioned solution is smooth in contrast to the *HCUSP* scheme without preconditioning where strong oscillations are present in the stagnation region. Note, that both solutions use exactly the same basic dissipation operator, $\mathcal{L}_{AD}$. However, the formulation (7.2.1) is obviously ill-conditioned for low mach numbers whereas the eigenvalues of the preconditioned convective flux and the artificial dissipation given by (7.2.2) or (7.2.3) are much better. This is confirmed by the convergence history for this flow case. We note, that the accuracy of the non-preconditioned scheme is limited even though the artificial viscosity scales with the local convective sound speed and not with the speed of sound. Thus, we see that preconditioned distributes the scales properly among all the components of the artificial viscosity and is more than just a rescaling of the coefficient in the artificial viscosity from the speed of sound to the convection speed.

Numerical accuracy is further assessed by plotting the pressure drag as a function of the freestream Mach number. Figure 3c compares the influence of the conservative and nonconservative preconditioning as well as the grid density. Note that the drag should approach zero for subsonic flows as the grid becomes more dense. It is seen that the results with and without preconditioning are essentially the same for freestream Mach numbers above 0.5. At low Mach numbers, the numerical solution without preconditioning degrades rapidly. The convergence rate of the residual behaves in a similar fashion. Without preconditioning one can not obtain a converged solution for flows with a mach number of less than about 0.01. The influence of the the basic dissipation model upon the pressure drag is presented in Figure 3d. The basic dissipation model without preconditioning is denoted as *JST*. This model scales the artificial dissipation terms with the largest eigenvalue, $u + a$. Hence, for the *JST* dissipation model it is no surprise that this model gives larger numerical errors for low Mach number flows compared with the *HCUSP* scheme without preconditiong.

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Figure 3a: Influence of free-stream Mach number on the inviscid flow around NACA0012 airfoil with the preconditioned *HCUSP* scheme.

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Figure 3b: Effect of preconditioning on convergence and flow solution iwith *HCUSP* near the stagnation point for almost incompressible flow around NACA0012 airfoil

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Figure 3c: Influence of grid density and preconditioning technique on zero-lift drag of inviscid flow over NACA0012 airfoil and on iteration numbers to achieve 5 orders of magnitude residual reduction (*HCUSP* scheme with preconditioning parameters $\alpha = 0, \delta = 0, M_0 = 0.5$).

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Figure 3d: Influence of preconditioning and dissipation model on zero-lift drag of inviscid flow over NACA0012 airfoil and on iteration numbers to achieve 5 orders of magnitude residual reduction (*HCUSP* with preconditioning parameters $\alpha = 0, \delta = 0, M_0 = 0.5$).

14.2 Flow over 2-D Multi-element Airfoil

The next test case considered here is that of a 3-element airfoil configuration that has been investigated both experimentally and theoretically [46]. Present computations are performed at a chord Reynolds number of 9 million and an angle of attack of 16.2 degrees using a variant of *Code2* [38]. The Mach number for this test case is 0.2. A 20-block structured grid shown in Figure 4a, is used for the computations. This test case has also been investigated with the original non-preconditioned version of the flow solver used in this work, as reported by Vatsa et. al. [41].

The convergence histories for this case are presented in Figure 4b, where the results from original non-preconditioned and current preconditioned (conservative) schemes are compared. It is observed that the residuals in the original code slow down considerably at approximately 4-orders; whereas the residuals in the preconditioned scheme exhibit much better convergence. Note that the free-stream Mach number for this case (M=0.2) is generally not considered too low for compressible codes. However, in this case several pockets of slow-moving flow exist in the cove regions of the slat and the main airfoil sections [41]; these pockets slow the convergence of standard compressible codes. The preconditioned system has a much better condition number for the eigenvalues and does not experience slowdown due to such disparities in the flow speed.

The computed pressure distributions for this case are compared in Figure 4c. As expected, very little difference is observed in the two sets of computed results. Furthermore these results compare quite favorably with the experimental data.

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Figure 4a: Partial view of block-structured grid for 3-element high lift airfoil configuration

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Figure 4b: Effect of preconditioning on convergence history for the the 3-element high-lift airfoil configuration

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Figure 4c: Effect of preconditioning on pressure distributions for the 3-element high-lift airfoil configuration

14.3 Flow over 3D Wing

The three-dimensional calculations have been carried out with a cell-vertex method [22] which corresponds closely to *Code1* used for the 2D flows. For identification purposes we shall refer to this as *Code1* within this section. The method employs central differencing with artificial viscosity, multi-stage time stepping, residual smoothing and multigrid. The present cases were run with a five stage scheme, a Courant number of 7.5 and 3-level full multigrid with W-cycling. Preconditioning is obtained with the Choi-Merkle matrix, that is $\delta = 1, \alpha = 0$ and the dissipation is formulated according to (7.1.3).

We consider the incompressible viscous flow over the ONERA-M6 wing at an angle of attack of $\alpha = 2°$, *Re*=11 million. The effect of turbulence is modelled according to the Baldwin and Lomax model. The grid used for the calculations has C-H topology and consists of 224x48x32 cells, these are 224 cells in streamwise, 48 cells in normal and 32 cells in spanwise directions. Figure 5 shows the grid in the plane of symmetry. The residual convergence for computations at $M_\infty = 0.10, 0.01, 0.001$ is presented in Figure 6a. It is shown that the rate of convergence is independent of the free stream Mach number. Without preconditioning, on the other hand, no convergence could be obtained for Mach numbers less than 0.1. Figure 6b compares the surface pressure distribution at spanwise section $\eta = 0.8$. There is no effect of the freestream Mach number visible. Mach number independence is further confirmed by Figure 6c where the total lift and pressure drag coefficients are plotted versus the Mach number. These values are constant if the Mach number approaches zero. The numerical accuracy of the incompressible solution is illustrated in Figure 7. Here, the variation of total force coefficients under grid refinement is shown. As for the 2D cases pressure drag is more sensitive than lift. Note, that the convergence rate towards the infinitely dense mesh is very similar when comparing with the results in two dimensions.

We now consider a different generalization to three space dimensions as presented in [38]. We shall refer to this code as *Code2* within this section. This code is based on (p, u, v, w, T) variables as was the above code. However, now the variables are given at the cell centers rather than cell vertices. We again consider viscous flow over the ONERA M6 wing. For this case, the Reynolds number (based on mean aerodynamic chord) is 11.7 million, and the angle of attack is chosen as $\alpha = 3.06°$. A grid consisting of 193x49x33 points is used for these computations.

Figure 8a shows the effect of preconditioning at a Mach number of 0.1 on the convergence history of the numerical algorithm. For these computations, an FMG strategy in multigrid scheme is used, in which 50 iterations on the coarse grid are followed by 300 iterations on the fine grid. It is obvious from this figure that a significant improvement is achieved in the convergence rate when preconditioning is used. Furthermore, the original, non-preconditioned scheme failed to converge at a lower Mach number of 0.01.

The effect of the free-stream Mach number is considered in the next series. Figures 8b,8c show the effect of free-stream Mach number on convergence rates and surface pressure distribution, respectively. The residuals in Fig. 8b have been normalized with their respective initial values in order to remove scaling effects due to differences in free-stream Mach number. Note that over a Mach number range of 0.01 to 0.2, the convergence rates for the preconditioned scheme are very similar; with the asymptotic convergence rates for the two lowest Mach numbers being almost identical. Although not shown here, similar results have been obtained at an even lower Mach number of 0.001.

The pressure distributions (at a span location of 80%) shown in Fig. 8c indicate that results for Mach 0.01 and 0.1 are identical within plotting accuracy. Even at Mach 0.2, except for a slightly higher value of the pressure peak in the leading-edge region, the effect of Mach number is found to be negligible. These results demonstrate that the preconditioned system approaches the incompressible limit in a smooth and systematic manner, without any penalty in convergence rate.

The effects of the conservative and non-conservative form of the artificial viscosity formulations are shown in Figures 8d,8e. The non-conservative formulation appears to converge better on the coarser grid, but is almost identical in performance to the conservative formulation on the fine grid. The effect of these two forms of artificial viscosity on pressure distribution is found to be negligible, which is expected to be the case at low subsonic speeds.

The final test case presented here deals with transonic flow over the ONERA M6 wing. The test conditions for this case are identical to the first test case except for the free stream Mach number, which is chosen as 0.84. The Navier-Stokes solutions for this case were obtained both for the conservative and non-conservative preconditioners. A baseline solution with no preconditioning was obtained for comparisons.

The computed pressure distributions for this case are compared in Figure 9a at 80% span location. It is clear from this figure that the pressure distribution from the conservative preconditioning is virtually indistinguishable from the baseline (unpreconditioned) case. However, the non-conservative form of preconditioning produces noticeable differences in pressure distribution in the vicinity of the shocks.

The convergence histories for this case are shown in Figure 9b, where it is observed that the baseline scheme stalls when the residual drops by about 4-orders in magnitude. The convergence properties of the two preconditioned schemes are very similar and much faster compared with the baseline scheme. Note that the slow convergence of the basic scheme cannot be attributed to low free stream velocities. However, the computational domain consists of low speed flows in the stagnation and boundary-layer regions, where the convective speed of propagation are much lower. It is believed that preconditioning helps reduce the imbalance of convective and acoustic speeds in these regions, thereby improving the overall convergence rate for such problems.

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Figure 5: Wing surface and grid in symmetry plane

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Figure 6a: *Code1*: Effect of freestream Mach number on residual convergence of 3D wing

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Figure 6b: *Code1*: Effect of freestream Mach number on pressure distribution of 3D wing

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Figure 6c: *Code1*: Effect of freestream Mach number on force coefficients of 3D wing

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Figure 7: *Code1*: Effect of grid density on force coefficients of 3D wing

-bb-error = =

Figure 8a: Code2: Effect of preconditioning on convergence history for ONERA M6 wing, $M_\infty = 0.01$

-bb-error = =

Figure 8b: *Code2*: Effect of Mach number on convergence history of the preconditioned scheme for ONERA M6 wing

-bb-error = =

Figure 8c: *Code2*: Effect of Mach number on ONERA M6 wing pressure distributions at 80% span location for preconditioned scheme

-bb-error = =

Figure 8d: *Code2*: Effect of conservative/non-conservative formulations of artificial dissipation on convergence history for ONERA M6 wing

-bb-error = =

Figure 8e: *Code2*: Effect of conservative/non-conservative formulations of artificial dissipation on ONERA M6 wing pressure distributions

-bb-error = =

Figure 9a: *Code2*: Comparison of pressure distribution for ONERA M6 wing at transonic speeds

-bb-error = =

Figure 9b: *Code2*: Comparison of convergence histories for ONERA M6 wing at transonic speeds

15 Concluding Remarks

The present report presents analysis and comparisons of preconditioning methods for low speed flows from an application-type point of view. Recent preconditioners given by Turkel and Choi-Merkle are covered and generalized into a single formulation. The report describes all the technical details which are neccessary in order to extend existing compressible flow codes for the low speed range.

The effects of the preconditioning matrices, artificial dissipation and grid density on accuracy and convergence are investigated. Tests of inviscid and viscous problems demonstrate that preconditioning has now matured to the point where real applications are possible.

There are several interesting questions that have not been touched by the present work, though. Among these is our observation that the lower cut-off value for the term βM_r^2 in the preconditioning matrices presently depends on the freestream velocity rather than on the local flow. A futher study of these phemonema would need to account for local variations near stagnation regions and cannot be treated with a constant coefficient model. Finally, future work is neccessary in order to extend the preconditioning approach towards the use of multiequation turbulence models and chemically reacting flows. In these cases additional stiffness due to the presence of strong source terms must be taken into account in an appropriate way.

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A Appendix

We define the sets of variables

$$w_0 = (p, u, v, S)$$

$$w_c = w_1 = (\rho, \rho u, \rho v, E)$$

$$w_2 = (p, \rho u, \rho v, E)$$

$$w_3 = (p, \rho u, \rho v, E - h_\infty \rho)$$

$$w_4 = (p, u, v, T)$$

Let a be the speed of sound, $q^2 = u^2 + v^2$ and $Q^2 = \frac{\rho q^2}{2}$. Given the preconditioning P_0 in w_0 variables we can compute the preconditioner P_i in w_i coordinates by

$$P_i = \frac{\partial w_i}{\partial w_0} P_0 \frac{\partial w_0}{\partial w_i}, \qquad P_0 \text{ given by (3.2)}$$

We have the following transformation matrices connecting these variables:

$$\frac{\partial w_0}{\partial w_1} = \begin{pmatrix} \frac{\gamma-1}{2}q^2 & (1-\gamma)u & (1-\gamma)v & \gamma-1 \\ \frac{-u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ \frac{-v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{\gamma-1}{2}q^2 - a^2 & (1-\gamma)u & (1-\gamma)v & \gamma-1 \end{pmatrix} \frac{\partial w_1}{\partial w_0} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 & \frac{-1}{a^2} \\ \frac{u}{a^2} & \rho & 0 & \frac{-u}{a^2} \\ \frac{v}{a^2} & 0 & \rho & \frac{-v}{a^2} \\ \frac{1}{\gamma-1} + \frac{M^2}{2} & \rho u & \rho v & \frac{-M^2}{2} \end{pmatrix}$$

$$\frac{\partial w_0}{\partial w_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-u}{(\gamma-1)Q^2} & \frac{v^2-u^2}{2Q^2} & \frac{-uv}{Q^2} & \frac{u}{Q^2} \\ \frac{-v}{(\gamma-1)Q^2} & \frac{-uv}{Q^2} & \frac{u^2-v^2}{2Q^2} & \frac{v}{Q^2} \\ 1 - \frac{2}{(\gamma-1)M^2} & \frac{-2u}{M^2} & \frac{-2v}{M^2} & \frac{2}{M^2} \end{pmatrix} \qquad \frac{\partial w_2}{\partial w_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u}{a^2} & \rho & 0 & \frac{-u}{a^2} \\ \frac{v}{a^2} & 0 & \rho & \frac{-v}{a^2} \\ \frac{1}{\gamma-1} + \frac{M^2}{2} & \rho u & \rho v & \frac{-M^2}{2} \end{pmatrix}$$

Let $G = \frac{u^2+v^2}{2} + h_\infty$, $L = \frac{1}{\gamma-1} + \frac{M^2}{2} - \frac{h_\infty}{a^2}$ and $E' = E - h_\infty$.

$$\frac{\partial w_0}{\partial w_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-u}{(\gamma-1)\rho G} & \frac{G-u^2}{\rho G} & \frac{-uv}{\rho G} & \frac{u}{\rho G} \\ \frac{-v}{(\gamma-1)\rho G} & \frac{-uv}{\rho G} & \frac{G-v^2}{\rho G} & \frac{v}{\rho G} \\ 1 - \frac{a^2}{(\gamma-1)G} & \frac{-ua^2}{G} & \frac{-va^2}{G} & \frac{a^2}{G} \end{pmatrix} \qquad \frac{\partial w_3}{\partial w_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u}{a^2} & \rho & 0 & \frac{-u}{a^2} \\ \frac{v}{a^2} & 0 & \rho & \frac{-v}{a^2} \\ L & \rho u & \rho v & \frac{-M^2}{2} + \frac{h_\infty}{a^2} \end{pmatrix}$$

$$\frac{\partial w_0}{\partial w_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\gamma & 0 & 0 & \frac{\gamma p}{T} \end{pmatrix} \qquad \frac{\partial w_4}{\partial w_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{(\gamma-1)T}{\gamma p} & 0 & 0 & \frac{T}{\gamma p} \end{pmatrix}$$

To transform the preconditioned residual in (p,u,v,T) to/from conservative variables we need the following Jacobians which are a direct result of those given above.

$$\frac{\partial w_1}{\partial w_4}=\begin{pmatrix} \frac{\rho}{p} & 0 & 0 & \frac{-\rho}{T} \\ \frac{\rho u}{p} & \rho & 0 & \frac{-\rho u}{T} \\ \frac{\rho v}{p} & 0 & \rho & \frac{-\rho v}{T} \\ \frac{E}{p} & \rho u & \rho v & \frac{-\rho q^2}{2T} \end{pmatrix} \quad \frac{\partial w_4}{\partial w_1}=\begin{pmatrix} \frac{(\gamma-1)q^2}{2} & (1-\gamma)u & (1-\gamma)v & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{1}{\rho}(\frac{(\gamma-1)q^2}{2}-T) & \frac{(1-\gamma)u}{\rho} & \frac{(1-\gamma)v}{\rho} & \frac{(\gamma-1)}{\rho} \end{pmatrix}$$

We also have $\Gamma_i = \frac{\partial w_i}{\partial w_0} P_0 \frac{\partial w_0}{\partial w_c}$. So $P_i = \Gamma_i \frac{\partial w_c}{\partial w_i}$. In particular $P_c = \frac{\partial w_1}{\partial w_4}\Gamma_4$ and $P_4 = \Gamma_4 \frac{\partial w_1}{\partial w_4}$. For the (p,u,v,T) variables we explicitly have

$$\Gamma_4 = \begin{pmatrix} \beta M_r^2 \delta & 0 & 0 & 0 \\ \frac{-u}{\rho}(1+\alpha\delta) & \frac{1}{\rho} & 0 & 0 \\ \frac{-v}{\rho}(1+\alpha\delta) & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)T}{\gamma p}\left[\beta M_r^2 + \frac{q^2}{2} - \frac{a^2}{\gamma-1}\right] & \frac{(1-\gamma)Tu}{\gamma p} & \frac{(1-\gamma)Tv}{\gamma p} & \frac{(\gamma-1)T}{\gamma p} \end{pmatrix}$$

$$-(\gamma-1)(\delta-1)\begin{pmatrix} \frac{\beta M_r^4}{2} & -\frac{\beta M_r^2 u}{a^2} & -\frac{\beta M_r^2 v}{a^2} & \frac{\beta M_r^2}{a^2} \\ -\frac{\alpha u M_r^2}{2\rho} & \frac{\alpha u^2}{\rho a^2} & \frac{\alpha uv}{\rho a^2} & -\frac{\alpha u}{\rho a^2} \\ -\frac{\alpha v M_r^2}{2\rho} & \frac{\alpha uv}{\rho a^2} & \frac{\alpha v^2}{\rho a^2} & -\frac{\alpha v}{\rho a^2} \\ \frac{\beta M_r^2 T}{\gamma p}(\frac{(\gamma-1)M_r^2}{2}-1) & \frac{(1-\gamma)T\beta M_r^2 u}{\gamma p a^2} & \frac{(1-\gamma)T\beta M_r^2 v}{\gamma p a^2} & \frac{(\gamma-1)T\beta M_r^2}{\gamma p a^2} \end{pmatrix}$$

and

$$\Gamma_4^{-1} = \begin{pmatrix} \frac{1}{\beta M_r^2} & 0 & 0 & 0 \\ \frac{(1+\alpha)u}{\beta M_r^2} & \rho & 0 & 0 \\ \frac{(1+\alpha)v}{\beta M_r^2} & 0 & \rho & 0 \\ \frac{1}{\beta M_r^2}(\frac{E+p}{\rho}+\alpha q^2)-1 & \rho u & \rho v & \frac{\gamma p}{(\gamma-1)T} \end{pmatrix}$$

$$+(\delta-1)\begin{pmatrix} \frac{1-\gamma}{a^2} & 0 & 0 & \frac{\gamma p}{a^2 T} \\ \frac{(1-\gamma)u}{a^2} & 0 & 0 & \frac{\gamma p u}{a^2 T} \\ \frac{(1-\gamma)v}{a^2} & 0 & 0 & \frac{\gamma p v}{a^2 T} \\ \frac{(1-\gamma)M_r^2}{2}-1 & 0 & 0 & \frac{\gamma p}{T}(\frac{1}{\gamma-1}+\frac{M_r^2}{2}) \end{pmatrix}$$

Given in this form, $\delta = 1$ simplifies these expressions and does not change the eigenvalues (but does change the eigenvectors) of PA. However, for P_4 (given in (3.8)) $\delta = 0$ is simpler. Hence, if $\delta = 0$ it is more efficient to evaluate $\Gamma_4\vec{x} = P_4(\frac{\partial w_4}{\partial w_1}\vec{x})$ then to use the above formula. Alternatively, one can use P_4 directly rather than Γ_4 (see Appendix II). If we examine the form of Γ_4^{-1} in more detail we see that in the first matrix the first column has terms that go to infinity like $\frac{1}{\beta M^2} \sim \frac{1}{q^2}$ while in the matrix that is multiplied by $\delta - 1$ the terms are all bounded as the Mach number goes to zero. Hence, the terms involving δ are lower order perturbations in Γ_4^{-1}. Hence, we expect that the choice of δ is not crucial.