Part I

Multistage Central Difference Schemes for the Euler and Navier-Stokes Equations

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1 Introduction

The subject of Computational Fluid Dynamics (CFD) is a multidisciplinary one involving fluid mechanics, numerical analysis, and computer science. The evolution of this subject over the last three decades has resulted in a broad range of capabilities to compute the aerodynamics of flight vehicles. At cruise flight conditions, a variety of approximate techniques are applied by the aircraft industry in the design process of a flight vehicle. Under the assumptions of inviscid and irrotational flow, versatile and reliable panel methods and nonlinear potential equation solvers are used. To determine viscous effects, either an integral or finite-difference approach is employed to solve the boundary-layer equations. When the interaction between the viscous and inviscid regions of the flow is not negligible, the computational procedures for these regions are coupled in either the direct (i.e., specify surface pressure) or the inverse mode (i.e., specify surface shear stress in the case of a solid wall). Although these techniques are efficient and provide reasonable estimates of viscous effects in many instances, they can be difficult to implement for three-dimensional flows, due to the variety of strong viscous-inviscid interactions, such as aircraft wing-body juncture flow, that can occur.

In the past few years, substantial progress has been made towards the goal of improving the mathematical models of the aerodynamic prediction techniques used in aircraft design. The Euler equations permit rotational effects (i.e., vortical structures) and nonisentropic shock waves, which can be present, in general, for flows over aerodynamic configurations. The Navier-Stokes equations model weak and strong interactions between viscous and inviscid flow regions without special consideration. Both the Euler and the time-averaged Navier-Stokes equations are currently being introduced into the design process. There are several reasons for this progress. A primary one is that the accuracy and efficiency of the numerical algorithms for solving these equations have been improved considerably. Another reason is the significant advancements in the memory capacity and processing times for computers. Although various technologies in computers and computer science will continue to help decrease processing times, this does not obviate the need for a strong effort to increase the robustness, accuracy, and efficiency of the flow solvers to allow their use in the analysis of complex fluid dynamics phenomena as well as in the design of aircraft.

A fairly extensive range of numerical algorithms has been developed to solve the Euler and Navier-Stokes equations during the past decade. These algorithms can be classified according to the type of time-stepping scheme and the type of spatial discretization scheme. Both explicit and implicit time-stepping methods have been constructed. Explicit schemes require less storage and a lower number of operations for time integration than implicit ones; however, they have a more severe limit on the allowable time step. If the temporal and spatial differencing are decoupled, both types of schemes are amenable to a variety of convergence acceleration techniques for steady-state problems. The explicit multistage Runge-Kutta scheme of Jameson, Schmidt, and Turkel [32] and the implicit Approximate Factorization (AF) scheme of Beam and Warming [9] are two examples of schemes that employ this decoupling. The multistage schemes in conjunction with local time stepping and other convergence enhancements [35] as well as the AF scheme with local time stepping and diagonalization of the implicit operator [75] have proven to be efficient schemes for the Euler equations.

For the spatial derivatives in the flow equations, central and one-sided differencing have been considered. To appreciate the selection of one type of differencing over another, it is important to understand the dominating design criterion for central and upwind schemes. In constructing a central difference scheme, the principal underlying guideline is to minimize the arithmetic operation count, and at the same time, maintain the highest possible accuracy. The multistage schemes and Lax-Wendroff schemes [73, 15] are the most widely used explicit algorithms with central spatial differencing at the present time. The AF scheme is the most frequently used implicit scheme with centered differencing. One of the main objectives for an upwind scheme is to capture flow discontinuities such as shock waves with the minimum number of mesh cells. In order to accomplish this, many of these schemes utilize the signs of the slopes of characteristics to determine the direction of propagation of information, and thus, the type of differencing for approximating spatial derivatives. Two procedures for the construction of upwind schemes for hyperbolic systems of conservation laws are the flux vector splitting of van Leer [117] and the flux difference splitting of Roe [83]. Such schemes have become quite popular due to their shock capturing capability. Generally, they represent shock waves with two interior cells rather than the three or four usually needed by central difference schemes, but they can require as much as two times the computational effort.

Multistage time-stepping schemes with a finite-volume formulation for spatial discretization and both structured and unstructured meshes are now being used for solving the Euler equations for flows over complex configurations, including an airplane [34, 38]. Members of this class of algorithms have also been extended to allow the solution of the compressible Navier-Stokes equations in both two and three dimensions [95, 118]. With the inclusion of convergence acceleration methods such as local time stepping and constant coefficient implicit residual smoothing, which extends the explicit time step limit, these solvers have proven to be reasonably effective. Significant improvements in performance have been achieved principally by using the multistage scheme as a driver of a multigrid method. This method, which involves a sequence of successively coarser meshes, has enhanced not only the convergence rate but also the robustness of the single-grid scheme. In [66], a three-stage Runge-Kutta scheme with multigrid was successfully applied to the two-dimensional (2-D) Navier-Stokes equations. Then, both Swanson and Turkel [97] and Martinelli and Jameson [68] demonstrated that the type of convergence behavior indicated in [66] could be substantially improved. The multigrid procedure was used to solve flow over a wing in [45]. Significant improvements in the performance of [45] were obtained [77, 119] by closely following and extending the ideas developed in the 2-D solvers [68, 97, 98].

In the current work, an efficient and versatile class of central difference, finite-volume multigrid schemes for the two-dimensional compressible Euler and Navier-Stokes equations is presented. The elements of these schemes are the basis for a family of computer codes (FLOMG series) developed by the authors. These codes are now being used in both industry and universities. They have been applied to numerous fluid dynamics problems over the last several years, and they have been employed as an analysis code in airfoil design procedures (i.e., [65]).

In Sections 2 and 3 of this report the flow equations are given and the finite-volume formulation for spatial discretization is described. Three alternatives for numerical approximation of viscous stress and heat flux terms are discussed. Then, the influence of grid stretching on numerical accuracy is determined. Section 4 discusses artificial dissipation. After outlining the historical development of a form for dissipation, the scalar dissipation model that has been frequently used with the present schemes is given in Section 4.2. The selection of boundary-point difference operators is an important consideration for the dissipation model. Suitable operators are given, and we show how local mode analysis can often provide an evaluation for a proposed boundary-point difference operator. Analysis based on considering the dissipative character of the discrete system of equations is also performed. In Section 4.5 the intimate connection between the formulation for an upwind scheme and a central difference scheme is examined. A foundation is established for a matrix dissipation model. Next, a theorem is proven that provides a sufficient condition for a central difference scheme with the dissipation form of Section 4.2 to be total variation diminishing (*TVD*). Imposing the requirements of the theorem we obtain a limiter function for the *TVD* property. The discussion on limiters also includes the symmetric limited positive (*SLIP*) scheme of Jameson [40]. In Section 4.8 the matrix dissipation model used with the present schemes is described. A matrix dissipation can allow the capturing of a shock wave with a single interior point. In Section 4.9 we discuss the convective upwind and split pressure *CUSP* scheme of Jameson [40] that also permits shocks with one interior point.

In Section 5 the discrete boundary conditions are discussed. Section 6 defines the class of explicit multistage time-stepping schemes that we consider and summarizes their properties. Then, the stability of Runge-Kutta schemes for systems of equations is examined. Subsequently, a time-step estimate for pseudo-time integration of the flow equations is given. Since the temporal and spatial discretizations are decoupled, these explicit schemes are amenable to convergence acceleration techniques. The techniques used here, which include local time stepping, implicit residual smoothing, and multigrid, are addressed in Section 7. In the initial part of this section we indicate how the discrete system of flow equations is preconditioned with local time stepping. In Section 7.2 there is first a discussion of constant coefficients for implicit residual smoothing. Some basic properties of residual smoothing are also presented. Then, a form for variable coefficients for implicit residual smoothing, which is based on stability analysis of a 2-D linear wave equation, is introduced. From this form two different formulas for variable smoothing coefficients evolve, and these formulas are compared. With these coefficients one still requires, in general, a time-step estimate that depends on a diffusion limit. In the last part of Section 7.2 we show how one can use such coefficients to construct new ones that can allow the removal of the diffusion restriction. Section 7.3 delineates the salient features of the multigrid method. It also includes some discussion of multigrid for hyperbolic equations. In Section 8 turbulence modeling is discussed. Both an algebraic equilibrium model and a half-equation nonequilibrium model are considered. Details for the present implementation of the turbulence models are given. Then, some numerical results are presented in Section 9. Different artificial dissipation schemes are compared on the basis of accuracy, convergence, and computational effort. In Section 10 some concluding remarks are made.

2 Mathematical Formulation

In this section we first define the integral form of the full Navier-Stokes equations. To complete the general mathematical formulation we then give boundary conditions for the infinite domain problem. In Section 3 we will discuss the discrete analogue of the full Navier-Stokes equations, and we will introduce the reduced form of these equations that is frequently solved in aerodynamic applications. Then, in Section 5 boundary conditions for the truncated (finite) domain problem will be given.

2.1 Equations

Let ρ, u, v, p, T, E, and H denote respectively the density, velocity components in the x and y Cartesian directions, pressure, temperature, specific total internal energy, and specific total enthalpy. The two-dimensional, unsteady Navier-Stokes equations, neglecting body forces and heat sources, can be written in conservative form in a Cartesian coordinate system as,

$$\frac{\partial}{\partial t}\iint_{\mathcal{S}} \mathbf{W} d\mathcal{V} + \int_{\partial\mathcal{S}} \mathcal{F}\cdot\mathbf{n} d\mathcal{S} = 0, \tag{2.1.1}$$

where

$$\mathbf{W} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \qquad \mathcal{F} = \begin{bmatrix} \rho q \\ \rho u q + p e_x + \bar{\tau}\cdot e_x \\ \rho v q + p e_y + \bar{\tau}\cdot e_y \\ \rho H q + \bar{\tau}\cdot q - Q \end{bmatrix},$$

and

$$\begin{aligned}
\mathbf{q} &= u\mathbf{e}_x + v\mathbf{e}_y, \\
\bar{\tau} &= \sigma_x \mathbf{e}_x\mathbf{e}_x + \tau_{xy}\mathbf{e}_x\mathbf{e}_y + \tau_{yx}\mathbf{e}_y\mathbf{e}_x + \sigma_y\mathbf{e}_y\mathbf{e}_y, \\
\sigma_x &= -\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\mu\frac{\partial u}{\partial x}, \\
\tau_{xy} &= \tau_{yx} = -\lambda\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\
\sigma_y &= -\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2\mu\frac{\partial v}{\partial y}, \\
\mathbf{Q} &= k\nabla T = k\left(\frac{\partial T}{\partial x}\mathbf{e}_x + \frac{\partial T}{\partial y}\mathbf{e}_y\right), \\
E &= e + \frac{1}{2}(u^2 + v^2), \\
H &= E + \frac{p}{\rho},
\end{aligned}$$

$\mathbf{e}_x$, $\mathbf{e}_y$ are unit vectors of the Cartesian coordinate system (x, y), and $\mathbf{n}$ is an outward pointing unit vector normal to the curve $\partial\mathcal{S}$ enclosing the region $\mathcal{S}$. In this paper, the working fluid is air, and it is assumed to be thermally and calorically perfect. That is, the equation of state is

$$p = \rho R T,$$

where $R = \bar{c}_p - \bar{c}_v$, and the specific heats $\bar{c}_p$, $\bar{c}_v$ are constant. The quantities μ and λ are the first and second coefficients of viscosity, respectively, and λ is taken to be $-\frac{2}{3}\mu$ (Stokes hypothesis). A simple power law or Sutherland's law can be used to determine the molecular viscosity coefficient μ. The coefficient of thermal conductivity (k) is evaluated using the constant Prandtl number assumption. The effect of turbulence is taken into account by using the eddy-viscosity hypothesis (see Section 8 on turbulence modeling).

2.2 Physical Boundary Conditions

In the continuum case we consider either an external or an internal flow problem defined for an infinite domain. Thus, we need to define appropriate conditions at wall boundaries, which we assume to be solid. Later, in the discrete case, we define finite domains. It will then be necessary to define suitable inflow and outflow boundary conditions.

For inviscid flows, the tangency (or nonpenetration) condition

$$\mathbf{q} \cdot \mathbf{n} = 0, \tag{2.2.1}$$

where $\mathbf{q}$ is the velocity vector and $\mathbf{n}$ is the unit vector normal to the surface, must be satisfied. Now, consider the vector momentum equation

$$\rho \frac{D\mathbf{q}}{Dt} = -\nabla p, \tag{2.2.2}$$

where $D\mathbf{q}/Dt$ denotes the substantial derivative of $\mathbf{q}$, and ∇ is the gradient operator. Clearly, along the surface boundary the substantial derivative of $\mathbf{q} \cdot \mathbf{n}$ must vanish. Therefore,

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla \right) (\mathbf{q} \cdot \mathbf{n}) = 0. \tag{2.2.3}$$

If one subtracts the inner product of the unit normal and (2.2.2) from (2.2.3), then

$$\rho \mathbf{q} \cdot (\mathbf{q} \cdot \nabla)\, \mathbf{n} = \mathbf{n} \cdot \nabla p. \tag{2.2.4}$$

Now, consider the transformation

$$(x, y) \to (\xi, \eta)\,,$$

taking $\eta(x, y) =$ constant to coincide with the surface boundary, and noting that the contravariant velocity component

$$V = -\frac{y_\xi}{J^{-1}} u + \frac{x_\xi}{J^{-1}} v$$

where the subscripts mean differentiation and J is the transformation Jacobian, is zero because of (2.2.1). Then, one obtains from (2.2.4) that

$$p_\eta = \frac{1}{\left(x_\xi^2 + y_\xi^2\right)} \left[\left(x_\xi x_\eta + y_\xi y_\eta\right) p_\xi + \left(y_\eta u - x_\eta v\right) \left(\rho v x_{\xi\xi} - \rho u y_{\xi\xi}\right) \right].$$

For viscous flows, the nonpenetration condition (2.2.1) and the no-slip condition

$$\mathbf{q} \cdot \mathbf{t} = 0,$$

where $\mathbf{t}$ is the unit vector tangent to the surface, must be satisfied. In addition, a boundary condition is required to determine the surface temperature. For this condition we either set the wall temperature to a specified value or we impose the adiabatic condition

$$\mathbf{Q} \cdot \mathbf{n} = 0,$$

where $\mathbf{Q}$ is the heat flux vector given in (2.1.1).

3 Spatial Discretization

A finite-volume approach is applied to discretize the equations of motion. The computational domain is divided into quadrilateral cells, fixed in time, and for each cell the governing equations can be nondimensionalized and written in integral form as follows:

$$\frac{\partial}{\partial t}\iint_{\Omega} \mathbf{W} dx dy + \int_{\partial\Omega} (\mathbf{F} dy - \mathbf{G} dx) = \frac{\sqrt{\gamma} M}{Re} \int_{\partial\Omega} (\mathbf{F}_v dy - \mathbf{G}_v dx), \tag{3.1}$$

where Ω is a generic cell (or cell area) with $\partial\Omega$ its boundary. In the scaling factor for the viscous terms on the right hand side of (3.1), the quantities γ, M, and Re are the specific heat ratio, Mach number, and Reynolds number, respectively, with M and Re defined in terms of nominal conditions. These factors arise because of our choice of nondimensionalization of the equations. The flux vectors are defined by

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{bmatrix},$$

and

$$\mathbf{F}_v = \begin{bmatrix} 0 \\ \sigma_x \\ \tau_{xy} \\ u\sigma_x + v\tau_{xy} - k\frac{\partial T}{\partial x} \end{bmatrix}, \qquad \mathbf{G}_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \sigma_y \\ u\tau_{yx} + v\sigma_y - k\frac{\partial T}{\partial y} \end{bmatrix}.$$

The independent variables x, y, and t are nondimensionalized as

$$x = \frac{\tilde{x}}{\tilde{l}_{\text{ref}}}, \qquad y = \frac{\tilde{y}}{\tilde{l}_{\text{ref}}}, \qquad t = \tilde{t}\,\frac{\tilde{u}_{\text{ref}}}{\tilde{l}_{\text{ref}}},$$

where $\tilde{u}_{\text{ref}} = \sqrt{\tilde{p}_{\text{ref}}/\tilde{\rho}_{\text{ref}}}$, and the tilde in this section means dimensional variable. Examples of a reference length are the chord for an airfoil and the throat height for a nozzle flow. The thermodynamic variables (p, ρ, T) and the transport coefficients (μ, k) are nondimensionalized by their corresponding quantity evaluated at some reference condition. The velocity components are scaled by $\tilde{u}_{\text{ref}}$, and the total quantities E and H are scaled by $\tilde{u}^2_{\text{ref}}$. For external flows, the nominal conditions are based on free-stream values, and for internal flows, they are based on stagnation values.

Let the computational region be partitioned with quadrilaterals, and apply (3.1) to each quadrilateral. This process is equivalent to performing a mass, momentum, and energy balance on each cell. By decoupling the temporal and spatial terms, a system of ordinary differential equations is obtained. In particular, consider an arbitrary quadrilateral (i.e., $ABCD$ in Figure 1) and approximate the line integrals of (3.1) with the midpoint rule. Let the indices (i, j) identify a cell. Then, taking $\mathbf{W}_{i,j}$ as the cell-averaged solution vector, equation (3.1) can be written in semidiscrete form as

$$\frac{d}{dt}(\Omega_{i,j}\mathbf{W}_{i,j}) + \mathcal{L}\mathbf{W}_{i,j} = 0, \tag{3.2}$$

where $\Omega_{i,j}$ is the area of the cell, and $\mathcal{L}$ is a spatial discretization operator defined by

$$\mathcal{L} = \mathcal{L}_C + \mathcal{L}_D + \mathcal{L}_{AD}, \tag{3.3}$$

with the subscripts C, D, and AD referring to convection, diffusion, and artificial dissipation.

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Figure 1: Finite-volume discretization.

The convective fluxes at the cell faces are obtained by an averaging process. Moreover, summing over the faces,

$$\mathcal{L}_C\mathbf{W}_{i,j} = \sum_{l=1}^{4} (\mathcal{F}_C)_l \cdot \mathbf{S}_l, \tag{3.4}$$

with the flux tensor, which is associated with convection, given by

$$(\mathcal{F}_C)_l = \mathbf{F}_l\mathbf{e}_x + \mathbf{G}_l\mathbf{e}_y,$$

and for each cell face l, the directed length $\mathbf{S}_l$ is expressed as

$$\mathbf{S}_l = (\Delta y)_l\,\mathbf{e}_x - (\Delta x)_l\,\mathbf{e}_y, \tag{3.5}$$

where the proper signs of $(\Delta x)_l$ and $(\Delta y)_l$ produce an outward normal to the cell face. The augmented convective flux tensor is evaluated as

$$\begin{aligned}
(\mathcal{F}_C)_l &= \frac{1}{2}(\mathbf{W}^{-}\mathbf{q}^{-} + \mathbf{W}^{+}\mathbf{q}^{+})_l + P_l, \qquad (3.6)\\
P_l &= \begin{bmatrix} 0 & (p_{\text{avg}})_l\mathbf{e}_x & (p_{\text{avg}})_l\mathbf{e}_y & ((p\mathbf{q})_{\text{avg}})_l \end{bmatrix}^T,\\
(p_{\text{avg}})_l &= \frac{1}{2}(p^{-} + p^{+})_l, \qquad ((p\mathbf{q})_{\text{avg}})_l = \frac{1}{2}(p^{-}\mathbf{q}^{-} + p^{+}\mathbf{q}^{+})_l,
\end{aligned}$$

and the superscripts minus and plus indicate quantities taken from the two cell centers adjacent to the edge l. The symbol $\mathbf{q}$ denotes the velocity vector. In this section the subscript avg always refers to the simple average defined in (3.6) for a given edge l.

The contribution of the diffusive fluxes in (3.2) is evaluated as

$$\mathcal{L}_D\mathbf{W}_{i,j} = \sum_{l=1}^{4} (\mathcal{F}_D)_l \cdot \mathbf{S}_l,$$

where $\mathbf{S}_l$ is given by (3.5), and

$$(\mathcal{F}_D)_l = (\mathbf{F}_v)_l\,\mathbf{e}_x + (\mathbf{G}_v)_l\,\mathbf{e}_y.$$

There are first-order spatial derivatives in the flux vectors $\mathbf{F}_v$ and $\mathbf{G}_v$. In the present finite-volume method, these derivatives are determined using Green's theorem. For example, consider the cell face BC in Figure 1. The contributions u_x and u_y to the viscous flux across BC are approximated by their mean values

$$\begin{aligned}
(u_x)_{i,j+\frac{1}{2}} &= (u_x)_{BC} = \frac{1}{\Omega'}\iint_{\Omega'} u_x dx dy\\
&= \frac{1}{\Omega'}\int_{\partial\Omega'} u dy,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
(u_y)_{i,j+\frac{1}{2}} &= (u_y)_{BC} = \frac{1}{\Omega'}\iint_{\Omega'} u_y dx dy\\
&= -\frac{1}{\Omega'}\int_{\partial\Omega'} u dx,
\end{aligned} \tag{3.8}$$

where Ω' is the area of an appropriate auxiliary cell. We have considered three alternatives for computing the diffusion-type terms. A comparison is now made between these three choices. The first two approximations for a diffusive flux are obtained with finite-volume methods, while the third one is determined with a frequently used method based on a local transformation of coordinates.

For the comparison we consider the molecular transport processes associated with cell face BC for the x-momentum equation only. Let

$$\Phi_{BC} = \left[(\mathcal{F}_D)_{BC} \cdot \mathbf{S}_{BC}\right]_2 = (\sigma_x \Delta y - \tau_{yx} \Delta x)_{BC} .$$

In one finite-volume method we consider the integration path $A'B'C'D'$ shown in Figure 1, which is the path used in [95] and [74]. Applying the midpoint rule for the required line integrals, we get

$$\begin{aligned} \Phi_{BC} &= \frac{\mu_{\text{avg}}}{\Omega'} \left[\phi_1 u_{i,j+1} + \phi_2 u_{i,j} + \phi_3 u_B + \phi_4 u_C\right] \\ &\quad + \frac{\mu_{\text{avg}}}{\Omega'} \left[\phi_5 v_{i,j+1} + \phi_6 v_{i,j} + \phi_7 v_B + \phi_8 v_C\right], \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \phi_1 &= \frac{4}{3}\Delta y_{BC} \Delta y_{B'C'} + \Delta x_{BC} \Delta x_{B'C'}, & \phi_2 &= \frac{4}{3}\Delta y_{BC} \Delta y_{D'A'} + \Delta x_{BC} \Delta x_{D'A'}, \\ \phi_3 &= \frac{4}{3}\Delta y_{BC} \Delta y_{A'B'} + \Delta x_{BC} \Delta x_{A'B'}, & \phi_4 &= \frac{4}{3}\Delta y_{BC} \Delta y_{C'D'} + \Delta x_{BC} \Delta x_{C'D'}, \\ \phi_5 &= \frac{2}{3}\Delta y_{BC} \Delta x_{B'C'} - \Delta x_{BC} \Delta y_{B'C'}, & \phi_6 &= \frac{2}{3}\Delta y_{BC} \Delta x_{D'A'} - \Delta x_{BC} \Delta y_{D'A'}, \\ \phi_7 &= \frac{2}{3}\Delta y_{BC} \Delta x_{A'B'} - \Delta x_{BC} \Delta y_{A'B'}, & \phi_8 &= \frac{2}{3}\Delta y_{BC} \Delta x_{C'D'} - \Delta x_{BC} \Delta y_{C'D'} \end{aligned}$$

$$u_B = \frac{1}{4}\left(u_{i,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1}\right), \quad u_C = \frac{1}{4}\left(u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i-1,j+1}\right),$$

with v_B and v_C defined similarly, and

$$\Delta x_{BC} = x_C - x_B, \qquad \Delta y_{BC} = y_C - y_B,$$

$$\mu_{\text{avg}} = \frac{1}{2}\left(\mu_{i,j} + \mu_{i,j+1}\right), \qquad \Omega' = \frac{1}{2}\left(\Omega_{i,j} + \Omega_{i,j+1}\right).$$

In [67] Martinelli introduced a different integration path, which is delineated as $BFCE$ in Figure 2, for calculating the viscous terms. Integrating around the boundary $BFCE$ with the trapezoidal rule, one obtains

$$\begin{aligned} \Phi_{BC} &= \frac{\mu_{\text{avg}}}{2\Omega''} \left[\left(\frac{4}{3}\Delta y_{BC}^2 + \Delta x_{BC}^2\right)(u_{i,j+1} - u_{i,j}) - \left(\frac{1}{3}\Delta x_{BC} \Delta y_{BC}\right)(v_{i,j+1} - v_{i,j})\right] \\ &+ \frac{\mu_{\text{avg}}}{2\Omega''} \left[\left(\frac{4}{3}\Delta y_{EF} \Delta y_{BC} + \Delta x_{EF} \Delta x_{BC}\right)(u_B - u_C)\right] \\ &+ \frac{\mu_{\text{avg}}}{2\Omega''} \left[\left(\frac{2}{3}\Delta x_{EF} \Delta y_{BC} - \Delta y_{EF} \Delta x_{BC}\right)(v_B - v_C)\right], \end{aligned} \tag{3.10}$$

where

$$\Delta x_{EF} = x_{i,j+1} - x_{i,j}, \qquad \Delta y_{EF} = y_{i,j+1} - y_{i,j},$$

and Ω'' is the area of the region enclosed by $BFCE$. The area Ω'' is given by

$$\Omega'' = \frac{1}{2}\left(\Delta x_{CB} \Delta y_{EF} - \Delta x_{EF} \Delta y_{CB}\right).$$

All other quantities in (3.10) are defined the same as for (3.9). The form of Φ_{BC} given by (3.10) is much more compact, requiring fewer arithmetic operations than that given by (3.9).

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Figure 2: Alternative integration path for physical diffusive fluxes.

A third approach for computing the diffusion-type terms is based on a local transformation from Cartesian coordinates (x, y) to arbitrary curvilinear coordinates (ξ, η). Derivatives with respect to x and y are expanded according to the chain rule for partial differentiation. The resulting relation for Φ_{BC} is as follows:

$$\begin{aligned} \Phi_{BC} \;=\; & \frac{\mu_{\text{avg}}}{\Omega'}\left[\left(\frac{4}{3}\Delta y_{CB} y_\xi + \Delta x_{CB} x_\xi\right) u_\eta + \left(\frac{2}{3}\Delta y_{CB} x_\xi - \Delta x_{BC} y_\xi\right) v_\eta\right]_{BC} \\ & + \frac{\mu_{\text{avg}}}{\Omega'}\left[\left(\frac{4}{3}\Delta y_{BC} y_\eta + \Delta x_{BC} x_\eta\right) u_\xi + \left(\frac{2}{3}\Delta y_{BC} x_\eta - \Delta x_{BC} y_\eta\right) v_\xi\right]_{BC}, \qquad (3.11) \end{aligned}$$

where

$$x_\xi = \Delta x_{CB}\ , \ y_\xi = \Delta y_{CB}, \ \ x_\eta = \Delta x_{EF}\ , \ y_\eta = \Delta y_{EF}.$$

With a uniformly spaced computational domain ($\Delta\xi = \Delta\eta = 1$), this representation for Φ_{BC} is the same as that of (3.10), except for the area factor. In the case of a Cartesian mesh, the expressions for Φ_{BC} in (3.10 and (3.11), as well as that in (3.9), are equivalent. If the streamwise-like differences associated with the viscous flux quantities are neglected, which is the thin-layer Navier-Stokes assumption, only the terms inside the first set of brackets are retained.

Note that with the thin-layer formulation, there are viscous contributions to the fluxes at faces BC and DA only. The following vector approximates the integrand of the right hand side of (3.1) at cell faces BC and DA:

$$\Phi_l = \begin{bmatrix} 0 \\ \tau_1 \\ \tau_2 \\ u_{avg}\tau_1 + v_{avg}\tau_2 + \tilde{Q} \end{bmatrix}_l ,$$

where

$$\tau_1 = \frac{\mu_{\text{avg}}}{\Omega_{\text{avg}}}\left(\phi_1 u_\eta - \phi_2 v_\eta\right), \quad \tau_2 = \frac{\mu_{\text{avg}}}{\Omega_{\text{avg}}}\left(\phi_3 v_\eta - \phi_2 u_\eta\right), \quad \tilde{Q} = \frac{\mu_{\text{avg}}}{Pr\ \Omega_{\text{avg}}}\left(\frac{\gamma}{\gamma - 1}\right)\phi_4 T_\eta,$$

$$\phi_1 = \frac{4}{3}y_\xi^2 + x_\xi^2, \qquad \phi_2 = \frac{1}{3}x_\xi y_\xi, \qquad \phi_3 = \frac{4}{3}x_\xi^2 + y_\xi^2, \qquad \phi_4 = x_\xi^2 + y_\xi^2.$$

Unless otherwise indicated we solve the thin-layer form of the equations.

In applying the three methods for approximating the diffusive terms we have not observed significant differences in the numerical solutions. From the comparison of the methods this is an expected result when solving the Navier-Stokes (thin-layer or full) equations on sufficiently smooth meshes (i.e., meshes without kinks or sudden jumps in mesh intervals). In the present work the integration path of Martinelli [67] is used in the finite-volume method for computing the viscous fluxes. With this choice the mean values of the viscous stresses for a given cell edge are obtained at the midpoint of the edge, even when there is a kink in the grid. This is not true for the path used in (3.9). Also, with the integration path of (3.10) there are fewer arithmetic operations required than with the path of (3.9).

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Figure 3: One-dimensional discretization for three point cell-centered scheme.

Theoretical estimates of the order of accuracy of the cell-averaged scheme are now introduced based on one-dimensional analysis using Taylor series expansions. Consider the coordinate grid around the location denoted by the index i, as displayed in Figure 3. Let ϕ be a test function. Then the numerical values of the first and second derivatives of this function are given by

$$(\phi_x)_{\text{num}} = \frac{1}{2}\phi_x \frac{\Delta x_+ + \Delta x_-}{\Delta x} + \frac{1}{4}\phi_{xx}\frac{\Delta x_+^2 - \Delta x_-^2}{\Delta x} + O(\Delta x^2), \tag{3.12}$$

$$\begin{aligned}(\phi_{xx})_{\text{num}} &= \frac{1}{2}\phi_{xx}\frac{\Delta x_+ \Delta x_-}{\Delta x} + \frac{1}{6}\phi_{xxx}\frac{\Delta x_+^2 - \Delta x_-^2}{\Delta x} \\ &\quad + \frac{1}{24}\phi_{xxxx}\frac{\Delta x_+^3 + \Delta x_-^3}{\Delta x} + O(\Delta x^3),\end{aligned} \tag{3.13}$$

where the derivatives in the expansions are evaluated at i. The approximations of (3.12) and (3.13) are zeroth-order accurate on arbitrarily stretched meshes. However, assuming a constant stretching factor of the grid (i.e., $\beta = \Delta x_{++}/\Delta x = \text{const.}$), we obtain the following relations:

$$\begin{aligned} \Delta x_{--} &= \Delta x\,\frac{1}{\beta}, \qquad \Delta x_{-} = \frac{1}{2}\Delta x\left(1+\frac{1}{\beta}\right), \\ \Delta x_{++} &= \Delta x\,\beta, \qquad \Delta x_{+} = \frac{1}{2}\Delta x\,(1+\beta)\,, \end{aligned} \tag{3.14}$$

For viscous flows, grids with constant stretching factor β are often used. If these grids are refined by doubling the number of points,

$$\beta_f = \sqrt{\beta_c} < 1 + \frac{\beta_c - 1}{2}, \quad \beta \geq 1,$$

where the subscripts refer to fine and coarse grids. In order to estimate the error reduction when refining the stretched mesh, we take

$$\beta \approx 1 + C_\beta \Delta x. \tag{3.15}$$

Then, if the quantities in (3.14) and (3.15) are substituted into (3.12) and (3.13),

$$(\phi_x)_{\text{num}} = \phi_x + \frac{1}{4}\phi_x\,(C_\beta\Delta x)^2 + \frac{1}{2}C_\beta\phi_{xx}\Delta x^2 + O(\Delta x^2), \tag{3.16}$$

$$(\phi_{xx})_{\text{num}} = \phi_{xx} + \frac{1}{4}\phi_{xx}\,(C_\beta\Delta x)^2 + \frac{1}{3}C_\beta\phi_{xxx}\Delta lx^2 + \frac{1}{12}\phi_{xxxx}\Delta x^2 + O(\Delta x^3), \tag{3.17}$$

for $\Delta x << 1$. Thus, second-order accuracy is achieved for the inviscid and viscous terms in the flow equations on smoothly stretched meshes. Additional discussion of accuracy is found in [111].

4 Artificial Dissipation

For inviscid flows the basic finite-volume scheme just described does not contain any dissipative terms. In order to prevent oscillations near shock waves or stagnation points, artificial dissipation terms are added to the governing discrete equations. The introduction of appropriate dissipation in the vicinity of shock waves allows an entropy condition to be satisfied. In gas dynamics such a condition can be the second law of thermodynamics, which states that the physical entropy cannot decrease. This condition guarantees the uniqueness of weak solutions (i.e., solutions containing shock waves); and thus, it ensures a physically correct solution (see [31] for further discussion). Another type of dissipation term is also added to the discrete flow equations. These terms are included to provide background dissipation, which is important for converging the numerical scheme that will be used to compute flow solutions. They also prevent odd-even point decoupling (i.e., creation of sawtooth or plus-minus waves with wavelength of two times the mesh spacing). For viscous flows, dissipative properties are present due to diffusive terms. However, due to the nonlinearity of the equations of motion, the physical dissipation may not be sufficient to guarantee stability, especially in the case of the highly stretched meshes generally used to resolve the steep gradients in shear layers. Thus, to maintain the stability and robustness of the numerical procedure, artificial dissipation is also included in viscous regions.

In this section we first discuss some historical information regarding the form of the artificial (or numerical) dissipation model used with many central difference schemes. Through this discussion we indicate how the model evolved, and in so doing, we provide a rudimentary understanding of the model. Then, the basic dissipation formulation, along with various modifications that we have investigated, is discussed. For the dissipation model boundary-point difference stencils are required. Several stencils are considered and analyzed. We next examine the intimate connection between the formulation for an upwind scheme and a central difference scheme, establishing a foundation for a matrix dissipation model. After discussing this connection, we prove a theorem that provides a simple sufficient condition to determine when a central difference scheme with dissipation terms comprised of second and fourth differences is total variation diminishing (*TVD*). In order to satisfy automatically the *TVD* property, a limiter function is selected. Appropriate flux limiter functions consistent with the central difference dissipation model are discussed. We also discuss the symmetric limited positive (*SLIP*) scheme of Jameson [40]. This scheme starts with a first-order diffusive flux and introduces anti-diffusion controlled by a limiter function with symmetric properties. It satisfies the local extremum diminishing (*LED*) property and produces a pure upwind scheme for the one-dimensional Euler equations at shock waves. A scheme is defined to be *LED* if local maxima do not increase and local minima do not decrease. The *LED* property reduces to the TVD property in one dimension. In a subsequent section we present the matrix dissipation model that we currently use. This model relies upon a characteristic decomposition of a flux vector. We also discuss the convective upwind split pressure (*CUSP*) scheme of Jameson [40], which employs an alternative decomposition of a flux vector.

4.1 Development of Dissipation Form

To facilitate the presentation of the historical notes, we consider the one-dimensional system of hyperbolic equations

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0,$$

where $\mathbf{W}$ and $\mathbf{F}$ are three-component state and flux vectors, respectively. Let the one-dimensional domain be partitioned by intervals defined by $\Delta x = x_{i+1/2} - x_{i-1/2}$, where the indices refer to interface points for adjacent intervals. Suppose we apply the Lax-Wendroff scheme

$$\mathbf{W}^{n+1} = \mathbf{W}^{n} - \tau(\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}), \tag{4.1.1}$$

where the superscript indicates time level, $\tau = \Delta t/\Delta x$, and the interface flux is

$$\mathbf{F}_{i+1/2} = \mathbf{F}^{LW}_{i+1/2} = \frac{1}{2}(\mathbf{F}_i + \mathbf{F}_{i+1}) - \frac{1}{2}\tau A^2_{i+1/2}(\mathbf{W}_{i+1} - \mathbf{W}_i), \tag{4.1.2}$$

with A representing the flux Jacobian matrix (an element $A_{jk} = \partial F_j/\partial W_k$). Unless noted otherwise all quantities are evaluated at time level n. In the initial work of computing flows with shock waves using the Lax-Wendroff scheme, the solutions contained oscillations in the vicinity of the shock wave. Then, von Neumann and Richtmyer [122] introduced an additional dissipation term to remove such oscillations. With the inclusion of this term one rewrites (4.1.2) as

$$\mathbf{F}_{i+1/2} = \mathbf{F}^{LW}_{i+1/2} - d^{(2)}_{i+1/2}(\mathbf{W}_{i+1} - \mathbf{W}_i),$$

or in the continuum

$$\mathbf{F} = \mathbf{F}^* - \Delta x d^{(2)} \frac{\partial \mathbf{W}}{\partial x} = \mathbf{F}^* - \mathbf{D}^{(2)},$$

where F^* is the physical flux function. The term $d^{(2)}$ is often called an artificial (or numerical) viscosity, and it plays the role of a control function. As pointed out by Hirsch [31] the form of $\mathbf{D}^{(2)}$ considered by von Neumann and Richtmyer [122] can be written for a system as

$$\mathbf{D}^{(2)} = \varepsilon^{(2)} \Delta x^2 \Psi \left| \frac{\partial \mathbf{W}}{\partial x} \right| \frac{\partial \mathbf{W}}{\partial x}, \tag{4.1.3}$$

where the coefficients $\Psi \geq 0$ and can depend on the mesh index i, and each element of $\mathbf{D}^{(2)}$ depends on the corresponding element of $\mathbf{W}$. Now, suppose we compute the flux difference $(F_{i+1/2} - F_{i-1/2})$ in (4.1.1) using (4.1.3). Then, in the case of the continuum, the total dissipation is given by

$$\mathbf{D}^{(2)}_{\text{tot}} = \varepsilon^{(2)} \Delta x^3 \frac{\partial}{\partial x} \left[\Psi \left| \frac{\partial \mathbf{W}}{\partial x} \right| \frac{\partial \mathbf{W}}{\partial x} \right].$$

One might characterize this dissipation term as third order. However, a Δx^{-1} appears in (4.1.1), so effectively we have a second-order term.

In 1975 MacCormack and Baldwin [60] appended a dissipation term for shock capturing to the 1969 scheme of MacCormack (a two-step Lax-Wendroff type scheme [61]). This term was introduced to remove oscillations at shock waves due to the spatial differencing of the MacCormack scheme. It is proportional to a second difference of the pressure and is given by

$$\mathbf{D}^{(2)} = \varepsilon^{(2)} \Delta x^3 \frac{|u| + c}{4p} \left| \frac{\partial^2 p}{\partial x^2} \right| \frac{\partial \mathbf{W}}{\partial x}. \tag{4.1.4}$$

In smooth regions of a flow field the product of Δx^{-1} and the dissipative flux balance $\mathbf{D}^{(2)}_{\text{tot}}$ is third order, while it is first order in the neighborhood of a shock wave.

As we have indicated, numerical dissipation is not only important in capturing discontinuities, it is also generally required to maintain stability and to provide the necessary background dissipation for convergence. In 1976 Beam and Warming [9] added to the explicit side of their implicit approximate factorization (AF) scheme, what they called, a fourth-order dissipation term, so as to damp high-frequency error components. In the context of our discussion with the Lax-Wendroff scheme, this term would appear as

$$\mathbf{D}^{(4)}_{\text{tot}} = -\varepsilon^{(4)} \Delta x^4 \frac{\partial^4 \mathbf{W}}{\partial x^4}. \tag{4.1.5}$$

It seems much more appropriate to define the order of the dissipation relative to the spatial discretization of the physical terms in the equation. So we consider $\Delta x^{-1} \mathbf{D}^{(4)}_{\text{tot}}$, and thus the dissipation term is third order. At any rate, a fourth-difference dissipation is included. This term and the second-difference term of (4.1.4) provide the basic ingredients for constructing a complete adaptive dissipation model. The second-difference term of (4.1.4) allows shock capturing without oscillations, while the linear fourth-difference term of (4.1.5) provides the important background dissipation. The one critical element missing is a switching function, that would turn on the appropriate dissipation form in a region and turn off the one that is not the desirable type (i.e., near shocks (4.1.4) should dominate with (4.1.5) negligible, while in smooth regions (4.1.5) should dominate with (4.1.4) negligible). In the next section we describe the dissipation model that added a switching function to the two basic types of terms just discussed.

4.2 Dissipation Model

In order to permit a complete description of the dissipation model, we now consider the two-dimensional Euler equations. The dissipation is based on the model introduced by Jameson, Schmidt, and Turkel [32]. This model defined a suitable switching function, at least for transonic and low supersonic flow, to allow blending of the second and fourth differences. According to the nonlinear model of [32], the quantity $\mathcal{L}_{AD}\mathbf{W}_{i,j}$ in (3.2) is expressed as

$$\mathcal{L}_{AD}\mathbf{W}_{i,j} = -(D_\xi^2 - D_\xi^4 + D_\eta^2 - D_\eta^4)\mathbf{W}_{i,j}, \tag{4.2.1}$$

where (ξ, η) are arbitrary curvilinear coordinates,

$$\begin{aligned} D_\xi^2\mathbf{W}_{i,j} &= \nabla_\xi \left[(\lambda_{i+1/2,j}\, \varepsilon^{(2)}_{i+1/2,j})\Delta_\xi\right] \mathbf{W}_{i,j}, & (4.2.2)\\ D_\xi^4\mathbf{W}_{i,j} &= \nabla_\xi \left[(\lambda_{i+1/2,j}\, \varepsilon^{(4)}_{i+1/2,j})\Delta_\xi \nabla_\xi \Delta_\xi\right] \mathbf{W}_{i,j}, & (4.2.3) \end{aligned}$$

i, j are indices (for a cell center) associated with the ξ and η directions, and Δ_ξ, ∇_ξ are forward and backward difference operators in the ξ direction. In the η direction the definitions are similar. The variable scaling factor λ is defined as

$$\lambda_{i+1/2,j} = \frac{1}{2}\left[(\lambda_\xi)_{i,j} + (\lambda_\xi)_{i+1,j} + (\lambda_\eta)_{i,j} + (\lambda_\eta)_{i+1,j}\right], \tag{4.2.4}$$

where λ_ξ and λ_η are the largest eigenvalues in absolute value (i.e., spectral radii) of the flux Jacobian matrices associated with the Euler equations. These spectral radii are given by

$$\begin{aligned} \lambda_\xi &= |u y_\eta - v x_\eta| + c\sqrt{y_\eta^2 + x_\eta^2}, \qquad (4.2.5)\\ \lambda_\eta &= |v x_\xi - u y_\xi| + c\sqrt{x_\xi^2 + y_\xi^2}, \end{aligned}$$

where u and v are Cartesian velocity components, and c is the speed of sound. The coefficients $\varepsilon^{(2)}$ and $\varepsilon^{(4)}$ use the pressure as a sensor for shocks and stagnation points, and they are defined as

$$\begin{aligned} \varepsilon^{(2)}_{i+1/2,j} &= \kappa^{(2)} \max(\nu_{i-1,j}, \nu_{i,j}, \nu_{i+1,j}, \nu_{i+2,j}),\\ \nu_{i,j} &= \left|\frac{p_{i-1,j} - 2p_{i,j} + p_{i+1,j}}{p_{i-1,j} + 2p_{i,j} + p_{i+1,j}}\right| \qquad (4.2.6)\\ \varepsilon^{(4)}_{i+1/2,j} &= \max\left[0, (\kappa^{(4)} - \varepsilon^{(2)}_{i+1/2,j})\right] \end{aligned}$$

where typical values for the constants $\kappa^{(2)}$ and $\kappa^{(4)}$ are in the ranges $\frac{1}{4}$ to $\frac{1}{2}$ and $\frac{1}{64}$ to $\frac{1}{32}$, respectively. We shall refer to (4.2.1) and (4.2.6) as the Jameson, Schmidt, Turkel (*JST*) scheme (or dissipation model). Also, we shall designate ν the *JST* switch. It should be mentioned that in [32] the coefficient $\varepsilon^{(2)}_{i+1/2,j} = \kappa^{(2)} \max(\nu_{i,j}, \nu_{i+1,j})$. The switching function ν can be interpreted as a limiter, in the sense that it activates the second-difference contribution at extrema and switches off the fourth-difference term. Moreover, at shock waves the dissipation is first order, and a first-order upwind scheme is produced for a scalar equation. In smooth regions of the flow field the dissipation is third order.

Thus, we have two different dissipation mechanisms at work. The switch determines which one is active in any given region. For smooth flows, ν is small and the dissipation terms consists of a linear fourth difference that damps the high frequencies which the central difference scheme does not damp. This is useful mainly for achieving a steady state and is less important for time dependent problems. In the neighborhood of large gradients in the pressure, ν becomes large and switches on the second difference viscosity while simultaneously reducing the fourth difference dissipation. This is mainly needed to introduce an entropy condition to reduce overshoots near discontinuities and choose the correct shock relationships. For subsonic steady state flow this can be turned off by choosing $\epsilon^{(2)} = 0$.

The isotropic scaling factor of (4.2.4) is generally satisfactory for inviscid flow problems when typical inviscid flow meshes (i.e., cell aspect ratio $O(1)$) are used. The factor can create too much numerical dissipation in cases of meshes with high-aspect-ratio cells. This is an important consideration for high Reynolds number viscous flows, where a mesh providing appropriate spatial resolution can have cell aspect ratios $O(10^3)$. In an effort to improve this situation, as well as to obtain sharper shock resolution on a given grid, Swanson and Turkel [97] replaced the isotropic scaling factor of (4.2.4) with the anisotropic one

$$\lambda_{i+1/2,j} = \frac{1}{2}\left[(\lambda_\xi)_{i,j} + (\lambda_\xi)_{i+1,j}\right]. \tag{4.2.7}$$

A similar scaling is used in the η direction. This idea was motivated by the scaling of dissipation occurring in dimensionally split upwind schemes (i.e., flux vector split scheme [117], approximate Riemann solver [83]). We often refer to this scaling as individual eigenvalue scaling. While the accuracy is improved with (4.2.7), especially with respect to shock resolution, this type of scaling in the streamwise (ξ) direction can be too severe for a standard multigrid algorithm. Moreover, the effectiveness of the multigrid driving scheme in damping high frequencies in the ξ direction can be significantly diminished, resulting in a much slower convergence rate.

An alternative to the individual eigenvalue scaling was proposed by Martinelli [67] and considered by Swanson and Turkel [97]. This modified scaling factor, which is a function of mesh cell aspect ratio, is defined as

$$\lambda_{i+1/2,j} = \frac{1}{2}[(\bar{\lambda}_\xi)_{i,j} + (\bar{\lambda}_\xi)_{i+1,j}], \tag{4.2.8}$$

where

$$\begin{aligned} (\bar{\lambda}_\xi)_{i,j} &= \phi_{i,j}(r)\,(\lambda_\xi)_{i,j}, \\ \phi_{i,j}(r) &= 1 + r^{\zeta}_{i,j}, \end{aligned} \tag{4.2.9}$$

r is the ratio λ_η/λ_ξ, and the exponent ζ is generally taken to be between $\frac{1}{2}$ and $\frac{2}{3}$. In the normal direction (η), one defines

$$(\bar{\lambda}_\eta)_{i,j} = \phi_{i,j}(r^{-1})\ (\lambda_\eta)_{i,j}.$$

Thus, the scaling factor of (4.2.8) is bounded from below by (4.2.7) and bounded from above by (4.2.4). As demonstrated in [68] and [97], it produces a significant improvement in accuracy for high-aspect-ratio meshes and permits good convergence rates with a multigrid method. The present scheme uses this scaling factor.

We now examine the impact of the dissipation form on the energy of a system of flow equations. For simplicity, consider the one-dimensional time dependent Euler equations with numerical dissipation terms as given by

$$J^{-1}\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \xi} = D_\xi^2\mathbf{W} - D_\xi^4\mathbf{W}, \tag{4.2.10}$$

where

$$D_\xi^2\mathbf{W} = \frac{\partial}{\partial \xi}\left(\bar{\varepsilon}^{(2)}\frac{\partial \mathbf{W}}{\partial \xi}\right), \qquad D_\xi^4\mathbf{W} = \frac{\partial}{\partial \xi}\left(\bar{\varepsilon}^{(4)}\frac{\partial^3 \mathbf{W}}{\partial \xi^3}\right),$$

$\bar{\varepsilon}^{(2)} = \lambda\varepsilon^{(2)}$, $\bar{\varepsilon}^{(4)} = \lambda\varepsilon^{(4)}$, and J^{-1} is the inverse transformation Jacobian. Suppose we form the inner products of $\mathbf{W}^T$, with T denoting transpose, and both sides of (4.2.10) and integrate over a domain Ω. After integration by parts and neglecting boundary terms we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\mathbf{W}\|^2 = \text{flux term} + J\left[I^{(2)} - I^{(4)}\right],$$

where

$$\begin{aligned}
\|\mathbf{W}\|^2 &= \int_\Omega \mathbf{W}^2 d\xi, \\
I^{(2)} &= -\int_\Omega \bar{\varepsilon}^{(2)}\left(\frac{\partial \mathbf{W}}{\partial \xi}\right)^2 d\xi, \\
I^{(4)} &= -\int_\Omega \bar{\varepsilon}^{(4)}\left(\frac{\partial^2 \mathbf{W}}{\partial \xi^2}\right)^2 d\xi + \int_\Omega \frac{\partial \bar{\varepsilon}^{(4)}}{\partial \xi}\left(\frac{\partial \mathbf{W}^T}{\partial \xi}\right)\cdot\left(\frac{\partial^2 \mathbf{W}}{\partial \xi^2}\right) d\xi.
\end{aligned}$$

The second-difference dissipation term $I^{(2)}$ only decreases the L^2 norm of the solution vector (i.e., it decreases the energy of the system), and thus, is strictly dissipative. The fourth-difference dissipation term $I^{(4)}$ contains a dispersive part as well as a dissipative contribution.

An alternative form for the third-order dissipation term (the last term in (4.2.10)) is

$$D_\xi^4\mathbf{W} = \frac{\partial^2}{\partial \xi^2}\left(\bar{\varepsilon}^{(4)}\frac{\partial^2 \mathbf{W}}{\partial \xi^2}\right),$$

which in the discrete case can be written as

$$D_\xi^4\mathbf{W}_i = \nabla_\xi\Delta_\xi\left[(\lambda_i\ \varepsilon_i^{(4)})\nabla_\xi\Delta_\xi\right]\mathbf{W}_i. \tag{4.2.11}$$

This modified form produces only dissipative contributions. If one defines

$$\mathbf{Q}_i = \lambda_i \varepsilon_i^{(4)} \nabla_\xi \Delta_\xi \mathbf{W}_i,$$

then

$$\nabla_\xi \Delta_\xi \mathbf{Q}_i = \delta \mathbf{Q}_{i+1/2} - \delta \mathbf{Q}_{i-1/2},$$

where δ is the standard centered difference operator. Therefore, the form of (4.2.11) is conservative, provided λ and $\varepsilon^{(4)}$ are evaluated at the cell centers, rather than at the cell faces. In [97] numerical tests were performed with the dissipation terms of (4.2.3) and (4.2.11). For steady state there seems to be no consistent benefit, either in terms of convergence or accuracy, from using the form of (4.2.11). Based on these results we continue to use the form of (4.2.3). However, for unsteady flows (4.2.11) may offer an advantage, due to the absence of dispersive effects, which can cause phase errors.

Until now we have considered a scalar viscosity where the viscosity is based on differences of the same quantity that we advance in time (see for example 4.2.10). This has the disadvantage that the total enthalpy is no longer constant in the steady state even when it should be identically constant for the inviscid equations. For the steady state Euler equations the total enthalpy is constant since the energy equation is just a constant multiple of the continuity equation when H is constant. Hence, it was suggested in [32] that the dissipation for the energy equation be based on differences of the total enthalpy rather than the total energy. Thus, a typical situation in one dimension is to replace (4.2.10) for the energy equation by

$$J^{-1}\frac{\partial \rho E}{\partial t} + \frac{\partial(\rho H u)}{\partial \xi} = D_\xi^2(\rho H) - D_\xi^4(\rho H), \tag{4.2.12}$$

where $\rho H = \rho E + p$. In [16] it was shown that this indeed yields a constant total enthalpy but that the entropy tends to be less accurate than if the dissipation term for the energy equation is based on differences of ρE rather than ρH. Thus, both choices have their advantages and disadvantages.

4.3 Boundary Treatment of Dissipative Terms

In a cell-centered finite-volume method, the first and last cells in each coordinate direction are auxiliary cells where the flow equations are usually not solved. The solution in these cells is found by a combination of the given physical boundary conditions and numerical boundary conditions. Hence, there is no difficulty in evaluating the second-difference dissipation term at the first or last interior cell in a given coordinate direction. In the case of the fourth-difference dissipation term, the treatment must be modified at the boundaries of the physical domain, since only one layer of auxiliary cells is considered. Moreover, the standard five-point difference stencil must be replaced at the first two interior mesh cells relative to a wall boundary. This means that one-sided or one-sided biased stencils are used at these cells. The dissipative character of these stencils is important because it influences both stability and accuracy. For example, if the dissipation is too large at a solid boundary, an artificial boundary layer is created in an inviscid flow, and the effective Reynolds number for a viscous flow is altered.

In this section we first define the two types of discrete boundary-point operators (difference stencils) used with the present scheme for solid surfaces. We next evaluate these operators by applying a local mode analysis. In addition, we show how this local mode analysis can provide an evaluation of candidate boundary-point operators, once a basis for comparison has been established. A more complete analysis for the boundary-point operators is based on the dissipation matrix for the system of difference equations approximating the governing flow equations. Sometimes one can characterize the dissipation matrix analytically. In general, the eigenvalues of the dissipation matrix must be determined. This approach for analyzing the dissipation stencils is also discussed.

We want to consider the total dissipation resulting from a numerical flux balance for a mesh cell in a particular coordinate direction. Let w_j and d_j denote a component of the solution vector $\mathbf{W}$ and the corresponding total dissipation, respectively. The index j indicates the mesh cell being considered. Let $d_{j+1/2}$ and $d_{j-1/2}$ represent the dissipative fluxes at the cell interfaces $j+1/2$ and $j-1/2$ (see Figure 4). At a cell interface, say $j+1/2$, let $(\Delta w)_{j+1/2}$ denote the difference between the solution for the adjacent cells $(w_{j+1} - w_j)$. For simplicity assume $\lambda\varepsilon^{(4)} = 1$. Then, for any cell j,

$$d_j = d_{j+1/2} - d_{j-1/2}, \tag{4.3.1}$$

where the dissipative fluxes are

$$\begin{aligned} d_{j+1/2} &= (\Delta w)_{j+3/2} - 2(\Delta w)_{j+1/2} + (\Delta w)_{j-1/2}, \\ d_{j-1/2} &= (\Delta w)_{j+1/2} - 2(\Delta w)_{j-1/2} + (\Delta w)_{j-3/2}. \end{aligned}$$

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Figure 4: Boundary-point dissipation.

Thus

$$d_j = (\Delta w)_{j+3/2} - 3(\Delta w)_{j+1/2} + 3(\Delta w)_{j-1/2} - (\Delta w)_{j-3/2}, \tag{4.3.2}$$

or

$$d_j = w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2},$$

Consider the first two interior cells adjacent to a solid boundary, as depicted in Figure 4. The total dissipation for these cells is denoted by d_2 and d_3. At $j = 2$, we must determine a value for $(\Delta w)_{1/2}$, since it is undefined. Also, in this formulation of the boundary point dissipation stencil, it is desired that there be no functional dependence on w_1, which is outside the domain. Hence, we must also provide a value for $(\Delta w)_{3/2}$. If we take

$$(\Delta w)_{1/2} = (\Delta w)_{5/2}, \qquad (\Delta w)_{3/2} = (\Delta w)_{5/2}, \tag{4.3.3}$$

then (4.3.2) gives

$$\begin{aligned} d_2 &= w_4 - 2w_3 + w_2, & (4.3.4)\\ d_3 &= w_5 - 4w_4 + 5w_3 - 2w_2. & (4.3.5) \end{aligned}$$

These boundary stencils are fairly standard ones, and we use them for inviscid flow calculations. An alternative form, which we have found to reduce the sensitivity to solid surface normal mesh spacing for viscous flow calculations without compromising stability or convergence, is obtained by replacing $(\Delta w)_{1/2}$ with

$$(\Delta w)_{1/2} = 2(\Delta w)_{3/2} - (\Delta w)_{5/2},$$

and leaving $(\Delta w)_{3/2}$ unchanged. This form is given by

$$\begin{aligned} d_2 &= w_4 - 3w_3 + 3w_2 - w_1, & (4.3.6)\\ d_3 &= w_5 - 4w_4 + 6w_3 - 4w_2 + w_1. & (4.3.7) \end{aligned}$$

In the case of turbulent flows, this boundary dissipation formulation is advantageous under the condition that the mesh is fine enough to adequately represent the laminar sublayer region of the boundary layer (i.e., at least two points are inside the sublayer). For coarse meshes this treatment of the dissipation can be less accurate than the zeroth-order extrapolation of (4.3.3).

4.3.1 Local Mode Analysis

We now consider local mode analysis to evaluate the relative damping behavior of boundary-cell difference operators. For comparison purposes, we first characterize the interior fourth difference. Taking a Fourier transform of (4.3.2) one obtains

$$z_j(\theta) = 4(\cos\theta - 1)^2,$$

where $z_j(\theta)$ is the Fourier symbol of the transformed d_j, and θ is the product of the wave number and the mesh spacing. Then,

$$z_j(\theta) \sim \theta^4 \quad \text{for small } \theta, \quad z_j(\pi) = 16.$$

The dissipation of long wavelengths is dictated by the behavior of $z_j(\theta)$ at small θ, and the dissipation of short wavelengths is governed by $z_j(\pi)$. As mentioned previously, we have assumed in this simple analysis that $\lambda\varepsilon^{(4)} = 1$. In practice, the coefficient $\kappa^{(4)}$, which is used in the evaluation of $\varepsilon^{(4)}$, for the fourth-difference dissipation affects the behavior of the boundary dissipation stencil. This coefficient is chosen such that the highest frequency is highly damped, according to a stability analysis using the interior point stencil. This is important, as will be discussed later, for a multigrid method. Near a boundary, the dissipation should behave in a similar manner. In the present dissipation model, the same value of $\kappa^{(4)}$ used for interior points of the domain is also used near a boundary.

A general form of the difference stencils at $j = 2, 3$ can be written as

$$d_j = \alpha w_{j+2} - \beta w_{j+1} + (\beta + \gamma - \alpha) w_j - \gamma w_{j-1}.$$

The associated Fourier symbol is given by

$$\begin{aligned} z_j(\theta) &= [\beta + \gamma - 2\alpha(1 + \cos\theta)]\,(1 - \cos\theta) \\ &\quad + i(\gamma - \beta + 2\alpha\cos\theta)\sin\theta. \end{aligned}$$

For small θ this symbol is replaced by

$$\begin{aligned} z_j(\theta) &= [\beta + \gamma - 4\alpha]\,\frac{\theta^2}{2} + \alpha\frac{\theta^4}{2} \\ &\quad + i(2\alpha - \beta + \gamma)\theta - i\alpha\theta^3, \end{aligned} \tag{4.3.8}$$

and at $\theta = \pi$ it reduces to

$$z_j(\pi) = 2(\beta + \gamma). \tag{4.3.9}$$

In the case of (4.3.4) and (4.3.5),

$$z_2(\theta) \approx \frac{\theta^4}{2} - i\theta^3 \quad \text{for small } \theta, \quad z_2(\pi) = 4, \tag{4.3.10}$$

$$z_3(\theta) \approx \frac{\theta^2}{2} - i\theta^3 \quad \text{for small } \theta, \quad z_3(\pi) = 12. \tag{4.3.11}$$

Note that $z_2(\theta)$ and $z_3(\theta)$ are not real. Thus, there is both dissipation and dispersion near the boundary. For the stencils of (4.3.6) and (4.3.7) we have

$$z_2(\theta) \approx \frac{\theta^4}{2} - i\theta^3 \quad \text{for small } \theta, \quad z_2(\pi) = 8, \tag{4.3.12}$$

and

$$z_3(\theta) \approx \theta^4 \quad \text{for small } \theta, \quad z_3(\pi) = 16. \tag{4.3.13}$$

By comparing (4.3.10) and (4.3.12), which correspond to the stencils of (4.3.4) and (4.3.6), respectively, we see that both stencils behave the same for the long wavelengths, while (4.3.12) is twice as dissipative for the short wavelengths. At $j = 3$ the stencil corresponding to (4.3.13) is fourth order on the long wavelengths, whereas the stencil associated with (4.3.11) is only second order. In addition, the symbol of (4.3.13) is more dissipative on the short wavelengths. Thus, the improved accuracy and high-frequency damping observed for the stencils of (4.3.6) and (4.3.7) in practice is substantiated with this simple analysis.

We can now show how the simple local mode analysis can be combined with the evaluations just considered to quickly evaluate candidate dissipation stencils. Let us consider a different set of boundary-point stencils. If we take Δw to represent either the component ρu or ρv of the solution vector $\mathbf{W}$ and impose the antisymmetry constraint

$$(\Delta w)_{1/2} = (\Delta w)_{5/2},$$

as for viscous flows, then (4.3.2) gives

$$\begin{aligned} d_2 &= w_4 - 5w_3 + 7w_2 - 3w_1, && (4.3.14) \\ d_3 &= w_5 - 4w_4 + 6w_3 - 4w_2 + w_1. && (4.3.15) \end{aligned}$$

The Fourier symbols of (4.3.14), using (4.3.8) and (
4.3.9), are

$$z_2(\theta) \approx 2\theta^2 \quad \text{for small } \theta, \quad z_2(\pi) = 16, \tag{4.3.16}$$

and the symbols for (4.3.15) are the same as given in (4.3.13). Comparing (4.3.16) with (4.3.12), one sees that the highest frequency is damped better with the proposed stencil, but that the proposed stencil is only second order on the long wavelengths, while the stencil of (4.3.6) is fourth order. This indicates that better accuracy is obtained with (4.3.6) and (4.3.7). The improved accuracy has been verified with numerical experiments (i.e., computed skin-friction solutions for turbulent airfoil flows on 160×32 meshes and compared them with high-density-mesh results).

4.4 Matrix Analysis

In order to determine the numerical dissipativity of a discrete system of equations such as (3.2), we examine the associated dissipation matrix. For simplicity, consider the one-dimensional system

$$\frac{d\mathbf{w}}{dt} = D^{(4)}\mathbf{w}, \tag{4.4.1}$$

where $\mathbf{w}$ is a discrete solution vector, and $D^{(4)}$ is a dissipation matrix corresponding to fourth-difference terms. Taking the inner product $\mathbf{w}^T$, the transpose of $\mathbf{w}$, with each side of (4.4.1), we obtain

$$\frac{1}{2}\frac{d\mathbf{w}^2}{dt} = \mathbf{w}^T D^{(4)}\mathbf{w}.$$

If the quadratic form $\mathbf{w}^T D^{(4)}\mathbf{w}$ is nonpositive definite, then the matrix $D^{(4)}$ is strictly dissipative. Moreover, the energy of the system is nonincreasing. Now suppose there are boundaries at $j = 3/2$ and $j = \mathrm{jl} + 1/2$, and $j = 2$ and $j = \mathrm{jl}$ are the indices for the first and last interior points, respectively. Apply the boundary point stencils of (4.3.4) and (4.3.5) at the first two interior cell centers at both boundaries and the standard stencil everywhere else. The resulting dissipation matrix is given by

$$D^{(4)} = \begin{bmatrix} -1 & 2 & -1 & & & & & & \\ 2 & -5 & 4 & -1 & & & & & \\ -1 & 4 & -6 & 4 & -1 & & & & \\ 0 & -1 & 4 & -6 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -6 & 4 & -1 & 0 \\ & & & & -1 & 4 & -6 & 4 & -1 \\ & & & & & -1 & 4 & -5 & 2 \\ & & & & & & -1 & 2 & -1 \end{bmatrix}, \qquad (4.4.2)$$

and the corresponding solution vector is given by

$$\mathbf{w} = \begin{bmatrix} w_2 & w_3 & w_4 & w_5 & \dots & w_{\mathrm{jl}-2} & w_{\mathrm{jl}-1} & w_{\mathrm{jl}} \end{bmatrix}^T .$$

Then

$$\mathbf{w}^T D^{(4)} \mathbf{w} = -\sum_{j=3}^{\mathrm{jl}-1} \left(w_{j+1} - 2w_j + w_{j-1}\right)^2 \leq 0. \qquad (4.4.3)$$

Thus, $D^{(4)}$ is strictly dissipative. This is the same result obtained by Rizzi [23]. For a 10×10 matrix with the form of (4.4.2), Pulliam [76] obtains two zero eigenvalues. Ideally, one would like $D^{(4)}$ to have no zero eigenvalues, since, as pointed out by Pulliam, zero eigenvalues can possibly produce undamped modes that cause instabilities.

Pulliam [76] recommends applying a stencil with the weights of (4.3.5) at the first interior cell, and a standard stencil, with the weights of (4.3.7), at the second interior cell. Then

$$D^{(4)} = \begin{bmatrix} -5 & 4 & -1 & & & & & & \\ 4 & -6 & 4 & -1 & & & & & \\ -1 & 4 & -6 & 4 & -1 & & & & \\ 0 & -1 & 4 & -6 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -6 & 4 & -1 & 0 \\ & & & & -1 & 4 & -6 & 4 & -1 \\ & & & & & -1 & 4 & -6 & 4 \\ & & & & & & -1 & 4 & -5 \end{bmatrix}, \qquad (4.4.4)$$

and

$$\begin{aligned} \mathbf{w}^T D^{(4)} \mathbf{w} &= -\sum_{j=3}^{\mathrm{jl}-1} \left(w_{j+1} - 2w_j + w_{j-1}\right)^2 \\ &\quad -(w_3 - 2w_2)^2 - (w_{\mathrm{jl}-1} - 2w_{\mathrm{jl}})^2 \leq 0. \end{aligned} \qquad (4.4.5)$$

Again the dissipation matrix is strictly dissipative. Moreover, as indicated by Pulliam, a 10×10 matrix with the structure of (4.4.4) has zero eigenvalues. However, we have found that for a cell-centered, finite-volume formulation this boundary-point treatment of the dissipation, although appropriate at inflow and outflow boundaries, is generally too dissipative at solid boundaries. Thus the stencils of (4.3.4) and (4.3.5) are preferred at a wall boundary.

Now consider the stencils with the weights of (4.3.6) and (4.3.7). The dissipation matrix is given by

$$D^{(4)} = \left[\begin{array}{ccccccccc} -3 & 3 & -1 & & & & & & \\ 4 & -6 & 4 & -1 & & & & & \\ -1 & 4 & -6 & 4 & -1 & & & & \\ 0 & -1 & 4 & -6 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -6 & 4 & -1 & 0 \\ & & & & -1 & 4 & -6 & 4 & -1 \\ & & & & & -1 & 4 & -6 & 4 \\ & & & & & & -1 & 3 & -3 \end{array}\right], \tag{4.4.6}$$

and

$$\begin{aligned} \mathbf{w}^T D^{(4)} \mathbf{w} &= -\sum_{j=3}^{\mathrm{jl}-1} (w_{j+1} - 2w_j + w_{j-1})^2 + w_2(w_3 - w_2) - (w_3 - w_2)^2 \\ &\quad -(w_{\mathrm{jl}-1} - w_{\mathrm{jl}})^2 + w_{\mathrm{jl}}(w_{\mathrm{jl}} - w_{\mathrm{jl}-1}) \leq 0. \end{aligned} \tag{4.4.7}$$

From the quadratic form of (4.4.7) it does not follow directly that $D^{(4)}$ is nonpositive definite. This is generally the case with the quadratic form. If we determine the eigenvalues of a 10×10 matrix with the structure of (4.4.6), we find that one is zero, and the others are negative. The matrix $D^{(4)}$ is, therefore, nonpositive definite. Even though there is one zero eigenvalue, the present scheme performs well using the boundary-point operators associated with (4.3.6) and (4.3.7) at solid boundaries when solving viscous flow problems.

4.5 The Upwind Connection

Upwind schemes for solving hyperbolic systems of conservation laws (i.e., Euler equations of gas dynamics) generally rely upon characteristic theory to determine the direction of propagation of information; and thus, the direction required for one-sided differencing approximations of the spatial derivatives. With such schemes shock waves can be captured without oscillations. Thus, a successful artificial dissipation model for a central difference scheme should imitate an upwind scheme in the neighborhood of shocks. We now review the connection between these two types of schemes.

Consider the one-dimensional scalar wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

with a constant. The first-order upwind scheme can be written as

$$u_j^{n+1} = u_j - a\frac{\Delta t}{\Delta x}\begin{cases} u_{j+1} - u_j, & a < 0 \\ u_j - u_{j-1}, & a > 0, \end{cases} \tag{4.5.1}$$

where all discrete quantities are evaluated at time level $n\Delta t$ unless otherwise denoted. The scheme of 4.5.1 can be rewritten as

$$u_j^{n+1} = u_j - a\frac{\Delta t}{2\Delta x}(u_{j+1} - u_{j-1}) + |a|\frac{\Delta t}{2\Delta x}(u_{j+1} - 2u_j + u_{j-1}), \tag{4.5.2}$$

which now contains a central difference term and a second-difference dissipation term. Now consider the system

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0, \tag{4.5.3}$$

where $\mathbf{u}$ is a N-component vector. The system case can be converted to a scalar one by diagonalizing the $N \times N$ matrix A with a similarity transformation

$$\Lambda = T^{-1}AT,$$

where the columns of T are the right eigenvectors of A. After diagonalizing (4.5.3) and applying the scheme of (4.5.2), the first-order upwind scheme is given by

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j - a\frac{\Delta t}{2\Delta x}(\mathbf{u}_{j+1} - \mathbf{u}_{j-1}) + |A|\frac{\Delta t}{2\Delta x}(\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1}), \tag{4.5.4}$$

where

$$|A| = T|\Lambda|T^{-1}, \qquad \Lambda = \mathrm{Diag}\left[|\lambda_1| \cdots |\lambda_N|\right]. \tag{4.5.5}$$

We note that since A has only three distinct eigenvalues, by the Cayley-Hamilton theorem we can express $|A|$ as a quadratic polynomial in A. The generalization to a system of conservation laws is straightforward; namely,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = 0$$

with $\mathbf{f}$ being a N-component flux vector, and

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j - \frac{\Delta t}{2\Delta x}(\mathbf{f}_{j+1} - \mathbf{f}_{j-1}) + \frac{\Delta t}{2\Delta x}\left[\left|A_{j+1/2}\right|(\mathbf{u}_{j+1} - \mathbf{u}_j) - \left|A_{j-1/2}\right|(\mathbf{u}_j - \mathbf{u}_{j-1})\right], \tag{4.5.6}$$

where the Jacobian matrix $A = \partial\mathbf{f}/\partial\mathbf{u}$, and $|A|$ is defined as for (4.5.4). The matrix $\left|A_{j+1/2}\right|$ can be computed as an arithmetic average or a Roe average [83]. For transonic steady flows the differences are negligible; therefore, we use the simpler arithmetic average. For hypersonic flows Yee [124] found that the Roe average yields better results. For time-dependent problems the Roe average also seems to give slightly better results.

4.6 The TVD Property

Consider the one-dimensional scalar conservation law

$$\frac{\partial}{\partial t}[u(x,t)] + \frac{\partial}{\partial x}[f(u(x,t))] = 0, \tag{4.6.1}$$

where

$$-\infty < x < \infty, \quad t \geq 0.$$

Let $v(t) = \{v_j(t)\}$ be the approximate solution of (4.6.1) and consider the semidiscrete equation

$$\begin{aligned}\frac{d}{dt}v_j(t) + \frac{1}{2\Delta x}[f_{j+1} - f_{j-1}] &= \frac{1}{2\Delta x}\left[Q_{j+1/2}\ \Delta v_{j+1/2} - Q_{j-1/2}\Delta v_{j-1/2}\right] \\ &\quad -\frac{\kappa^{(4)}}{\Delta x}\left[R_{j+1/2}\Delta^3 v_{j+1/2} - R_{j-1/2}\Delta^3 v_{j-1/2}\right] \qquad (4.6.2)\end{aligned}$$

with

$$\Delta v_{j+1/2} = (\Delta v)_{j+1/2} = v_{j+1}(t) - v_j(t);$$

Δ^3 is a third-difference operator defined as

$$\Delta^3 v_{j+1/2} = (\Delta^3 v)_{j+1/2} = v_{j+2}(t) - 3v_{j+1}(t) + 3v_j(t) - v_{j-1}(t)$$

The terms on the right hand side of (4.6.2) represent second- and fourth-difference numerical dissipation terms, with $\kappa^{(4)}$ a constant. Define

$$s_{j+1/2} = \operatorname{sgn}\left(\Delta v_{j+1/2}\right), \qquad (4.6.3)$$

where sgn represents the signum function. Then, a modification of the theorem of Tadmor [105] is given by the following.

Theorem 4.1 *The semidiscrete scheme (4.6.2) is TVD if*

$$(a) \qquad \left[2 - s_{j+1/2}(s_{j-1/2} + s_{j+3/2})\right]\ Q_{j+1/2} \geq s_{j+1/2}(s_{j+3/2} - s_{j-1/2})\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}$$

and

$$(b) \qquad R_{j+1/2} = 0 \qquad \text{when} \qquad s_{j+3/2} - 2s_{j+1/2} + s_{j-1/2} \neq 0.$$

Proof. Shifting the indices by one in (4.6.2) and subtracting (4.6.2) from the resulting equation, we obtain

$$\begin{aligned}\frac{d}{dt}\Delta v_{j+1/2} &+ \frac{1}{2\Delta x}\left[\Delta f_{j+3/2} - \Delta f_{j-1/2}\right] \\ &= \frac{1}{2\Delta x}\left[Q_{j+3/2}\Delta v_{j+3/2} - Q_{j-1/2}v_{j-1/2}\right] \qquad (4.6.4) \\ &- \frac{\kappa^{(4)}}{\Delta x}\left[R_{j+3/2}\Delta^3 v_{j+3/2} - 2R_{j+1/2}\ \Delta^3 v_{j+1/2} + R_{j-1/2}\Delta^3 v_{j-1/2}\right].\end{aligned}$$

Multiply (4.6.4) by $s_{j+1/2}$ and sum over all j. Note that $s_{j+1/2} = \pm 1$, so $s^2_{j+1/2} = 1$, and

$$s_{j+1/2}\ \Delta v_{j+1/2} = \left|\Delta v_{j+1/2}\right|.$$

We then obtain

$$\begin{aligned}\frac{d}{dt}\sum_j \left|\Delta v_{j+1/2}\right| &= -\frac{1}{2\Delta x}\sum_j s_{j+1/2}\left(s_{j-1/2}-s_{j+3/2}\right)\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}\left|\Delta v_{j+1/2}\right| \\ &\quad +\frac{1}{2\Delta x}\sum_j s_{j+1/2}\left(s_{j+3/2}-2s_{j+1/2}+s_{j-1/2}\right)Q_{j+1/2}\left|\Delta v_{j+1/2}\right| \\ &\quad -\frac{\kappa^{(4)}}{\Delta x}\sum_j \left(s_{j+3/2}-2s_{j+1/2}+s_{j-1/2}\right)\ R_{j+1/2}\Delta^3 v_{j+1/2}.\end{aligned} \tag{4.6.5}$$

We stress that the last term in (4.6.5) will not help for *TVD*. Its purpose is to eliminate high frequencies and accelerate convergence to a steady state. Hence, we want this contribution to be zero. This can be accomplished if we demand either

$$s_{j+3/2}-2s_{j+1/2}+s_{j-1/2}=0 \tag{4.6.6}$$

or

$$R_{j+1/2}=0, \tag{4.6.7}$$

i.e., condition (b). We are left with

$$\begin{aligned}\frac{d}{dt}\,(\mathrm{TV}) &= -\frac{1}{2\Delta x}\sum_j -s_{j+1/2}(s_{j+3/2}-s_{j-1/2})\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}\left|\Delta v_{j+1/2}\right| \\ &\quad -s_{j+1/2}(s_{j+3/2}-2s_{j+1/2}+s_{j-1/2})Q_{j+1/2}\ \left|\Delta v_{j+1/2}\right|,\end{aligned} \tag{4.6.8}$$

where TV denotes the total variation as given by

$$\mathrm{TV}=\sum_j \left|\Delta v_{j+1/2}\right|.$$

Thus, a sufficient condition that the total variation not increase is that each term in the summation of (4.6.8) must be positive. This means that

$$-s_{j+1/2}\left(s_{j+3/2}-2s_{j+1/2}+s_{j-1/2}\right)Q_{j+1/2}\geq s_{j+1/2}\left(s_{j+3/2}-s_{j-1/2}\right)\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}. \tag{4.6.9}$$

Since $s^2_{j+1/2}=1$, (4.6.9) is equivalent to condition (a) of the theorem. Defining

$$\chi_j = 1 - s_{j-1/2}\ s_{j+1/2}$$

(4.6.9) can be written in the form given by Tadmor; namely,

$$(\chi_j+\chi_{j+1})Q_{j+\frac{1}{2}}\geq(\chi_j-\chi_{j+1})\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}. \tag{4.6.10}$$

We now consider four conditions and give for each one an appropriate $Q_{j+1/2}$ and $R_{j+1/2}$, according to Theorem 4.1, for the scheme of (4.6.2) to be *TVD*.

Remark 4.2 *(i) When no extrema are present locally ($s_{j-1/2} = s_{j+1/2} = s_{j+3/2}$), both conditions of the theorem are trivially satisfied for all $Q_{j+1/2}$ and $R_{j+1/2}$. For such regions we want $Q_{j+1/2} = O(\Delta x)$ for second-order accuracy and $R_{j+1/2}$ to be chosen so that high frequencies are damped.*

(ii) If $s_{j-1/2} = s_{j+3/2} = -s_{j+1/2}$ (a local oscillation at $x_{j+1/2}$), we require

$$Q_{j+1/2} \geq 0, \qquad R_{j+1/2} = 0.$$

(iii) If $s_{j-1/2} = s_{j+1/2} = -s_{j+3/2}$ (a local extremum at x_{j+1}),

$$Q_{j+1/2} \geq \frac{-\Delta f_{j+1/2}}{\Delta v_{j+1/2}}, \qquad R_{j+1/2} = 0.$$

(iv) If $s_{j-1/2} = -s_{j+1/2} = -s_{j+3/2}$ (a local extremum at x_j),

$$Q_{j+1/2} \geq \frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}, \qquad R_{j+1/2} = 0.$$

It follows from these inequalities that $Q_{j+1/2}$ can be negative. As far as total variation is concerned, central differences are not nondissipative. That is, they can either increase or decrease the total variation. In cases where central differences decrease the total variation, $Q_{j+1/2}$ can be negative. For systems of equations, especially in multidimensions, this behavior can sometimes lead to difficulties. Hence, we demand the stronger condition that

$$Q_{j+1/2} \geq \left|\frac{\Delta f_{j+1/2}}{\Delta v_{j+1/2}}\right|.$$

For systems this condition is replaced by

$$Q_{j+1/2} \geq \left|A_{j+1/2}\right|, \tag{4.6.11}$$

where $A = \partial f/\partial u$, and the average at $j + 1/2$ is constructed by the technique of Roe [83].

4.7 Flux Limiters

In this section switching functions are introduced that force the scheme to automatically satisfy the inequalities previously presented. These switches are required to be smooth so that limit cycles are not experienced when marching in time to obtain a steady-state solution. The discrete switching functions are defined as

$$Q_{j+1/2} = \psi \cdot \left|a_{j+1/2}\right| \quad \text{with} \quad 0 \leq \psi \leq 1, \qquad R_{j+1/2} = \Gamma \cdot \left|a_{j+1/2}\right|, \tag{4.7.1}$$

where $\psi = 1$ near extrema, so that the inequalities are satisfied, and $\psi = O(\Delta x)$ in smooth regions of the flow field. Conversely, $\Gamma = 0$ at extrema, and $\Gamma = 1$ in the smooth regions.

The functions ψ and Γ of (4.7.1) can be defined in terms of a limiter function. Let

$$r = \frac{v_j - v_{j-1}}{v_{j+1} - v_j} = \frac{\Delta v_{j-1/2}}{\Delta v_{j+1/2}}. \tag{4.7.2}$$

The van Leer flux limiter is given by

$$\varphi_j(r) = \frac{r+|r|}{1+|r|} = \begin{cases} \frac{2r}{1+r}, & r > 0 \\ 0, & r \leq 0. \end{cases} \tag{4.7.3}$$

From Sweby [102] it is straightforward to see that for any flux limiter

$$\psi_j(r) = 1 - \varphi_j(r).$$

Since we want $\psi_j(r)$ to be positive, this relation is redefined as

$$\psi_j(r) = |1 - \varphi_j(r)|.$$

Then, for the van Leer limiter,

$$\psi_j(r) = \left|\frac{1-r}{1+|r|}\right| = \begin{cases} \frac{|1-r|}{1+r}, & r > 0 \\ 1, & r \leq 0, \end{cases}$$

so $0 \leq \psi_j \leq 1$ and define

$$\psi_{j+1/2} = \max(\psi_j, \psi_{j+1}).$$

We now show that the inequality (4.6.9) is satisfied with this ψ. By (i) of Remark 4.2 the inequality is satisfied when $s_{j-1/2} = s_{j+1/2} = s_{j+3/2}$, and so in this case we only need $Q = O(\Delta x)$. Since $r > 0$ in this case,

$$\psi_j(r) = \frac{|1-r|}{1+r}.$$

In addition, for smooth regions of the flow field,

$$r = 1 + O(\Delta x),$$

and thus

$$\psi_{j+1/2} = O(\Delta x), \qquad Q_{j+1/2} = O(\Delta x).$$

Next, consider the case $s_{j-1/2} = -s_{j+1/2}$. This implies $r_j < 0$, and so, $\psi_j = 1$. Moreover,

$$s_{j-1/2} = -s_{j+1/2} \Rightarrow \psi_j = 1 \Rightarrow \psi_{j+1/2} = 1.$$

Similarly, if $s_{j+3/2} = -s_{j+1/2}$,

$$s_{j+1/2} = -s_{j+3/2} \Rightarrow \psi_{j+1} = 1 \Rightarrow \psi_{j+1/2} = 1.$$

It also follows, using (4.7.1), that the inequalities in (ii) and (iii) of Remark 4.2 are satisfied. Also, for these two cases, setting $\Gamma_{j+1/2} = 1 - \psi_{j+1/2}$ guarantees that $\Gamma_{j+1/2} = 0$. So

$$\Gamma_{j+1/2} = 1 - \max(\psi_j, \psi_{j+1}).$$

The flux limiters ψ and Γ can be connected to those used in upwind schemes. Sweby [102] considers an upwind Lax-Wendroff scheme. In particular, for the one-dimensional wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

the numerical solution is obtained with

$$u_j^{n+1} = u_j - \nu(u_j - u_{j-1}) - \Delta_- \left[\frac{\nu(1-\nu)}{2}\varphi_j(u_{j+1} - u_j)\right] \tag{4.7.4}$$

where

$$\nu = a\frac{\Delta t}{\Delta x}, \qquad 0 \le \ \varphi(r) \le 2, \qquad \frac{\varphi(r)}{r} = \varphi\left(\frac{1}{r}\right),$$

and the backward difference operator Δ_- is defined by

$$\Delta_- u_j = u_j - u_{j-1}.$$

If (4.7.4) is rewritten as a central difference scheme, then

$$\begin{aligned} u_j^{n+1} \ &= \ u_j - \frac{\nu}{2}(u_{j+1} - u_{j-1}) + \frac{\nu}{2}\left[(1-\varphi_j)\ (u_{j+1} - u_j) - (1-\varphi_{j-1})\ (u_j - u_{j-1})\right] \\ &\quad + \frac{\nu^2}{2}\left[\varphi_j(u_{j+1} - u_j) - \varphi_{j-1}(u_j - u_{j-1})\right]. \end{aligned} \tag{4.7.5}$$

By dropping the ν^2 term in (4.7.5) and changing to a semidiscrete formulation, we get

$$\frac{d}{dt}u_j = -\frac{\nu}{2\Delta t}(u_{j+1} - u_{j-1}) + \frac{\nu}{2\Delta t}\left[(1-\varphi_j)(u_{j+1} - u_j) - (1-\varphi_{j-1})(u_j - u_{j-1}\right]. \tag{4.7.6}$$

If

$$\varphi_j = 1 - \psi_{j+1/2},$$

the second-difference dissipation term has the same form as the one presented in 4.6.

To complete the connection between the limiters for the central difference and upwind schemes, we compare their behavior. For the central difference case,

$$0 \le \psi \le 1, \qquad 0 \le \varphi \le 1,$$

whereas for the upwind case, $0 \le \varphi \le 2$. Furthermore, for the central difference limiter,

$$\psi\left(\frac{1}{r}\right) = \psi(r), \qquad \varphi\left(\frac{1}{r}\right) = \varphi(r)$$

means that it does not matter if $\Delta u_{j+1/2} > \Delta u_{j-1/2}$ or $\Delta u_{j+1/2} < \Delta u_{j-1/2}$ (i.e., the sign of a does not affect the scheme), as opposed to the upwind limiter where

$$\frac{\varphi(r)}{r} = \varphi\left(\frac{1}{r}\right).$$

-bb-error = =

Figure 5: Sweby diagram

In Figure 5 the Sweby diagram for the upwind van Leer flux limiter and the central difference van Leer flux limiter is shown. For $r < 1$, the limiters are the same. For $r > 1$, the upwind version continues until $\varphi = 2$, while the central difference version returns to zero. In Sweby's paper the flux limiters are not allowed to decrease when $r > 1$. However, there is no difficulty from such behavior. In both cases $\varphi(r) = 0$ when $r \leq 0$ so that the limiter is turned off for extrema. We note that other limiters besides that of van Leer can be used.

For systems of equations we use a scalar limiter. Using the matrix form of the dissipation it is easy to implement different limiters for different characteristic variables [48].

In terms of the pressure the switch becomes

$$\psi_j = \frac{1-r}{1+|r|}, \qquad r = \frac{\Delta p_{j-1/2}}{\Delta p_{j+1/2}}, \qquad a > 0$$

or

$$\psi_j = \frac{|p_{j+1} - p_j| - (p_j - p_{j-1}) \operatorname{sgn}(p_{j+1} - p_j)}{|p_{j+1} - p_j| + |p_j - p_{j-1}| + \epsilon}, \qquad a > 0$$

with

$$\psi_{j+\frac{1}{2}} = \max(\psi_j,\ \psi_{j+1}),$$

and $\epsilon = O(\Delta x^2)$ to prevent a zero denominator for constant pressure regions. Consider also the wave speed $a < 0$. Then,

$$\psi_j = \frac{p_{j+1} - 2p_j + p_{j-1}}{|p_{j+1} - p_j| + |p_j - p_{j-1}| + \epsilon} \begin{cases} \operatorname{sgn}(p_{j+1} - p_j), & a > 0 \\ \operatorname{sgn}(p_{j-1} - p_j), & a < 0. \end{cases}$$

We use a conservative approach and take

$$\psi_j = \frac{|p_{j+1} - 2p_j + p_{j-1}|}{|p_{j+1} - p_j| + |p_j - p_{j-1}| + \epsilon}. \tag{4.7.7}$$

Notice that this switch is very similar to the *JST* switch in (4.2.6). There is only a change in the denominator. However, with this change and the factor 1/2 in front of the second-difference dissipation term, the scalar equation becomes first-order upwind near shocks.

In order to obtain some flexibility, to avoid the difficulty of defining a suitable ϵ (i.e., one that does not activate the switch when small oscillations occur in the discrete field), and to improve convergence for steady-state calculations, we introduced [99] the switching function

$$\psi_j = \frac{|p_{j+1} - 2p_j + p_{j-1}|}{(1-\omega)(|p_{j+1} - p_j| + |p_j - p_{j-1}|) + \omega(p_{j+1} + 2p_j + p_{j-1})}, \tag{4.7.8}$$

where $0 \leq \omega \leq 1$, and the *TVD* switch of (4.7.7) is recovered when $\omega \ll 1$. Computationally, (4.7.8) is found to be more robust than (4.7.7). One possible explanation is that (4.7.7) is not affected by the amplitude of the wave. Hence, it is activated by noise in the far field which slows down the convergence.

We now no longer have a free parameter for the second-difference dissipation. We also usually use $\Gamma = \max(0, 1 - 2\psi)$ so that the fourth-difference dissipation is cut off for $\psi \geq 1/2$. The only free parameter is the coefficient $\kappa^{(4)}$ of the fourth-difference term.

4.7.1 SLIP Scheme

Until now we have a considered a combination of a low-order and high-order artificial viscosity based on a scalar switch. This switch, which is usually pressure based, senses where a shock is located. In smooth regions the sensor is about zero, while it increases near shocks. We found that the original *JST* switch in (4.2.6) was about 0.1 near a typical transonic shock while increasing to about 0.25 for a hypersonic shock. However, this switch has the disadvantage that one quantity, the pressure, controls the shock sensor. Hence, inside a boundary layer where the pressure is almost constant this sensor is not very effective. One can then use a sensor that is temperature based. However, it then becomes difficult to generalize the sensor so that it is useful both near shocks and inside boundary layers. Furthermore, because the choice of these variables is based on the physics, it is difficult to generalize to other cases. Finally, it forces all variables to be treated equal, even though some experience sharp changes through the discontinuity while others are continuous across the shock.

The requirement to choose a particular flow variable for a switch can be eliminated. One can instead limit independently each dependent variable in each coordinate direction. In [40] Jameson introduced a new class of limiters and implemented such a limiting process within a framework that he called the symmetric limited positive (*SLIP*) scheme. For the *SLIP* formulation Jameson constructed a family of limiter functions based on the function

$$R(u, v) = 1 - \left| \frac{u - v}{|u| + |v| + \epsilon} \right|^q, \tag{4.7.9}$$

where q is a positive number and ϵ has the dimensions of u. Note that $R(u, v) = 0$ whenever u and v have the opposite sign. Let w be an element of the solution vector for the governing flow equations. Note that according to our previous theory $R(\Delta w_{j+3/2}, \Delta w_{j-1/2})$, where $\Delta w_{j+3/2} = w_{j+2} - w_{j+1}$, would be replaced by $\psi_{j+1/2}$, where $\psi_{j+1/2}$ is the maximum of ψ_j over the nearest neighbors and ψ is given by (4.7.7). Define the limiter function $L(u, v)$ by

$$L(u,v) = R(u,v)\frac{u+v}{2}. \tag{4.7.10}$$

An alternative given by [108] is

$$L(u,v) = S(u,v)\min\left[\left|\frac{u+v}{2}\right|, \alpha|u|, \alpha|v|\right],$$

where

$$S(u,v) \quad = \quad sgn(u) + sgn(v) = \begin{cases} 1 & \text{if} \quad u > 0 \quad \text{and} \quad v > 0 \\ 0 & \text{if} \quad uv \leq 0 \\ -1 & \text{if} \quad v > 0 \quad \text{and} \quad v < 0. \end{cases}$$

Other possibilities are presented in [44, 107].

At the mesh cell interface $j + 1/2$, we define the left and right states for each dependent variable as

$$\begin{aligned} w_L &= w_j + \frac{1}{2}L(\Delta w_{j+3/2}, \Delta w_{j-1/2}), \\ w_R &= w_{j+1} - \frac{1}{2}L(\Delta w_{j+3/2}, \Delta w_{j-1/2}), \end{aligned} \tag{4.7.11}$$

and so

$$w_R - w_L = \Delta w_{j+1/2} - L(\Delta w_{j+3/2}, \Delta w_{j-1/2}).$$

For the artificial viscosity all differences will be based on $w_R - w_L$. In the neighborhood of shock waves $R(u, v)$ and hence $L(u, v)$ are close to zero. Moreover, $w_R - w_L = \Delta w_{j+1/2}$, resulting in a first-order scheme for the artificial viscosity. For smooth flow $R(u, v) = 1$, and $L(u, v) = (u + v)/2$. Hence, in a smooth region

$$\begin{aligned} w_R - w_L &= \Delta w_{j+1/2} - L(\Delta w_{j+3/2}, \Delta w_{j-1/2}) \\ &\simeq \Delta w_{j+1/2} - \frac{\Delta w_{j+3/2} + \Delta w_{j-1/2}}{2} = -\frac{1}{2}\Delta^3 w_{j+1/2}. \end{aligned} \tag{4.7.12}$$

Thus, in the smooth regions $w_R - w_L$ behaves as a third difference, while in the vicinity of shock waves it behaves as a first difference. Consequently, the *SLIP* scheme has similar properties to the *JST* scheme. One can obtain the relationship between the *SLIP* and *JST* by defining the diffusive flux $d_{j+1/2}$ as

$$d_{j+1/2} = \alpha_{j+1/2}(w_R - w_L) \tag{4.7.13}$$

with

$$\alpha_{j+1/2} = \kappa^{(2)}\lambda_{j+1/2},$$

where $\kappa^{(2)}$ is a parameter, and λ is the spectral radius of the associated flux Jacobian matrix (see Section 4.2).

One difference between the *JST* and *SLIP* schemes involves the parameters $\kappa^{(2)}$ and $\kappa^{(4)}$ for the second and fourth differences, respectively. Both $\kappa^{(2)}$ and $\kappa^{(4)}$ are free parameters in the *JST* scheme. As seen from (4.7.12) and (4.7.13) these parameters are automatically chosen as $\kappa^{(2)}$ and $\frac{1}{2}\kappa^{(2)}$ with the *SLIP* scheme. Furthermore, for the matrix viscosity (see Section 4.8) and the convective upstream split pressure (*CUSP*) scheme of Jameson [40] (which will be discussed in Section 4.9) $\kappa^{(2)} = \frac{1}{2}$, and so we no longer have any free parameters. The coefficient of the second difference is chosen as $\frac{1}{2}$ so that the scheme is fully upwind for supersonic flows. However, the fourth-difference viscosity is introduced only to accelerate the convergence to a steady state by eliminating the odd-even point decoupling. Hence, we wish $\kappa^{(4)}$ to be as small as possible for accuracy while still achieving a good convergence rate. It does not seem reasonable to connect the two components of the artificial viscosity. In addition, some tests indicate that indeed the resultant scheme is very diffusive. Using the scalar viscosity we have an effective $\kappa^{(4)}$ of $\frac{1}{4}$, which is large compared to the more typical values of $\kappa^{(4)} = \frac{1}{32}$. When used with the *CUSP* scheme, it is harder to compare since the artificial viscosity no longer scales with the largest eigenvalue; and so, there is no easy comparison with the coefficients of the *JST* artificial viscosity. However, for the preconditioned low Mach number flows all the viscosities scale with the convective speed. In this case the *SLIP* scheme still gives a third-order viscosity that is about twice as large as desired. Hence, we shall generalize the *SLIP* scheme by reintroducing a free parameter that essentially governs the level of the third-order viscosity in the smooth regions. The resulting scheme has the disadvantage that a free parameter must be chosen; however, it has the advantages of greater flexibility and increased accuracy. We now define a new L as

$$\hat{L}(u,v,w) = R(u,w)\left[(1-4\kappa^{(4)})v + 4\kappa^{(4)}\frac{u+w}{2}\right], \qquad (4.7.14)$$

where left and right state values are determined by

$$\begin{aligned} w_L &= w_j + \frac{1}{2}L(\Delta w_{j+3/2}, \Delta w_{j+1/2}, \Delta w_{j-1/2}), \\ w_R &= w_{j+1} - \frac{1}{2}L(\Delta w_{j+3/2}, \Delta w_{j+1/2}, \Delta w_{j-1/2}). \end{aligned}$$

When $\kappa^{(4)} = \frac{1}{4}$, the L of (4.7.14) reduces to the original L of (4.7.10). Now, at shock waves $R(u,w) = 0$, and we again have $w_R - w_L = \Delta w_{j+1/2}$. For smooth regions of the flow field we have $w_R - w_L = -2\kappa^{(4)}\Delta^3 w_{j+1/2}$.

One difficulty with *SLIP*, and indeed with any *TVD* switch, is that it limits the differences near minima and maxima independent of the amplitude of the function. Hence, in the far field where the solution is almost uniform the switch is activated by small noise levels. Since this occurs randomly it frequently prevents the convergence of the residual beyond three or four orders of magnitude. Indeed, for the scalar viscosity, this is why (4.7.7) is preferable to (4.7.8). Hence, one possibility is to prevent R(u,v), (4.7.9), from getting close to zero when both u and v are small. This can be accomplished by choosing ϵ in (4.7.9) proportional to some reference value of u.

Another possibility suggested by Jameson is to let the limiter depend on the sign of the eigenvalue. For the system of the Euler equations we replace this by looking at the direction of the flow (i.e., the sign of the local Mach number). They call this scheme the upstream symmetric limited positive (*USLIP*) scheme, and it is given by

$$d_{j+1/2} = \alpha_{j+1/2} \left\{ \begin{array}{ll} \Delta w_{j+1/2} - L\left(\Delta w_{j+1/2}, \Delta w_{j-1/2}\right) & \text{if} \quad M_{j+1/2} > 0 \\ \Delta w_{j+1/2} - L\left(\Delta w_{j+1/2}, \Delta w_{j+3/2}\right) & \text{if} \quad M_{j+1/2} < 0. \end{array} \right.$$

Near shocks $L(u,v) = 0$, and $w_R - w_L = \Delta w_j$ as before. For smooth flow $L(u,v) = (u+v)/2$; and so, for smooth flow

$$d_{j+1/2} = -\alpha_{j+1/2} \left\{ \begin{array}{ll} \Delta^2 w_j & \text{if} \quad M_{j+1/2} > 0 \\ \Delta^2 w_{j+1} & \text{if} \quad M_{j+1/2} < 0. \end{array} \right.$$

Thus, for smooth flow the total dissipative term $-\Delta^4 w$ of the *SLIP* scheme is replaced by the dispersive term $-\Delta^3 w$. The results of [108] indicate that the *USLIP* scheme is more accurate than the *SLIP* scheme, but it is not clear how the convergence rates compare, since we no longer have a fourth-difference dissipation to remove high-frequency errors.

4.8 Matrix Dissipation Model

As indicated in Section 4.5 high resolution of shock waves without oscillations can be achieved by closely imitating an upwind scheme in the neighborhood of a shock wave. A key feature of upwind schemes is a matrix evaluation of the numerical dissipation. With this evaluation the dissipative terms of each discrete equation (associated with a given coordinate direction) are scaled by the appropriate eigenvalues of the flux Jacobian matrix rather than by the spectral radius, as in the *JST* scheme. Such a matrix dissipation also allows high resolution of wall bounded shear layers (see [101]). We now present the modifications of the *JST* dissipation model required to produce the matrix dissipation model that we currently use.

Consider the two-dimensional, time-dependent Euler equations in the form

$$\frac{\partial (J^{-1}\mathbf{W})}{\partial t} + \frac{\partial \mathbf{F}}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} = 0, \tag{4.8.1}$$

where $\mathbf{F}$ and $\mathbf{G}$ are flux vectors, $\mathbf{W}$ is the solution vector, and (ξ, η) are arbitrary curvilinear coordinates. Define A and B to be the flux Jacobian matrices $\partial\mathbf{F}/\partial\mathbf{W}$ and $\partial\mathbf{G}/\partial\mathbf{W}$, respectively. By extending the scheme given in (4.5.6) to two dimensions, it follows that the $|A|$ and $|B|$ must be the scaling factors in a matrix dissipation model. Now, consider the *JST* dissipation model. The necessary modification to the contributions for the ξ direction of the artificial dissipation term defined by (4.2.1) is to substitute $|A|$ for the eigenvalue scaling factor, λ, in (4.2.2) and (4.2.3). For the η direction, ξ and $|A|$ are replaced by η and $|B|$, respectively. We next define explicitly the form for the matrix $|A|$. Let

$$\Lambda = \text{Diag}\ [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_3]$$

with

$$\lambda_1 = q + \sqrt{a_1^2 + a_2^2}\; c, \qquad \lambda_2 = q - \sqrt{a_1^2 + a_2^2}\; c, \qquad \lambda_3 = q,$$
$$a_1 = J^{-1}\xi_x, \quad a_2 = J^{-1}\xi_y, \quad q = a_1 u + a_2 v.$$

Then,

$$\begin{aligned}|A| = |\lambda_3| I + \left(\frac{|\lambda_1| + |\lambda_2|}{2} - |\lambda_3|\right)\left[\frac{\gamma - 1}{c^2} E_1 + \frac{1}{a_1^2 + a_2^2} E_2\right] \\ + \frac{|\lambda_1| - |\lambda_2|}{2}\left(\frac{1}{\sqrt{a_1^2 + a_2^2}\; c}\right)\left[E_3 + (\gamma - 1)E_4\right],\end{aligned} \tag{4.8.2}$$

where

$$E_1 = \begin{bmatrix} \phi & -u & -v & 1 \\ u\phi & -u^2 & -uv & u \\ v\phi & -uv & -v^2 & v \\ H\phi & -uH & -vH & H \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a_1 q & a_1^2 & a_1 a_2 & 0 \\ -a_2 q & a_1 a_2 & a_2^2 & 0 \\ -q^2 & q a_1 & q a_2 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} -q & a_1 & a_2 & 0 \\ -uq & u a_1 & u a_2 & 0 \\ -vq & v a_1 & v a_2 & 0 \\ -Hq & H a_1 & H a_2 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1\phi & -a_1 u & -a_1 v & a_1 \\ a_2\phi & -a_2 u & -a_2 v & a_2 \\ q\phi & -qu & -qv & q \end{bmatrix},$$

H is the total enthalpy, and $\phi = (u^2 + v^2)/2$. We note that for the matrices E_j each row is a scalar multiple of the other rows (except for zero rows). Hence, to find the product $E_j\mathbf{W}$ we need only find the product of one element of $E_j\mathbf{W}$ and the other rows are then scalar multiples of that element. Because of the special form of $|A|$ for any λ_1, λ_2, and λ_3, an arbitrary vector $\mathbf{x}$ can be multiplied by $|A|$ very quickly. That is, we calculate $\left|A_{j+1/2}\right|(\mathbf{W}_{j+1} - \mathbf{W}_j)$ directly rather than calculate $\left|A_{j+1/2}\right|$ and multiply a matrix times a vector. The matrix $|B|$ is computed in the same way as $|A|$ by simply replacing ξ with η.

In practice one cannot choose $\lambda_1, \lambda_2, \lambda_3$ as given above. Near stagnation points λ_3 approaches zero while near sonic lines λ_1 or λ_2 approach zero. A zero artificial viscosity would create numerical difficulties. Hence, we limit these values as

$$\begin{aligned} |\tilde{\lambda}_1| &= \max(|\lambda_1|, V_n \rho(A)), \qquad \rho(A) = |q| + c\sqrt{a_1^2 + a_2^2}, \\ |\tilde{\lambda}_2| &= \max(|\lambda_2|, V_n \rho(A)), \qquad |\tilde{\lambda}_3| = \max(|\lambda_3|, V_\ell \rho(A)), \end{aligned}$$

where the linear eigenvalue λ_3 can be limited differently than the nonlinear eigenvalues. The parameters V_n and V_ℓ have been determined numerically. Various values have been evaluated by comparing their corresponding computed solutions on the basis of the following: 1) Sharpness of shock waves captured (without producing oscillations), 2) Convergence rate of numerical scheme. Based on this evaluation a good choice for V_n was found to be 0.25 and for V_ℓ to be smaller. However, in [101] accurate coarse-grid solutions for a low-speed, high Reynolds number (5×10^5) laminar flow over a flat plate were not obtained with $V_\ell = 0.2$. Accurate coarse-grid (i.e., five to ten points in boundary layer) results were computed with a $V_\ell = 0.01$ for the direction normal to the plate and a $V_\ell = 0.2$ for the streamwise-like direction.

We have thus far replaced $\lambda_{i+1/2,j}$ in (4.2.2) and (4.2.3) by a matrix while leaving the limiters $\epsilon^{(2)}$ and $\epsilon^{(4)}$ as scalars. One can also introduce $\epsilon^{(2)}$ and $\epsilon^{(4)}$ into the diagonal matrix Λ. This allows different limiters to be chosen for different characteristic variables. For example, the limiter may be based on pressure for the nonlinear waves. However, the pressure is smooth through a contact discontinuity. Hence, a switch based on temperature may be more appropriate for the linear wave. One could also use different mesh scalings, $\phi(r)$, for the linear and nonlinear waves. One can also combine the matrix viscosity with the *SLIP* scheme described previously.

This can also be extended to time-dependent problems [48].

4.9 CUSP Scheme

In the previous sections we have described the use of an artificial viscosity based on either a scalar or matrix coefficient. Inspired by earlier work on flux-vector splitting [117] Liou et al. designed a scheme called Advection Upstream Splitting Method (*AUSM*) [56, 57] and later refined for large-scale 3-D viscous computations [81]. *AUSM* is based on a splitting of the flux function into convective and pressure contributions. In some sense, the pressure terms contribute to the acoustic waves while the velocity terms contribute to convective waves. Hence, it is reasonable that these flux terms be treated differently. Liou thus considers decompositions of the flux vector that is not based on a characteristic decomposition but on Mach-number scaled contributions of the left and the right states to the interface flux. This decomposition has the disadvantage that it is more difficult to develop for other sets of equations compared with a characteristic decomposition. A similar type scheme called Convective Upwind Split Pressure (*CUSP*) scheme was later introduced by Jameson [40] and subsequently modified by Tatsumi, Martinelli, and Jameson ([43, 44], [106, 107]). The *CUSP* scheme has several advantages. First, one can consider the scheme as another type of artificial viscosity, since it is defined as a sum of the central flux average plus a dissipative flux. Hence, it can be readily used with a variety of time-stepping schemes (e.g., multi-stage, *LU*, implicit, etc.). Second, CUSP can be used with multi-stage schemes which do not evaluate the artificial dissipation fluxes at every stage, in order to reduce computational work. Another advantage of *CUSP* is that it can be easily combined with preconditioning, since preconditioning is based on the inviscid flux form and not the artificial dissipation (see notes on preconditioning). Hence, we shall only describe the *CUSP* version of this type of scheme.

4.9.1 Definition of CUSP Scheme

In one dimension we consider the semidiscrete equation (see 4.2.10)

$$\frac{dw_j}{dt} + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = \frac{d_{j+1/2} - d_{j-1/2}}{\Delta x}. \tag{4.9.1}$$

The extension of the formulations given below to multidimensions is based upon the application of the 1-D scheme in the individual coordinate directions, thereby using appropriate definitions of metric terms, contravariant velocity components, speed of sound, etc. Previously, we introduced the scalar and matrix-valued viscosities by considering $d_{j+1/2}$ of the form

$$d_{j+1/2} = \frac{1}{2} Q_{j+1/2}(w_{j+1} - w_j). \tag{4.9.2}$$

The factor $\frac{1}{2}$ is introduced so that we get full upwinding when $Q_{j+1/2} = I$ (see 4.5.2). We note that for the scheme to be *TVD*, Q must be sufficiently large (see 4.6.10 and 4.6.11). For the matrix viscosity we chose $Q = |A|$ (modified near zero eigenvalues) while for the scalar viscosity we chose $Q = \sigma(A)I$ (σ is the spectral radius).

For the *CUSP* scheme we instead choose d as a linear combination of w and f. In one dimension we consider two choices for the state vector:

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad f = u \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} + \begin{pmatrix} 0 \\ p \\ up \end{pmatrix} = uw + f_p. \tag{4.9.3}$$

and

$$w_h = \begin{pmatrix} \rho \\ \rho u \\ \rho H \end{pmatrix}, \quad f = u \begin{pmatrix} \rho \\ \rho u \\ \rho H \end{pmatrix} + \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} = uw_h + f_p. \tag{4.9.4}$$

The first-order accurate *CUSP* is defined as

$$d_{j+1/2} = \frac{1}{2} \nu c (w_{j+1} - w_j) + \frac{\beta}{2}(f_{j+1} - f_j) \tag{4.9.5}$$

The factor c is included so that ν is dimensionless. We thus consider only scalar parameters instead of a matrix coefficient, but we have two free parameters, ν and β. The scheme is total enthalpy preserving if $w_h = (\rho, \rho u, \rho H)^T$ is chosen as the basis. This choice is denoted HCUSP by Jameson [43]. By using the arithmetic average, $\overline{u} = \frac{1}{2}(u_{j+1} + u_j)$, and the definition

$$\alpha c = \nu c + \beta \overline{u}. \tag{4.9.6}$$

one can rearrange 4.9.5,

$$d_{j+1/2} = \frac{1}{2} \alpha c (w_{j+1} - w_j) + \frac{\beta}{2}(f_{p_{j+1}} - f_{p_j}) + \frac{\beta}{2} \overline{w}(u_{j+1} - u_j). \tag{4.9.7}$$

Introducing the Roe matrix A_{RL}, we have $f_R - f_L = A_{RL}(w_R - w_L)$. This relation is exact if A_{RL} is computed from weighted averages of the left and the right states

$$\begin{aligned} u &= \frac{\sqrt{\rho_R}u_R + \sqrt{\rho_L}u_L}{\sqrt{\rho_R} + \sqrt{\rho_L}} \\ H &= \frac{\sqrt{\rho_R}H_R + \sqrt{\rho_L}H_L}{\sqrt{\rho_R} + \sqrt{\rho_L}} \\ \rho &= \sqrt{\rho_R \rho_L} \end{aligned} \tag{4.9.8}$$

Then the first-order dissipation is

$$d_{j+1/2} = \frac{1}{2}(\beta A_{RL} + \nu c I)(w_R - w_L). \tag{4.9.9}$$

We see from this formula that d is a linear function of A. Recall that $|A|$ is a quadratic function of A, by the Cayley-Hamilton Theorem. Hence, it is not possible to bound d by $|A|$. Thus, *CUSP* can not be a *TVD* scheme (see 4.6.10 and 4.6.11).

Assume that the subscript L denotes the interior point inside the shock zone, R is the state downstream of the shock, and the state LR is subsonic. Jameson [43] shows that the downstream point with the state R is in equilibrium if

$$f_R - f_L + \frac{\nu c}{1+\beta}(w_R - w_L) = 0, \tag{4.9.10}$$

Substituting the Roe matrix for the difference in f into (4.9.10) we get

$$\left(A_{RL} + \frac{\nu c}{1+\beta} I\right)(w_R - w_L) = 0. \tag{4.9.11}$$

Hence, $w_R - w_L$ is a eigenvector of A_{RL}, and $-(\nu c)/(1+\beta)$ is the corresponding eigenvalue. However, the eigenvalues of A_{RL} are known to be λ^+, λ^- and u. If λ is an eigenvalue of A, then using this formula for νc in (4.9.9) we have

$$d_{j+1/2} = \frac{1}{2}\left[-\lambda I + \beta(A_{RL} - \lambda I)\right](w_R - w_L).$$

In order to have a positive diffusion when $u > 0$, we require that λ be negative (i.e., $-(\nu c)/(1+\beta) = \lambda^-$). Thus,

$$\nu c = -(1+\beta)\lambda^-. \tag{4.9.12}$$

For $u < 0$ we obtain similarly

$$\nu c = (1-\beta)\lambda^+. \tag{4.9.13}$$

So, we have reduced our two free parameters to one free parameter by demanding a one point shock profile. More generally, Jameson shows that one obtains a shock profile with one interior point if the following two conditions hold:

1. When the flow is supersonic through the shock then one obtains a totally upwind flux

2. The artificial dissipation Q satisfies a generalized eigenvalue problem

$$(A_{RL} - \alpha_{RA} Q_{RA}) \left(w_R - w_{A)}\right) = 0$$

at the exit from the shock.

The second condition is satisfied by both the matrix viscosity and the *CUSP* scheme; however, the scalar viscosity does not satisfy the first condition. We again note that the *TVD* condition $Q \geq \frac{1}{2}|A|$ is satisfied by the scalar and matrix viscosities but not by *CUSP*.

What remains to be done is to choose suitable functions for β and νc which satisfy the above requirements. Jameson's choice for β is based upon the eigenvalues corresponding to the acoustic waves.

$$\beta = \begin{cases} +\max\left(0, \frac{u+\lambda^-}{u-\lambda^-}\right) & \text{if} \quad 0 \leq M < 1 \\ -\max\left(0, \frac{u+\lambda^+}{u-\lambda^+}\right) & \text{if} \quad -1 < M < 0 \\ \operatorname{sgn}(M) & \text{if} \quad |M| \geq 1. \end{cases} \tag{4.9.14}$$

The cutoffs, $\beta \geq 0$ for $u > 0$, and $\beta \leq 0$ for $u < 0$, ensure that the pressure terms are discretized centrally for small Mach numbers. Shock capturing with one interior point is obtained by taking

$$\nu c = \begin{cases} |u| & \text{if} \quad \beta = 0 \\ -(1+\beta)\lambda^- & \text{if} \quad \beta > 0 \text{ and } 0 < M < 1 \\ (1-\beta)\lambda^+ & \text{if} \quad \beta < 0 \text{ and } -1 < M < 0 \\ 0 & \text{if} \quad |M| \geq 1. \end{cases} \tag{4.9.15}$$

The dissipation coefficients are to be computed from Roe-averaged quantities (4.9.8). They provide full upwinding for supersonic flow, $\beta = \operatorname{sgn}(M)$, $\nu = 0$. The choice of $\nu c = |u|$ for $\beta = 0$ yields a continuous dissipation coefficient in the subsonic region and it does not smear slip lines with $|u|$ close to zero. This makes the CUSP formulation attractive for viscous flow calculations with boundary layers. However, viscous flows are usually discretized by using cells with large aspect ratio. It is well known that this situation requires larger dissipation scaling in the direction of the long cell sides than given by $|u|$. Following the discussion of section 4.2 we redefine the dissipation coefficients in the individual coordinate directions. For the ξ-direction we have

$$\nu c_\xi = r^+ \begin{cases} \max(|u|, \delta c r^-) & \text{if} \quad \beta = 0 \\ -(1+\beta)\lambda^- & \text{if} \quad \beta > 0 \text{ and } 0 < M < 1 \\ (1-\beta)\lambda^+ & \text{if} \quad \beta < 0 \text{ and } -1 < M < 0 \\ 0 & \text{if} \quad |M| \geq 1. \end{cases} \tag{4.9.16}$$

where r^+, r^- are functions of the spectral radii in the ξ and η directions, λ_ξ and λ_η, as follows:

$$r = \left(\frac{\lambda_\eta}{\lambda_\xi}\right)^\zeta, r^+ = \max(r, 1), r^- = \min(r, 1)$$

The dissipation coefficient in the η-direction is defined correspondingly.

4.9.2 Simplified Scheme

Several modifications of the *CUSP* scheme have been in use so far. Based upon the $w_h = (\rho, \rho u, \rho H)^T$ system Refs. [43] and [107] propose the dissipation coefficients

$$\begin{aligned}
\alpha &= \begin{cases} |M| & \text{if} \quad |M| \geq \epsilon \\ \frac{1}{2}\left(\epsilon + \frac{M^2}{\epsilon}\right) & \text{if} \quad |M| < \epsilon, \end{cases} \\
\beta &= \begin{cases} +\max\left(0, \frac{u+\lambda^-}{u-\lambda^-}\right) & \text{if} \quad 0 \leq M < 1 \\ -\max\left(0, \frac{u+\lambda^+}{u-\lambda^+}\right) & \text{if} \quad -1 < M < 0 \\ \operatorname{sgn}(M) & \text{if} \quad |M| \geq 1. \end{cases}
\end{aligned} \tag{4.9.17}$$

This choice does not allow exact shock capturing because (4.9.12) and (4.9.13) are not satisfied. Furthermore, Roe averaging has been replaced by arithmetic averaging in Ref. [43] and λ^-, λ^+ by $u - c, u + c$, respectively. This simplification saves a few square roots in the coding of the dissipative flux. Eq..(4.9.17) is then

$$\begin{aligned}
\alpha &= \begin{cases} |M| & \text{if} \quad |M| \geq \epsilon \\ \frac{1}{2}\left(\epsilon + \frac{M^2}{\epsilon}\right) & \text{if} \quad |M| < \epsilon, \end{cases} \\
\beta &= \begin{cases} \max(0, 2M - 1) & \text{if} \quad 0 \leq M < 1 \\ \min(0, 2M + 1) & \text{if} \quad -1 < M < 0 \\ \operatorname{sgn}(M) & \text{if} \quad |M| \geq 1. \end{cases}
\end{aligned} \tag{4.9.18}$$

4.9.3 Higher Order Scheme

Having determined νc and β, we see from (4.9.5) that the scheme is completely defined in terms of w and f_p. Recalling from previous sections, we now need to combine a first order artificial dissipation with a high order dissipation. As previously the high order dissipation is used in the smooth regions while the low order artificial viscosity is used in the neighborhood of shocks. To allow changing from one type to the other we can either use the *JST* switch (4.2.6) or the *SLIP* scheme. Formula (4.9.5), as given, is only first order accurate, as it depends only on $d_{j+1/2} = w_{j+1} - w_j$, and so the complete artificial viscosity behaves like a second difference. In order to improve this situation we need to use a *MUSCL* or *SLIP* approach to get the first difference to higher order accuracy. For example, we can replace (4.9.5) by

$$d_{j+1/2} = \frac{\nu c}{2}(w_R - w_L) + \frac{\beta}{2}\left[f_p(w_R) - f_p(w_L)\right],$$

where w_R, w_L are given by (4.7.11). This procedure was followed throughout the numerical examples shown below. Application of (4.7.11) to the $w_h = (\rho, \rho u, \rho H)^T$ variables still allows total enthalpy to be preserved in the higher order scheme. We note that *SLIP* requires considerably more computer time than the evaluation of the *JST* switch.

4.9.4 Analysis of CUSP Scheme

The eigenvalues of $\beta A_{RL} + \nu c I$ (see 4.9.9) are $\mu_1 c$, $\mu_2 c$, and $\mu_3 c$. Using the simplifications (4.9.18) the eigenvalues are:

$$\mu_1 = |M|,$$

$$\mu_2 = \begin{cases} |M| & \text{if} \quad |M| < \frac{1}{2} \\ \alpha + \beta & \text{if} \quad \frac{1}{2} \leq M \leq 1 \\ |M+1| & \text{if} \quad |M| > 1, \end{cases}$$

$$\mu_3 = \begin{cases} |M| & \text{if} \quad |M| < \frac{1}{2} \\ \alpha - \beta & \text{if} \quad \frac{1}{2} \leq M \leq 1 \\ |M-1| & \text{if} \quad |M| > 1 \end{cases}$$

We note that for $|M| < \frac{1}{2}$ all three eigenvalues of the artificial viscosity are equal, and so we have the equivalent of a scalar viscosity. The scalar viscosity now scales with $|M|c$ rather than $(|M| + 1)c$ for the *JST* scalar viscosity. This is more similar to the case of preconditioning where all the eigenvalues are approximately $|M|c$ for low speed flow. Hence, we expect that the *CUSP* dissipation should work properly for very low Mach numbers provided the central flux terms are augmented by a suitable preconditioning matrix.

In the subsonic range where $\beta = 0$ all the *CUSP* schemes do not satisfy (4.9.12), (4.9.13) necessary for shock capturing. That is, the cell-face Mach numbers in the shock structure have to be larger than about 0.5 in order to avoid post-shock oscillations. The motivation to design $\nu c = |u|$ for $\beta = 0$ has already been discussed above and is not repeated here. However, the choice of the function for β, (4.9.14) is not necessarily optimal. For example, $\beta = max(0, (u + \frac{1}{2}\lambda^-)/(u - \lambda^-))$ would allow shock capturing for Mach numbers down to about 1/3, but the subsonic dissipation would be twice as large, $\nu c = 2|u|$ for $\beta = 0$. Nevertheless, our own experience gained from a number of numerical applications suggests that there is no need for further modifications of β.

It is rather difficult to compare the effect of the parameter κ^4 of the *JST* and the combined *CUSP SLIP* scheme since these schemes also include eigenvalue information which is not the same in the two schemes. To isolate the effect we consider a low Mach number flow with preconditioning (see Part II for details). Now both switches are based on the convective eigenvalue, u. A typical value for the *JST* scheme is $\kappa^4 = \frac{1}{32}$. However, for a aspect ratio of one the Martinelli scalings adds another factor of two. The parameter $\alpha = 0$ in the preconditioning adds an additional factor of approximately 2.6. Hence, the effective constant multiplying the fourth difference is about $\frac{5}{32}$ which is somewhat smaller than that $\frac{1}{4}$ used with the original *CUSP-SLIP* scheme. For transonic flows it is more difficult to compare the levels of dissipation. However, it seems that the original *SLIP* yields to high a viscosity level and so the κ^4 introduced in (4.7.1) should be reduced to less than $\frac{1}{4}$. Numerical computations demonstrate the improved accuracy (though slower convergence) for standard transonic turbulent flows when κ^4 is reduced.

4.10 Other Variables

The equations that we are solving have the form

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0, \tag{4.10.1}$$

where the elements of $\mathbf{W}$ are the conservative variables. We are only interested in solving the steady state equations. However, the steady state equations are of a mixed hyperbolic-parabolic-elliptic type and highly nonlinear. To avoid these complications the normal procedure is to solve the time-dependent equations until a steady state is achieved. In practice, we monitor some residuals and stop the calculations when the norm of the residual has decreased by 4-5 orders of magnitude from its initial value. Since the time-dependent path is not of interest, we introduce acceleration techniques that destroy the true time-dependent information but enable the algorithm to reach the steady state much faster. Some examples of such techniques are local time-stepping, multigrid, and preconditioning.

As such, there is no real reason to update the conservative variables at each stage. Thus, we can replace (4.10.1) by

$$\frac{\partial \mathbf{\Psi}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0,$$

where $\mathbf{\Psi}$ is another set of variables that preserve the correct dimensions. Since

$$\frac{\partial \mathbf{\Psi}}{\partial t} = \frac{\partial \mathbf{\Psi}}{\partial \mathbf{W}} \frac{\partial \mathbf{W}}{\partial t} = \mathbf{P}^{-1} \frac{\partial \mathbf{W}}{\partial t},$$

we could consider this as a preconditioning of the original equations. If we use a scalar preconditioning this would now be based on $\mathbf{\Psi}$ variables rather than conservative variables. Hence, our steady state would be different than the original scheme. Thus, introducing $\mathbf{\Psi}$ variables would change the artificial dissipation, and hence, the accuracy of the scheme, in addition to changing the update procedure. We note that this applies both to scalar and matrix-valued artificial viscosities. For example, the Roe matrix is constructed based on the Jacobian of the flux terms relative to the conservative variables. One could equally well construct a Roe matrix based on the Jacobian of the flux vector relative to the $\mathbf{\Psi}$ variables. Hence, different $\mathbf{\Psi}$ yield different artificial viscosities (or Roe matrices), and hence, different approximations to the steady state solution. It is not known which $\mathbf{\Psi}$ is in some sense optimal.

Another possibility is a hybrid of the above schemes. Then, the scheme would be updated for the conservative variables (for example) as in the original scheme, but the artificial dissipation would be based on $\mathbf{\Psi}$ variables. One advantage of using an artificial dissipation rather than including these terms in the scheme is that it allows a complete decoupling between the updated variables and the artificial dissipation. One example of this was used in the original formulation of [32] where the artificial viscosity of the energy equation was based on the total enthalpy rather than the total energy (see 4.2.12), although the energy equation was updated for the total energy. This allowed the scheme to preserve a constant enthalpy in the steady state.

Another application is to split the flux used in the *CUSP* scheme so that only the total enthalpy remains in the flux rather than the energy that was used previously (see 4.9.3). Hence, we replace (4.9.3) by (4.9.4) The eigenvalues of the Jacobian of the one dimensional flux with respect to w_h are u, λ^-, λ^+ where

$$\lambda^\pm = \frac{\gamma+1}{2\gamma}u \pm \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)^2 + \frac{c^2-u^2}{\gamma}},$$

and

$$c = \sqrt{(\gamma-1)(H-\frac{u^2}{2})}$$

As just discussed the *H-CUSP* scheme was designed to only replace the artificial viscosity of the energy equation which is still updated for the total energy. We note that the eigenvalues $\lambda^\pm$ are smaller than the corresponding ones for the standard Euler equations for subsonic flow (though they coincide on the sonic line). Hence, if one were to modify the scheme to update the energy equation for the total enthalpy, then one could use larger time steps.

4.11 Summary

The central difference scheme requires an artificial viscosity in order to both prevent oscillations near shocks and damp high frequencies, enabling the iteration procedure to reach a steady state. In the Jameson, Schmidt, Turkel (*JST*) formulation these artificial viscosities are provided by second and fourth differences of the variables with a scalar coefficient included. This scalar coefficient depends on the largest eigenvalue (in each direction) to scale the size of the viscosity. In addition, the coefficient depends on the second difference of the pressure to sense shocks. In the neighborhood of shocks the fourth difference is turned off while the second difference prevents overshoots. In smooth regions of the flow the second difference (which leads to first order accuracy) is minimal while the fourth difference damps the high-frequency error.

This technique works quite well for transonic flow and was the main technique for many years. With the increasing popularity of upwind schemes it was seen that this scheme is less accurate than upwind schemes, especially on coarse meshes (see e.g., [4]). This led to the introduction of a matrix-valued coefficient in the artificial viscosity (dissipation) that mimics the effects of an upwind scheme, but within the context of a central difference scheme with an artificial dissipation coupled with a multistage time advancement. Later Jameson introduced the *CUSP* scheme, which is in between the matrix dissipation and the scalar dissipation schemes. With the *CUSP* scheme the dissipation is a function of the Mach number and becomes fully upwind in supersonic regions similar to the matrix dissipation. However, it avoids the need for a full-matrix coefficient while still obtaining one point shock profiles. Unfortunately, in order to achieve second-order accuracy extrapolations are needed that make this scheme more expensive than the matrix-viscosity method.

The idea of using the Mach number to mimic fully upwind methods has been useful in other applications besides the artificial viscosity. In Section 7.2.4 we discuss the use of the Mach number to adjust the parameters of the residual smoothing so that it becomes fully upwind in supersonic regions. Similarly one can construct a multigrid method (section 7.3) whose weighting factors depend on the Mach number in a manner that it becomes fully upwind in supersonic regions. We label this

Poor Man's Upwinding

Advantages:

- Cheap
- Fully upwind in supersonic flow

Disadvantage:

- Not fully upwind for subsonic flow

Applications

- Residual smoothing
- *CUSP* scheme
- Multigrid

In the opposite direction the Mach number is used in Part II to construct preconditionings that are useful in low Mach number regions. Then the Mach number can be used to turn off the preconditioning in supersonic regions.

5 Discrete Boundary Conditions

An important element in the development of an accurate and efficient algorithm for solving the Euler and Navier-Stokes equations is the selection of proper boundary conditions. The choice of conditions must be consistent with physical constraints of the problem of interest as well as the interior discrete formulation. Moreover, the physical conditions, in general, must be supplemented with a sufficient number of numerical relations to allow determination of all dependent variables. In addition to defining the conditions at wall boundaries, which can be solid or porous, the infinite domain problem must be adequately simulated in the case of external flows. This is usually done by delineating boundaries at some distance from the primary region of consideration, and then prescribing suitable boundary conditions for that location. In the case of a lifting airfoil, the outer boundary position must be sufficiently far so as to not compromise the development of the lift. For example, five airfoil chords would be too close whereas 20 chords would be satisfactory if the far-field vortex effect [109] is taken into account. Even for inviscid nonlifting flow over a circular cylinder, an outer boundary placed too close can cause inaccurate prediction of the flow over the aft portion of the cylinder.

At a solid boundary, a row of auxiliary cells is created exterior to the domain of the flow. Approximating the normal pressure gradient of (2.2.4) with a three-point centered difference at the surface, the auxiliary cell pressure is obtained. The density at this cell is equated to the density at the first point off the surface. The tangency condition is enforced by determining the Cartesian velocity components from

$$\begin{bmatrix} u \\ v \end{bmatrix}_{i,1} = \begin{bmatrix} \bar{x}_\xi & -\bar{y}_\xi \\ \bar{y}_\xi & \bar{x}_\xi \end{bmatrix}_w \begin{bmatrix} q_t \\ q_n \end{bmatrix}_{i,2},$$

where ξ is the coordinate aligned with the surface boundary, q_t and q_n are the tangential and normal velocity components, the subscript w means wall, the indices $(i,1)$ and $(i,2)$ refer to the centers of the auxiliary and the first interior cells, respectively. The overbar means the quantity is divided by $\sqrt{(x_\xi^2 + y_\xi^2)}$. Finally, the total internal energy is computed using the relation

$$\rho E = \frac{1}{\gamma - 1} p + \frac{1}{2} \rho \left(u^2 + v^2 \right).$$

In the case of viscous flows, the no slip condition is required, and this is imposed by treating the Cartesian velocity components as antisymmetric functions with respect to the solid surface. Thus,

$$u_{i,1} = -u_{i,2}, \qquad v_{i,1} = -v_{i,2}.$$

The surface values of pressure p and temperature T are computed using the reduced normal momentum and energy equations

$$\frac{\partial p}{\partial \eta} = 0, \qquad \frac{\partial T}{\partial \eta} = 0, \tag{5.1}$$

where η is the coordinate normal to the surface. As part of the boundary conditions we include the option to specify the wall temperature instead of imposing the adiabatic condition of (5.1).

In order to compute the unknown flow variables at the outer boundary of an external aerodynamics problem, we appeal to characteristic theory, some simplifying assumptions, and the concept of a point vortex. In Appendix A, we consider a point on the outer boundary and the two-dimensional Euler equations. Then, assuming a locally homentropic flow, we derive (for completeness) the one-dimensional equations of gas dynamics for the direction normal to the boundary. The elements of the solution vector are proportional to the local tangential velocity component and the Riemann invariants

$$R^+ = q_n + \frac{2c}{\gamma - 1}, \qquad R^- = q_n - \frac{2c}{\gamma - 1},$$

where the tangential and normal velocity components are defined as

$$q_t = \frac{x_\xi u + y_\xi v}{\sqrt{(x_\xi^2 + y_\xi^2)}}, \qquad q_n = \frac{-y_\xi u + x_\xi v}{\sqrt{(x_\xi^2 + y_\xi^2)}}.$$

We use this set of dependent variables, the homentropic assumption, and characteristic theory to determine the unknown flow variables.

To compute the discrete solution at the outer boundary points we introduce (as for the wall boundary) a row of auxiliary (boundary) cells exterior to the domain. Then, at a boundary cell, we compute the normal velocity component q_n and the speed of sound c from the relations

$$q_n = \frac{1}{2}\left(R^+ + R^-\right), \qquad c = \frac{(\gamma - 1)}{4}\left(R^+ - R^-\right),$$

where the characteristic variables R^+ and R^- are appropriately determined. Assume that the flow normal to the boundary is subcritical. If inflow occurs, the characteristic variables corresponding to the incoming characteristics are specified. Since we actually have a two-dimensional system, an additional quantity must be given. It follows directly that one should specify the entropy s (the flow has been assumed to be locally homentropic). In practice, for convenience, we define $s^* = p/\rho^\gamma$, which has the same functional dependence as entropy, and use this variable in place of entropy. So, for an inflow situation, we set

$$q_t = q_{t\infty}, \qquad R^+ = R^+_\infty, \qquad s^* = s^*_\infty, \tag{5.2}$$

and extrapolate R^- from the interior. If outflow occurs at the boundary, there is only one incoming characteristic (corresponding to R^+); and thus, we set $R^+ = R^+_\infty$, and extrapolate from the interior q_t, R^-, and s^*. In the particular case of supersonic flow, all characteristics are incoming if there is inflow, and they are outgoing if there is outflow. Therefore, we specify the dependent variables with their free-stream values if inflow occurs, and we use extrapolation to determine the boundary flow variables if outflow occurs.

At a distance far enough away from a two-dimensional lifting body, one can view the body as a point vortex with strength proportional to the circulation associated with the lift. One can then compute the components of the induced velocity at the far-field boundary due to the vortex. Moreover, as given in [109], the effective velocity components at the far-field boundary are computed as

$$u = u_\infty \cos\alpha + F\sin\phi \qquad v = v_\infty \sin\alpha - F\cos\phi, \tag{5.3}$$

where

$$\begin{aligned} F &= \frac{c_l\, c\, \bar{\beta}}{4\pi\, R}\left[1 - M_\infty^2 \sin^2(\phi - \alpha)\right]^{-1}, \\ \bar{\beta} &= \sqrt{1 - M_\infty^2}, \end{aligned}$$

the subscript ∞ refers to free-stream values, α is the angle of attack, R and ϕ are the magnitude and angle of the position vector originating from a reference point at the body (i.e., quarter-chord point for airfoil) and extending to the far-field boundary point, c is the body length, and c_l is the lift coefficient. The polar angle ϕ is defined to be positive in the counterclockwise direction relative to a reference line (i.e., coinciding with chord for airfoil) emanating from the leading edge of the body and proceeding downstream. The Cartesian velocity components u and v of (5.3) are used to compute the local tangential and normal velocity components required in the boundary conditions.

-bb-error = =

Figure 6: Physical domain for airfoil calculations.

Consider the particular case of a C-type mesh wrapped around an airfoil, and denote the outer boundary of a finite domain as $\Sigma_1 + \Sigma_2$, as depicted in Figure 6. For such airfoil computations, the boundary cells at Σ_1 are treated as just described. This is also done for Σ_2 when the flow is inviscid. In the viscous flow problem, a portion of the boundary Σ_2 can in general be wake flow. If the same boundary conditions applied at Σ_2 for inviscid flows are used for viscous flows, instabilities can occur. One way to treat the boundary cells at Σ_2 is to specify the pressure and extrapolate the variables ρ, ρu, and ρv. This approach would satisfy the requirement of characteristic theory to specify one quantity. However, this results in pressure wave reflections, which can seriously delay the convergence of the numerical scheme. An alternative boundary-point treatment is to extrapolate all the dependent variables, allowing the outer and surface boundaries to determine a unique solution. We have used numerical experiments to demonstrate that solutions obtained applying these two treatments are essentially the same near the airfoil. Furthermore, if the outer boundary is far enough away (i.e., 20 chords), there is generally no effect on global quantities such as lift and drag. With the second approach noticeable improvement in the convergence behavior of the solution algorithm has been observed.

For internal flows with a subsonic inlet Mach number, we replace the specified flow quantities of (5.2) with the total pressure, total enthalpy, and flow inclination angle. These conditions are usually known for such problems. The Riemann variable R^- is extrapolated from the interior of the domain. At a subcritical exit boundary, the pressure is specified, while the Riemann variable R^+, the total enthalpy, and the velocity component parallel to the boundary are extrapolated from the interior. If there is supersonic flow at the inlet or exit boundary, the variables at that boundary are determined in the same manner as described for supersonic external flow problems.

6 Basic Time-Stepping Schemes

In the first part of this section the class of Runge-Kutta (R-K) schemes that we use for time integration is defined. There are parameters associated with these schemes. The requirements to determine the parameters are discussed. Then, stability analysis for the four-stage and five-stage schemes that we have applied is done by considering a linear wave equation. In the next section we present stability properties of R-K schemes for systems of fluid dynamic equations. This requires writing the Navier-Stokes equations in general curvilinear coordinates and defining associated Jacobian matrices. In the final section, with this framework in place, an estimate for the time step is given.

6.1 Runge-Kutta Schemes

For problems where the area of a mesh cell is independent of time, the semidiscrete system of (3.2) becomes

$$\frac{d}{dt}\,\mathbf{W}_{i,j} + \mathbf{R}(\mathbf{W}_{i,j}) = 0, \tag{6.1.1}$$

where $\mathbf{R}(\mathbf{W}_{i,j})$ is the residual function defined by

$$\mathbf{R}(\mathbf{W}_{i,j}) = \frac{1}{\Omega_{i,j}}\left[\mathcal{L}_C + \mathcal{L}_D + \mathcal{L}_{AD}\right]\mathbf{W}_{i,j}.$$

A variety of methods for the integration of ordinary differential equations (ODE's) can be used to advance the solution of (6.1.1) in time. Single-step multistage schemes (i.e., Runge-Kutta) are usually preferred rather than linear multistep schemes (i.e., Adams-Bashforth), which require more storage and introduce implementation difficulties when combined with a multigrid method. In [32] a four-stage Runge-Kutta (R-K) scheme, which belongs to the class of standard R-K schemes and is fourth-order accurate in time, is used to solve a system of ODE's corresponding to the Euler equations. This scheme can be written as

$$\begin{aligned}
\mathbf{W}^{(0)} &= \mathbf{W}^{(n)} && \text{(6.1.2a)}\\
\mathbf{W}^{(1)} &= \mathbf{W}^{(0)} - \frac{\Delta t}{2}\mathbf{R}^{(0)} && \text{(6.1.2b)}\\
\mathbf{W}^{(2)} &= \mathbf{W}^{(0)} - \frac{\Delta t}{2}\mathbf{R}^{(1)} && \text{(6.1.2c)}\\
\mathbf{W}^{(3)} &= \mathbf{W}^{(0)} - \Delta t\mathbf{R}^{(2)} && \text{(6.1.2d)}\\
\mathbf{W}^{(4)} &= \mathbf{W}^{(0)} - \frac{\Delta t}{6}\left(\mathbf{R}^{(0)} + 2\mathbf{R}^{(1)} + 2\mathbf{R}^{(2)} + \mathbf{R}^{(3)}\right) && \text{(6.1.2e)}\\
\mathbf{W}^{(n+1)} &= \mathbf{W}^{(4)} && \text{(6.1.2f)}
\end{aligned}$$

where $\mathbf{R}^{(q)} = \mathbf{R}(\mathbf{W}^{(q)})$, the superscript n denotes the time level $n\Delta t$, and the mesh indices (i,j) associated with the solution vector $\mathbf{W}$ are suppressed for convenience. If one is interested only in steady flow problems, then the higher-order accuracy in time is not important, and other classes of multistage schemes can be considered. One can construct schemes with certain desirable stability and damping characteristics, leading to efficient steady-state solvers. For example, as discussed by Jameson [35], the solution at the $(q+1)$th stage can be expressed as

$$\mathbf{W}^{(q+1)} = \mathbf{W}^{(0)} - \alpha_{q+1}\Delta t\mathbf{R}^{(q)}, \tag{6.1.3}$$

where the residual function

$$\mathbf{R}^{(q)} = \frac{1}{\Omega}\left[\sum_{r=0}^{q}\beta_{qr}\,\mathcal{L}_C\mathbf{W}^{(r)} + \sum_{r=0}^{q}\delta_{qr}\,\mathcal{L}_D\mathbf{W}^{(r)} + \sum_{r=0}^{q}\gamma_{qr}\,\mathcal{L}_{AD}\mathbf{W}^{(r)}\right], \tag{6.1.4}$$

and the consistency conditions

$$\sum\beta_{qr} = 1, \qquad \sum\delta_{qr} = 1, \qquad \sum\gamma_{qr} = 1$$

must be satisfied. The basic parameters α_p, where $p = q+1$ $(q = 0,\ldots,m-1)$, and the weighting factors β_{qr}, δ_{qr}, and γ_{qr} must be prescribed to define the m-stage time-stepping scheme. The desired stability and damping properties of the scheme provide the requirements for determining the basic parameters and weighting factors. Both hyperbolic and parabolic stability limits must be considered. The hyperbolic and parabolic limits are intervals along the imaginary and negative real axes, respectively, in the complex plane. One can choose the coefficients α_p so as to have the best possible hyperbolic or parabolic stability limit without special regard to the high-frequency damping characteristics of the scheme. However, if the scheme is to be used as a driver of a multigrid method, one must ensure that the scheme damps effectively the highest frequency error components. In [115] van der Houwen gives the parameters α_p that correspond to the maximum (or nearly so) attainable Courant-Friedrichs-Lewy (CFL) number. For schemes with an odd number of stages, van der Houwen proved that the largest possible stability interval along the imaginary axis of the complex domain is $(m-1)$. Vichnevetsky [121] conjectured that $(m-1)$ is also the optimal CFL number when m is even and demonstrated this for $m = 2, 4$. Sonneveld and van Leer [88] proved that this CFL number limit is valid when m is even. In [35]-[41] Jameson considers schemes with the α_p's of van der Houwen and defines appropriate weighting factors for the artificial dissipation evaluations to yield a good parabolic limit. Although the α_p's are obtained using only a hyperbolic stability limitation, they are still a good choice for a viscous flow solver, especially at high Reynolds numbers. That is, the convection (hyperbolic) limit on the time step remains the controlling stability factor for practical aerodynamic flows.

Several members of the class of schemes defined by (6.1.3) and (6.1.4) have been analyzed in [35] by considering the model problem

$$\frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} + \epsilon_4\Delta x^3\frac{\partial^4 w}{\partial x^4} = 0, \tag{6.1.5}$$

which is the one-dimensional linear wave equation with a constant coefficient third-order dissipation term. If the spatial derivatives in (6.1.5) are approximated with central differencing, then

$$\Delta t\frac{dw}{dt} = -\frac{N}{2}(w_{j+1}^n - w_{j-1}^n) - \epsilon_4\frac{N}{a}(w_{j+2}^n - 4w_{j+1}^n + 6w_j^n - 4w_{j-1}^n + w_{j-2}^n), \tag{6.1.6}$$

where $N = a\Delta t/\Delta x$ is the Courant number. Taking the Fourier transform of (6.1.6), we obtain

$$\Delta t \frac{d\hat{w}}{dt} = z\ \hat{w}^n,$$

where the Fourier symbol

$$z = -\mathrm{i} N \sin\theta - 4\epsilon_4 \frac{N}{a}\left(1-\cos\theta\right)^2, \tag{6.1.7}$$

$\mathrm{i} = \sqrt{-1}$, and θ is the Fourier angle. If the residual function for any stage q is given by

$$R^{(q)} = \left(\mathcal{L}_C + \mathcal{L}_{AD}\right) w^{(q)},$$

(i.e. the entire residual is updated at every step) then the amplification factor for a m-stage scheme is

$$g(z) = 1 + \hat{f}(z(\theta)), \tag{6.1.8}$$

where

$$\hat{f}(\theta) = \kappa_1 + \kappa_2 z^2 + \cdots + \kappa_m z^m, \tag{6.1.9}$$

$\kappa_1 = \alpha_m$ with $\alpha_m = 1$ for consistency, and $\kappa_l = \kappa_{l-1}\alpha_{m-l+1}$ with $l = 2, 3, \ldots, m$. Since $g(z)$ is analytic, we are guaranteed by the maximum modulus theorem that all contours $|g(z)| < 1$ lie inside the curve $|g(z)| = 1$, which is the absolute stability curve. For this subclass of schemes the optimal polynomials, which are those satisfying the requirement that $|z| \leq (m-1)$ (see [35] and [88]), are defined as

$$g(z) = P_k(z) = \mathrm{i}^{k-1} T_{k-1}\left(\frac{-\mathrm{i}z}{k-1}\right) + \frac{\mathrm{i}^k}{2}\left[T_k\left(\frac{-\mathrm{i}z}{k-1}\right) - T_{k-2}\left(\frac{-\mathrm{i}z}{k-1}\right)\right],$$

where T_k is a Chebyshev polynomial, and $k \geq 2$. The coefficients κ_l for the four-stage and five-stage schemes given in [115] are

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{2}, \quad \kappa_3 = \frac{1}{6}, \quad \kappa_4 = \frac{1}{24}$$

and

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{2}, \quad \kappa_3 = \frac{3}{16}, \quad \kappa_4 = \frac{1}{32}, \quad \kappa_5 = \frac{1}{128}.$$

In the more general situation, the amplification factor is not a polynomial in z. For example, consider the subclass of schemes defined by (6.1.4) that are called (m, n) schemes. The m again refers to the number of stages, and n designates the number of evaluations of the dissipative contribution. Suppose,for example, we have a $(m, 2)$ scheme, where the numerical dissipation terms are evaluated on the first and second stages and frozen for the remaining stages (like the $(4, 2)$ scheme that we have used).

$$z_{\mathrm{r}} = \Re(z), \qquad z_{\mathrm{i}} = \mathrm{i}\Im(z).$$

Then, if $m \geq 3$, the $\hat{f}$ of (6.1.9) is replaced by

$$\begin{aligned} \hat{f} \;=\; & \kappa_1 - (\kappa_1\alpha_1 z_{\mathrm{r}} + \kappa_2 z_{\mathrm{i}}) z_{\mathrm{i}}^0 + (\kappa_2\alpha_1 z_{\mathrm{r}} + \kappa_3 z_{\mathrm{i}}) z_{\mathrm{i}}^1 - \cdots \\ & + (-1)^m (\kappa_{m-2}\alpha_1 z_{\mathrm{r}} + \kappa_{m-1} z_{\mathrm{i}}) z_{\mathrm{i}}^{m-3} - (-1)^m \kappa_m z z_{\mathrm{i}}^{m-2}. \end{aligned} \tag{6.1.10}$$

Consider the (5,3) scheme that we generally use, where the dissipation terms are evaluated on the first, third, and fifth stages, and frozen on the second and fourth stages. For this scheme the weighting factors for the dissipation terms, which are γ_{qr} in (6.1.4), are as follows:

$$\begin{array}{lllllllll}
\gamma_{00} = 1, & & & & \\
\gamma_{10} = 1, & \gamma_{11} = 0, & & & \\
\gamma_{20} = \Gamma_3, & \gamma_{21} = 0, & \gamma_{22} = \overline{\gamma}_3, & & \\
\gamma_{30} = \Gamma_3, & \gamma_{31} = 0, & \gamma_{32} = \overline{\gamma}_3, & \gamma_{33} = 0, & \\
\gamma_{40} = \Gamma_3\Gamma_5, & \gamma_{41} = 0, & \gamma_{42} = \overline{\gamma}_3\Gamma_5, & \gamma_{43} = 0, & \gamma_{44} = \overline{\gamma}_5,
\end{array} \tag{6.1.11}$$

where $\Gamma_3 = (1-\overline{\gamma}_3)$, $\Gamma_5 = (1-\overline{\gamma}_5)$, $\overline{\gamma}_3 = .56$, and $\overline{\gamma}_5 = .44$. The symbol of the time-stepping operator $\hat{f}$ for this scheme is given by

$$\hat{f} = \alpha_5\left[1 - \alpha_4 z_1(1-\alpha_3 z_{\rm i}) - \alpha_4 z_3 z_1(\alpha_3 z_2 z_{\rm i} - \overline{\gamma}_3 z_{\rm r}) - \Gamma_5\overline{\gamma}_3 z_3 z_{\rm r}\right]$$

where

$$\begin{aligned}
z_1 &= z_{\rm i} + \overline{\gamma}_5 z_{\rm r}, \\
z_2 &= z_{\rm i} + \overline{\gamma}_3 z_{\rm r}, \\
z_3 &= \alpha_2(1-\alpha_1 z_{\rm i}).
\end{aligned}$$

In a number of flow computations we have applied four-stage and five-stage schemes (i.e., [97], [6], [98], [64]). For these schemes, the residual function is

$$\mathbf{R}^{(q)} = \frac{1}{\Omega}\left[\mathcal{L}_C\mathbf{W}^{(q)} + \mathcal{L}_D\mathbf{W}^{(0)} + \sum_{r=0}^{q}\gamma_{qr}\,\mathcal{L}_{AD}\mathbf{W}^{(r)}\right].$$

By evaluating the physical diffusion terms on the first stage only, we reduce the computational effort of the scheme. This incompatibility with the computation of the numerical dissipation terms has not caused any deterioration in the performance of the schemes. The evaluation of the numerical dissipation terms on certain stages and the weighting of these evaluations can provide two advantages. First, it can allow the parabolic stability limit to be extended and the high-frequency damping to be improved. Second, it can reduce the expense of calculating the dissipation terms. For some additional discussion on the weighting of dissipation, see [35].

The time-stepping parameters for the four-stage scheme are

$$\alpha_1 = \frac{1}{4}, \quad \alpha_2 = \frac{1}{3}, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = 1.$$

Since $\alpha_{m-1} = 1/2$, the scheme is second-order accurate in time. The numerical dissipation terms are evaluated as for (6.1.10). For the model problem of (6.1.5) the absolute stability curve for this scheme is presented in Figure 7a. The hyperbolic stability limit (as determined by the extent of the stability interval on the imaginary axis) is $2\sqrt{2}$. The parabolic stability limit (as determined by the extent of the stability interval on the negative real axis) is 4. The dashed line curve in Figure 7a represents the locus of the Fourier symbol, as defined in (6.1.7), which must lie inside the $|g| = 1$ curve for stability. Figure 7b shows the variation of the amplification factor g with the Fourier angle θ. The scheme exhibits good high-frequency damping, which is crucial for a rapidly convergent multigrid method. When analyzing the stability and damping properties it is important to include all the components of the scheme. For example, if we extend the stability limit of the algorithm through the introduction of an implicit residual smoothing procedure, which will be discussed in Section 7.2, there is some deterioration in the high-frequency damping, as revealed in Figures 7c and 7d.

In the case of the five-stage scheme, the basic parameters are

$$\alpha_1 = \frac{1}{4}, \quad \alpha_2 = \frac{1}{6}, \quad \alpha_3 = \frac{3}{8}, \quad \alpha_4 = \frac{1}{2}, \quad \alpha_5 = 1.$$

A very large parabolic stability limit is established by evaluating the artificial dissipation terms on the first, third, and fifth stages with the weightings of (6.1.11). From Figure 8a one can see that for this scheme the hyperbolic stability limit is 4, and the parabolic limit is about 9. As indicated in Figure 8b, this scheme also has good high-frequency damping. Figures 8c and 8d show the stability curves for this scheme when implicit residual smoothing is applied. In practice, with the implicit residual smoothing, we use a CFL number of 7.5 for this scheme. For the four-stage scheme, we use a CFL number of 5.0. So, for the expense of one additional evaluation of the implicitly smoothed residual function, we increase the CFL number by a factor of 1.5.

-bb-error = =

Figure 7a: Stability curves of four-stage R-K scheme with two evaluations of dissipation (CFL = 2.4, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/3, 1/2, 1).

-bb-error = =

Figure 7b: Amplification factor of four-stage R-K scheme with two evaluations of dissipation (CFL = 2.4, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/3, 1/2, 1).

-bb-error = =

Figure 7c: Stability curves of four-stage R-K scheme with implicit residual smoothing (CFL = 4.8, $\beta = 0.6$, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/3, 1/2, 1).

-bb-error = =

Figure 7d: Amplification factor of four-stage R-K scheme with implicit residual smoothing (CFL = 4.8, $\beta = 0.6$, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/3, 1/2, 1).

-bb-error = =

Figure 8a: Stability curves of five-stage R-K scheme with three evaluations of dissipation (CFL = 3.0, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/6, 3/8, 1/2, 1).

-bb-error = =

Figure 8b: Amplification factor of five-stage R-K scheme with three evaluations of dissipation (CFL = 3.0, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/6, 3/8, 1/2, 1).

-bb-error = =

Figure 8c: Stability curves of five-stage R-K scheme with implicit residual smoothing (CFL = 6.0, $\beta = 0.6$, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/6, 3/8, 1/2, 1).

-bb-error = =

Figure 8d: Amplification factor of five-stage R-K scheme with implicit residual smoothing (CFL = 6.0, $\beta = 0.6$, $\kappa^{(2)} = 0$, $\kappa^{(4)} = 1/32$, coefficients: 1/4, 1/6, 3/8, 1/2, 1).

6.2 Stability of Runge-Kutta Schemes for Systems

The compressible Navier-Stokes equations can be classified as hyperbolic-parabolic type, or as incomplete parabolic type [93]. As discussed in the previous section, we solve these equations numerically with a member of a class of multistage time-stepping schemes. By considering the hyperbolic (inviscid) and parabolic (viscous and numerical dissipation) operators separately, one can obtain sufficient conditions for stability. These conditions can be used as a starting point in estimating a time step for solving the full Navier-Stokes equations.

In order to estimate the restriction on the time step due to convection and diffusion, we consider the two-dimensional Navier-Stokes equations expressed in generalized coordinates (ξ, η). These equations can be written as

$$\frac{\partial \widetilde{\mathbf{W}}}{\partial t} + \frac{\partial \widetilde{\mathbf{F}}^{(\xi)}}{\partial \xi} + \frac{\partial \widetilde{\mathbf{F}}^{(\eta)}}{\partial \eta} = \frac{\partial \widetilde{\mathbf{F}}_v^{(\xi)}}{\partial \xi} + \frac{\partial \widetilde{\mathbf{F}}_v^{(\eta)}}{\partial \eta}, \tag{6.2.1}$$

where $\widetilde{\mathbf{W}} = J^{-1}\mathbf{W}$ with J^{-1} and $\mathbf{W}$ representing the inverse of the transformation Jacobian and the vector of conserved flow variables. The derivatives associated with the transformation from Cartesian coordinates (x, y) to (ξ, η) and the corresponding inverse transformation are related as

$$\begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} = \frac{1}{J^{-1}} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix},$$

and the inverse of the transformation Jacobian is given by

$$J^{-1} = \frac{\partial(x, y)}{\partial(\xi, \eta)} = x_\xi y_\eta - x_\eta y_\xi = \Omega.$$

The transformed inviscid flux vectors $\widetilde{\mathbf{F}}^{(\xi)}$ and $\widetilde{\mathbf{F}}^{(\eta)}$ are given by

$$\begin{aligned} \widetilde{\mathbf{F}}^{(\xi)} &= J^{-1}(\xi_x \mathbf{F}^{(\xi)} - \xi_y \mathbf{F}^{(\eta)}), \\ \widetilde{\mathbf{F}}^{(\eta)} &= J^{-1}(\eta_x \mathbf{F}^{(\xi)} - \eta_y \mathbf{F}^{(\eta)}), \end{aligned}$$

Using the notation of [72] the viscous flux vectors $\widetilde{\mathbf{F}}_v^{(\xi)}$ and $\widetilde{\mathbf{F}}_v^{(\eta)}$ can be expressed as

$$\begin{aligned} \widetilde{\mathbf{F}}_v^{(\xi)} &= \widetilde{B}^{(\xi,\xi)} \frac{\partial \widetilde{\mathbf{W}}}{\partial \xi} + \widetilde{B}^{(\xi,\eta)} \frac{\partial \widetilde{\mathbf{W}}}{\partial \eta}, \\ \widetilde{\mathbf{F}}_v^{(\eta)} &= \widetilde{B}^{(\eta,\eta)} \frac{\partial \widetilde{\mathbf{W}}}{\partial \eta} + \widetilde{B}^{(\eta,\xi)} \frac{\partial \widetilde{\mathbf{W}}}{\partial \xi}, \end{aligned}$$

where

$$\widetilde{B}^{(\alpha,\beta)} = B^{(\alpha,\beta)} M, \quad \alpha, \beta \in \{\xi, \eta\}.$$

and $B_1^{(\alpha,\beta)}$ is a viscous matrix obtained by rewriting the $\widetilde{F}_v$ vectors in terms of primitive variables. The matrix M, which transforms nonconservative (primitive) variables to conservative variables, and its inverse are given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -u\rho^{-1} & \rho^{-1} & 0 & 0 \\ -v\rho^{-1} & 0 & \rho^{-1} & 0 \\ \phi^2 & -(\gamma-1)u & -(\gamma-1)v & (\gamma-1) \end{bmatrix},$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ (\gamma-1)^{-1}\phi^2 & \rho u & \rho v & (\gamma-1)^{-1} \end{bmatrix},$$

with $\phi^2 = (\gamma-1)(u^2+v^2)/2$. The viscous matrixes $\widetilde{B}^{(\xi,\xi)}$, $\widetilde{B}^{(\eta,\eta)}$, $\widetilde{B}^{(\xi,\eta)}$, and $\widetilde{B}^{(\eta,\xi)}$ are defined according to

$$\widetilde{B}^{(\alpha,\beta)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}^{\alpha,\beta}_{x,x} & \mathcal{B}^{\alpha,\beta}_{x,y} & 0 \\ 0 & \mathcal{B}^{\alpha,\beta}_{y,x} & \mathcal{B}^{\alpha,\beta}_{y,y} & 0 \\ -\mathcal{B}^{\alpha,\beta}p\rho^{-2} & \mathcal{B}^{\alpha,\beta}_{x,x}u + \mathcal{B}^{\alpha,\beta}_{y,x}v & \mathcal{B}^{\alpha,\beta}_{y,y}v + \mathcal{B}^{\alpha,\beta}_{x,y}u & \mathcal{B}^{\alpha,\beta}\rho^{-1} \end{bmatrix},$$

with

$$\begin{aligned} \mathcal{B}^{\alpha,\beta}_{a,b} &= \frac{\sqrt{\gamma}M}{Re}\left[\lambda\frac{\partial\alpha}{\partial a}\frac{\partial\beta}{\partial b} + \mu\frac{\partial\alpha}{\partial b}\frac{\partial\beta}{\partial a} + \mu\nabla\alpha\cdot\nabla\beta\ \delta_{a,b}\right], \\ B^{\alpha,\beta} &= \frac{\sqrt{\gamma}M}{Re}\left(\frac{\gamma}{\gamma-1}\right)\frac{\mu}{Pr}\nabla\alpha\cdot\nabla\beta, \end{aligned}$$

∇ is the gradient operator, and $\delta_{a,b}$ is the Kronecker delta. Using the Jacobian matrices $\widetilde{A}^{(\alpha)} = \partial\widetilde{F}^{(\alpha)}/\partial\widetilde{W}$, where $\alpha \in \{\alpha,\beta\}$, we can rewrite (6.2.1) as

$$\begin{aligned} \frac{\partial\widetilde{\mathbf{W}}}{\partial t} + \widetilde{A}^{(\xi)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\xi} + \widetilde{A}^{(\eta)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\eta} &= \frac{\partial}{\partial\xi}\left(\widetilde{B}^{(\xi,\xi)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\xi} + \widetilde{B}^{(\xi,\eta)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\eta}\right) \\ &+ \frac{\partial}{\partial\eta}\left(\widetilde{B}^{(\eta,\xi)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\xi} + \widetilde{B}^{(\eta,\eta)}\frac{\partial\widetilde{\mathbf{W}}}{\partial\eta}\right), \end{aligned} \tag{6.2.2}$$

where

$$\widetilde{A}^{(\alpha)} = \begin{bmatrix} 0 & \alpha_x & \alpha_y & 0 \\ -uU^{(\alpha)} + \alpha_x\phi^2 & U^{(\alpha)} - (\gamma-2)\alpha_x u & \alpha_y u - (\gamma-1)\alpha_x v & (\gamma-1)\alpha_x \\ -vU^{(\alpha)} + \alpha_y\phi^2 & \alpha_x v - (\gamma-1)\alpha_y u & U^{(\alpha)} - (\gamma-2)\alpha_y v & (\gamma-1)\alpha_y \\ U^{(\alpha)}(\phi^2-\omega) & \alpha_x\omega - (\gamma-1)uU^{(\alpha)} & \alpha_y\omega - (\gamma-1)vU^{(\alpha)} & \gamma U^{(\alpha)} \end{bmatrix},$$

with $U^{(\alpha)} = \alpha_x u + \alpha_y v$ and $\omega = \gamma E - \phi^2$. Take the coefficient matrices of (6.2.2) to be locally constant in space and time and transform to primitive variables. Then, we obtain

$$\begin{aligned} \frac{\partial\mathbf{W}_P}{\partial t} + \bar{A}^{(\xi)}\frac{\partial\mathbf{W}_P}{\partial\xi} + \bar{A}^{(\eta)}\frac{\partial\mathbf{W}_P}{\partial\eta} &= \bar{B}^{(\xi,\xi)}\frac{\partial^2\mathbf{W}_P}{\partial\xi^2} + \bar{B}^{(\eta,\eta)}\frac{\partial^2\mathbf{W}_P}{\partial\eta^2} \\ &+ \left(\bar{B}^{(\xi,\eta)} + \bar{B}^{(\eta,\xi)}\right)\frac{\partial^2\mathbf{W}_P}{\partial\xi\partial\eta}, \end{aligned} \tag{6.2.3}$$

where

$$\mathbf{W}_P = \begin{bmatrix} \rho & u & v & p \end{bmatrix}^T,$$

$$\bar{A}^{(\alpha)} = M\tilde{A}^{(\alpha)}M^{-1} = \begin{bmatrix} U^{(\alpha)} & \rho\alpha_x & \rho\alpha_y & 0 \\ 0 & U^{(\alpha)} & 0 & \rho^{-1}\alpha_x \\ 0 & 0 & U^{(\alpha)} & \rho^{-1}\alpha_y \\ 0 & \gamma p\alpha_x & \gamma p\alpha_y & U^{(\alpha)} \end{bmatrix},$$

$$\bar{B}^{(\alpha,\beta)} = MB^{(\alpha,\beta)} = \rho^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}^{\alpha,\beta}_{x,x} & \mathcal{B}^{\alpha,\beta}_{x,y} & 0 \\ 0 & \mathcal{B}^{\alpha,\beta}_{y,x} & \mathcal{B}^{\alpha,\beta}_{y,y} & 0 \\ -(\gamma-1)\mathcal{B}^{\alpha,\beta}p\rho^{-1} & 0 & 0 & (\gamma-1)\mathcal{B}^{\alpha,\beta} \end{bmatrix}$$

The eigenvalues of the matrices $\bar{A}^{(\alpha)}$ are

$$U^{(\alpha)}, \quad U^{(\alpha)}, \quad U^{(\alpha)} + c\sqrt{\alpha_x^2+\alpha_y^2}, \quad U^{(\alpha)} - c\sqrt{\alpha_x^2+\alpha_y^2}.$$

The eigenvalues of the matrices $\bar{B}^{(\alpha,\alpha)}$ are

$$0, \quad \gamma\frac{\mu}{Pr}\Gamma_1, \quad \mu\Gamma_1, \quad (2\mu+\lambda)\Gamma_1,$$

where

$$\Gamma_1 = \frac{\sqrt{\gamma}M}{Re}\frac{1}{\rho}(\alpha_x^2+\alpha_y^2).$$

For the matrices $\bar{B}^{(\alpha,\beta)}$ they are

$$0, \quad \frac{\sqrt{\gamma}M}{Re}\left[2\frac{\gamma}{Pr}\frac{\mu}{\rho}(\nabla\alpha\cdot\nabla\beta)\right], \quad \frac{\sqrt{\gamma}M}{Re}\left[\frac{(\lambda+3\mu)}{\rho}(\nabla\alpha\cdot\nabla\beta) + \frac{(\lambda+\mu)}{\rho}|\nabla\alpha|\,|\nabla\beta|\right],$$
$$\frac{\sqrt{\gamma}M}{Re}\left[\frac{(\lambda+3\mu)}{\rho}(\nabla\alpha\cdot\nabla\beta) - \frac{(\lambda+\mu)}{\rho}|\nabla\alpha|\,|\nabla\beta|\right].$$

Abarbanel and Gottlieb [1] have shown that the matrices of (6.2.3) can be simultaneously symmetrized. They symmetrize the matrices with the similarity transformation determined by

$$S = \begin{bmatrix} \sqrt{\gamma}\rho c^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{\gamma}}\rho c & 0 & 0 & \frac{\sqrt{\gamma-1}}{\sqrt{\gamma}}\rho c \end{bmatrix}, \tag{6.2.4}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\gamma}}\rho^{-1}c^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{\gamma(\gamma-1)}}\rho c & 0 & 0 & \frac{\sqrt{\gamma}}{\sqrt{\gamma-1}}\rho^{-1}c^{-1} \end{bmatrix}. \tag{6.2.5}$$

Using this similarity transformation we can rewrite (6.2.3) as

$$\frac{\partial \mathbf{V}}{\partial t} + A^{(\xi)}\frac{\partial \mathbf{V}}{\partial \xi} + A^{(\eta)}\frac{\partial \mathbf{V}}{\partial \eta} = B^{(\xi,\xi)}\frac{\partial^2 \mathbf{V}}{\partial \xi^2} + B^{(\eta,\eta)}\frac{\partial^2 \mathbf{V}}{\partial \eta^2} + \left(B^{(\xi,\eta)} + B^{(\eta,\xi)}\right)\frac{\partial^2 \mathbf{V}}{\partial \xi \partial \eta}, \tag{6.2.6}$$

where

$$A^{(\alpha)} = S^{-1}\bar{A}^{(\alpha)}S, \quad B^{(\alpha,\beta)} = S^{-1}\bar{B}^{(\alpha,\beta)}S,$$

with $\alpha, \beta \in \{\xi, \eta\}$.

Define as a discrete computational domain $\{(\xi,\eta):\ 1 \leq \xi \leq \mathrm{L},\ 1 \leq \eta \leq \mathrm{L},\ \text{and}\ \Delta\xi = \Delta\eta = 1\}$, where $\mathrm{L} = \mathrm{N}$, the number of mesh intervals in either coordinate direction. Let a discrete vector function such as $\mathbf{V}(i\Delta\xi, j\Delta\eta) = \mathbf{V}(i,j)$ be denoted by $\mathbf{V}_{i,j}$. If the spatial derivatives of (6.2.6) are approximated with second-order central differences, we obtain the semidiscrete representation

$$\frac{d}{dt}\mathbf{V}_{i,j} + \mathcal{L}\mathbf{V}_{i,j} = 0, \tag{6.2.7}$$

where

$$\begin{aligned} \mathcal{L}\mathbf{V}_{i,j} \ = \ & +(A_1^{(\xi)})_{i,j}\mu_\xi\delta_\xi\mathbf{V}_{i,j} + (A_1^{(\eta)})_{i,j}\mu_\eta\delta_\eta\mathbf{V}_{i,j} - (B_1^{(\xi,\xi)})_{i,j}\delta_\xi^2\mathbf{V}_{i,j} \\ & - \left[(B_1^{(\xi,\eta)})_{i,j} + (B_1^{(\eta,\xi)})_{i,j}\right]\mu_\xi\delta_\xi\mu_\eta\delta_\eta\mathbf{V}_{i,j} - (B_1^{(\eta,\eta)})_{i,j}\delta_\eta^2\mathbf{V}_{i,j}. \end{aligned}$$

Then, taking the Fourier transform of (6.2.7), we get

$$\Delta t\frac{d}{dt}\hat{\mathbf{V}}_{i,j} = \ -\Delta t\hat{\mathcal{L}}\hat{\mathbf{V}}_{i,j} = Z\hat{\mathbf{V}}_{i,j},$$

where the caret indicates a transformed quantity,

$$Z = Z_C + Z_D, \tag{6.2.8}$$

with

$$\begin{aligned} Z_C \ &= \ -\mathrm{i}A_1^{(\xi)}\sin\theta_\xi - \mathrm{i}A_1^{(\eta)}\sin\theta_\eta \\ Z_D \ &= \ -4B_1^{(\xi,\xi)}\sin^2\frac{\theta_\xi}{2} - (B_1^{(\xi,\eta)} + B_1^{(\eta,\xi)})\sin\theta_\xi\sin\theta_\eta - 4B_1^{(\eta,\eta)}\sin^2\frac{\theta_\eta}{2}. \end{aligned}$$

and $\mathrm{i} = \sqrt{-1}$.

If all terms in the flow equations and the numerical dissipation are evaluated at each step of the R-K scheme, then the amplification matrix G for a m-stage scheme is a function of one variable. In particular,

$$G(Z) = I + \kappa_1 Z + \kappa_2 Z^2 + \cdots + \kappa_m Z^m,$$

where κ_l is a function of the coefficients of the R-K scheme (see (6.1.9) for definition). Let $\lambda_q(G)$ be any eigenvalue of $G(Z)$ and $\lambda_q(\hat{\mathcal{L}})$ be any eigenvalue of $\hat{\mathcal{L}}$. Also let $z = -\Delta t\lambda_q(\hat{\mathcal{L}})$. The $\lambda_q(G)$ are related to z as

$$\lambda_q(G) = 1 + \kappa_1 z + \kappa_2 z^2 + \cdots + \kappa_m z^m.$$

The stability of a scheme requires that the amplification matrix satisfy the condition $\| G^n \| \leq C$ for all n. We define the spectral radius σ of a matrix as equal to the largest eigenvalue in absolute value. If the matrix G is normal (i.e., $GG^* = G^*G$), then its norm is equivalent to the spectral radius $\sigma(G)$. Thus, for a normal matrix we require that $\sigma(G^n) = \sigma^n(G) \leq C$. This is equivalent to the von Neumann condition for stability that requires

$$\sigma(G) \leq 1.$$

Hence, if Z and G are normal matrices, then the condition for stability is

$$|g(z)| = |g\Big(-\Delta t \lambda_q(\hat{\mathcal{L}})\Big)| \leq 1 \qquad \text{for all } q. \tag{6.2.9}$$

In order to determine sufficient conditions for stability one considers separately the hyperbolic and parabolic operators associated with (6.2.1). We first examine stability for the Euler equations (i.e., $Re \to \infty$ in (6.2.1)). The following theorem gives a sufficient condition for stability:

Theorem 6.1 *Suppose the R-K scheme satisfies the property that*

$$[-\mathrm{i}\, N_{CFL}, \mathrm{i}\, N_{CFL}] \subset \{z \in \mathcal{C} : |g(z)| \leq 1\}, \tag{6.2.10}$$

where N_{CFL} is the CFL number for the R-K scheme. Assume smooth initial data such that the Cauchy problem for the Euler equations is well posed. If the Euler equations are solved with this R-K scheme and second-order centered difference approximations for the spatial derivatives, the condition

$$\Delta t\, [\sigma(\alpha A^{(\xi)} + \beta A^{(\eta)})] \leq N_{CFL}, \tag{6.2.11}$$

where $\mid \alpha \mid \leq 1$ and $\mid \beta \mid \leq 1$ is sufficient for stability of the linearized problem.

The proof follows directly. Since the matrices $A^{(\xi)}$ and $A^{(\eta)}$ are symmetric, the amplification matrix $G(Z)$ is normal. Furthermore, for a central difference scheme the Fourier transform of the first derivative is a pure imaginary number. Hence, $Z = -\mathrm{i}\Delta t \left(A^{(\xi)} \sin\theta_\xi + A^{(\eta)} \sin\theta_\eta\right)$, (remember that $\Delta\xi = \Delta\eta = 1$) and so the result follows.

Now consider the parabolic equation derived from (6.2.6) by eliminating the first-order spatial derivatives. The Fourier symbol of the difference operator is given by (6.2.8) when $A^{(\alpha)} \equiv 0$. Let N_D denote the diffusion number of the R-K scheme (i.e., it defines the stability interval along the negative real axis). Then, the maximum allowable time step is $N_D \Delta t_D$, where for the 1-D scalar diffusion equation with central differencing $\Delta t_D = \Delta x^2/(4\mu)$, with μ being the diffusion coefficient. A sufficient condition for stability is defined as follows:

Theorem 6.2 *Suppose the R-K scheme satisfies the property that*

$$[-N_D, 0] \subset \{z \in \mathcal{C} : |g(z)| \leq 1\}, \tag{6.2.12}$$

where N_D is the diffusion number for the R-K scheme. Assume smooth initial data such that the Cauchy problem for the viscous equations is well posed. If the viscous equations are solved with this R-K scheme and second-order centered spatial differencing, the condition

$$\max\left(\Delta t\, \sigma_D\right) \leq N_D, \tag{6.2.13}$$

with

$$\sigma_D = 4\,\sigma\left(B^{(\xi,\xi)}\sin^2\frac{\theta_\xi}{2} + B^{(\eta,\eta)}\sin^2\frac{\theta_\eta}{2} + \frac{1}{4}(B^{(\xi,\eta)} + B^{(\eta,\xi)})\sin\theta_\xi\sin\theta_\eta\right)$$

is sufficient for stability of the linearized equation.

Since

$$Z = -\Delta t\left[4\left(B^{(\xi,\xi)}\sin^2\frac{\theta_\xi}{2} + B^{(\eta,\eta)}\sin^2\frac{\theta_\eta}{2}\right) + (B^{(\xi,\eta)} + B^{(\eta,\xi)})\sin\theta_\xi\sin\theta_\eta\right]$$

is a negative real symmetric matrix, $G(Z)$ is again normal. Therefore, the proof is similar to the previous case. Note that if we neglect the cross-derivative terms, the inequality of (6.2.13) reduces to

$$\Delta t\left[4\,\sigma\Big(B^{(\xi,\xi)} + B^{(\eta,\eta)}\Big)\right] \le N_D.$$

Next, suppose we add numerical dissipation in the form of fourth-difference terms to the discrete viscous equations. Then,

$$\begin{aligned}\mathcal{L}\mathbf{V}_{i,j} \;\;=\;\; & -(B^{(\xi,\xi)})_{i,j}\delta_\xi^2\mathbf{V}_{i,j} - (B^{(\eta,\eta)})_{i,j}\delta_\eta^2\mathbf{V}_{i,j} \\ & - \left[(B^{(\xi,\eta)})_{i,j} + (B^{(\eta,\xi)})_{i,j}\right]\mu_\xi\delta_\xi\mu_\eta\delta_\eta\mathbf{V}_{i,j} \\ & + \varepsilon^{(4)}\sigma(\tilde{A}^{(\xi)})\delta_\xi^4\mathbf{V}_{i,j} + \varepsilon^{(4)}\sigma(\tilde{A}^{(\eta)})\delta_\eta^4\mathbf{V}_{i,j}\end{aligned}$$

and the Fourier symbol of (6.2.8) is replaced by

$$Z = Z_D + Z_{AD},$$

where Z_D is defined as in (6.2.8), and

$$Z_{AD} = -16\,\Delta t\,\varepsilon^{(4)}\sigma\left(\widetilde{A}^{(\xi)}\sin^4\frac{\theta_\xi}{2} + \widetilde{A}^{(\eta)}\sin^4\frac{\theta_\eta}{2}\right)I$$

with I being the identity matrix.

Lemma 6.3 *If we consider only the viscous terms and the fourth-difference dissipation terms, then the condition*

$$\max\left[\Delta t\,(\sigma_D + \sigma_{AD})\right] \le N_D, \tag{6.2.14}$$

with σ_D as given in (6.2.13) and

$$\sigma_{AD} = 16\,\varepsilon^{(4)}\sigma\left(\widetilde{A}^{(\xi)}\sin^4\frac{\theta_\xi}{2} + \widetilde{A}^{(\eta)}\sin^4\frac{\theta_\eta}{2}\right)$$

is sufficient for the linearized stability of the multistage scheme.

The proof is the same as the previous case. If cross-derivative terms are neglected, the inequality (6.2.14) becomes

$$\Delta t\left[4\,\sigma\Big(B^{(\xi,\xi)} + B^{(\eta,\eta)}\Big) + 16\,\varepsilon^{(4)}\sigma\Big(\widetilde{A}^{(\xi)} + \widetilde{A}^{(\eta)}\Big)\right] \le N_D.$$

6.3 Time Step Estimate

Now consider the situation where the R-K scheme satisfies simultaneously the properties of (6.2.10) and (6.2.12). Consider the one-dimensional case in the ξ direction. Then the R-K scheme depends on the matrix Z, where $Z = -\Delta t \left(\mathrm{i}A^{(\xi)} \sin\theta + 4B^{(\xi,\xi)} \sin^2 \theta_\xi/2 \right)$. Since $A^{(\xi)}$ and $B^{(\xi,\xi)}$ do not commute, the matrix Z, and thus the amplification matrix G, is no longer a normal matrix. Hence, the von Neumann condition on the largest eigenvalue of G is now a necessary but not sufficient condition for stability. Thus, there is no simple way to go from properties of Z to properties of G. The spectral mapping theorem relates the eigenvalues of Z to the eigenvalues of G. Since the eigenvalues do not tell the entire story, we need to use energy estimates based on norms. However, there is no simple relationship between the norm of the matrix Z and the norm of G (see [55] for some theory on the stability of R-K methods for non-normal matrices).

In practice, we use a simplified stability condition. There is no strict mathematical proof of its stability; nevertheless, it seems to work well. Consider

$$\Delta t_C = \frac{N_{CFL}}{\sigma_C}, \quad \Delta t_D = \frac{N_D}{\sigma_D},$$

where

$$\sigma_C = \sigma(\widetilde{A}^{(\xi)}) + \sigma(\widetilde{A}^{(\eta)}), \qquad \sigma_D = \sigma(\widetilde{B}^{(\xi,\xi)}) + \sigma(\widetilde{B}^{(\eta,\eta)}) + \sigma(\widetilde{B}^{(\xi,\eta)}).$$

Let

$$\frac{1}{\Delta t} = \frac{1}{\Delta t_C} + \frac{1}{\Delta t_D},$$

and so

$$\Delta t = \frac{N_{CFL}}{\sigma_C + (\frac{N_{CFL}}{N_D})\sigma_D}. \tag{6.3.1}$$

Since we have considered schemes where $N_{CFL} \approx N_D$, we compute the time step for each cell in the computational domain as

$$\Delta t = \frac{N_{CFL}\,\Omega}{\lambda_C + \lambda_D}, \tag{6.3.2}$$

where Ω is the cell area,

$$\lambda_C = \ \lambda_\xi + \lambda_\eta, \qquad \lambda_D = \ (\ \lambda_D)_\xi + (\lambda_D)_\eta + (\lambda_D)_{\xi\eta},$$

with λ_ξ and λ_η defined by (4.2.5), and

$$\begin{aligned}
(\lambda_D)_\xi &= \frac{\sqrt{\gamma}M}{Re}\frac{\gamma\mu}{\rho Pr}\Omega^{-1}(x_\eta^2 + y_\eta^2), \quad (\lambda_D)_\eta = \frac{\sqrt{\gamma}M}{Re}\frac{\gamma\mu}{\rho Pr}\Omega^{-1}(x_\xi^2 + y_\xi^2), \\
(\lambda_D)_{\xi\eta} &= \frac{\sqrt{\gamma}M}{Re}\frac{1}{\rho\Omega}\left[-(\lambda + 3\mu)(y_\eta y_\xi + x_\xi x_\eta) + (\lambda + \mu)\sqrt{(y_\eta^2 + x_\eta^2)(y_\xi^2 + x_\xi^2)}\ \right]
\end{aligned}$$

For the thin-layer Navier-Stokes equations we take $\lambda_D = (\lambda_D)_\eta$ where η is the direction normal to the boundary layer.

Remark 6.4 *So far we have considered only central differencing for the spatial approximations in arriving at a time-step estimate for an explicit R-K scheme. Since a numerical flux function for an upwind scheme can generally be expressed as the sum of a centered (physical) contribution and a numerical dissipation contribution, then certainly (6.3.1) is also a reasonable estimate for the time step when an upwind scheme is used.*

7 Convergence Acceleration Techniques

7.1 Local Time Stepping

The first technique employed to accelerate convergence of the basic explicit time-stepping scheme to a steady-state solution is local time stepping, where each cell is updated using its own time step. For simplicity, the one-dimensional Euler equations can be used to understand the meaning of local time stepping from the discrete point of view. Suppose we write the Euler equations in the form

$$\frac{\partial \mathbf{W}}{\partial t} + A\frac{\partial \mathbf{W}}{\partial x} = 0.$$

If the equations are discretized in an explicit sense, then

$$\Delta \tilde{\mathbf{W}} = \tilde{A}\tilde{\mathbf{W}}^n,$$

where the tilde indicates that the vector or matrix is for the complete discrete system. The block matrix $\tilde{A}$ is a function of $\Delta t_{\min}/\Delta x$, with $\Delta t_{\min}$ being the minimum time step permitted in the domain, and each element of $\tilde{A}$ is a 3×3 matrix. Let the explicit matrix $\tilde{A}$ for the system of difference equations be preconditioned by the diagonal matrix Λ given by

$$\Lambda = \mathrm{Diag}\left[\begin{array}{ccccc} \overline{\Delta t}_2 I & \overline{\Delta t}_3 I & \cdots & \overline{\Delta t}_{N-2} I & \overline{\Delta t}_{N-1} I \end{array}\right],$$

where

$$\overline{\Delta t}_i = \frac{\Delta t_i}{\Delta t_{\min}} \quad , \quad i = 2 \ , \ N-1,$$

Δt_i is the largest local time step allowed by stability, and I is a 3×3 identity matrix. Such a process results in a significant speed up in the transport of information and gives roughly a factor of two increase in the convergence rate of explicit schemes.

7.2 Residual Smoothing

The local stability range of the basic time-stepping scheme can be extended by applying a procedure called implicit residual smoothing. This technique was first introduced by Lerat [53] for the Lax-Wendroff scheme, and later devised by Jameson [33] for Runge-Kutta schemes. The constant coefficient approach of Jameson is discussed initially in this section. Some basic properties of the residual smoothing are also presented. Then, we discuss variable coefficients for implicit smoothing. The coefficients introduced by the present authors, as well as those of Martinelli [67], are derived and compared. In the final part of this section, we develop coefficients for the implicit residual smoothing that allow a time-step estimate independent of a physical diffusion limit.

7.2.1 Constant Coefficients

The constant coefficient implicit residual averaging of Jameson [33] can be applied in two dimensions using the factored form

$$(1 - \beta_\xi \nabla_\xi \Delta_\xi)(1 - \beta_\eta \nabla_\eta \Delta_\eta)\bar{\mathcal{R}}_{i,j}^{(m)} = \mathcal{R}_{i,j}^{(m)}, \tag{7.2.1}$$

where the quantity $\nabla\Delta$ is a standard second-difference operator; and thus,

$$\nabla_\xi \Delta_\xi \bar{\mathcal{R}}_{i,j}^{(m)} = \bar{\mathcal{R}}_{i-1,j}^{(m)} - 2\bar{\mathcal{R}}_{i,j}^{(m)} + \bar{\mathcal{R}}_{i+1,j}^{(m)}.$$

The quantity β is a smoothing coefficient, and (ξ, η) are the coordinates of a uniformly spaced computational domain. The residual of the unsmoothed scheme $\mathcal{R}_{i,j}^{(m)}$ is defined by

$$\mathcal{R}_{i,j}^{(m)} = \alpha_m \frac{\Delta t_{i,j}}{\Omega_{i,j}}[\mathcal{L}_C \mathbf{W}_{i,j}^{(m-1)} + \mathcal{L}_D \mathbf{W}_{i,j}^{(0)} + \mathbf{AD}^{(m)}], \qquad m = 1, 5 \tag{7.2.2}$$

and computed in the Runge-Kutta stage m, and $\mathbf{AD}^{(m)}$ is the total artificial dissipation at stage m, and $\bar{\mathcal{R}}_{i,j}^{(m)}$ is the final residual at stage m after the sequence of smoothings in the ξ and η directions. A tridiagonal system of equations is solved for each coordinate direction to obtain the unknown residuals $\bar{\mathcal{R}}_{i,j}^{(m)}$. To determine β_ξ and β_η, Jameson considers as a model problem equation (6.1.5) without the numerical dissipation. Then the semidiscrete equation (6.1.6) simply becomes

$$\Delta t \frac{dw}{dt} = -\Delta t \mathcal{L} w_j,$$

with

$$\Delta t \mathcal{L} w_j = \frac{N}{2}(w_{j+1}^n - w_{j-1}^n),$$

and the Fourier symbol of the difference operator $-\Delta t \mathcal{L}$ is given by

$$z = -\Delta t \hat{\mathcal{L}} = -\mathrm{i} N \sin\theta. \tag{7.2.3}$$

Let $\Delta \overline{w}_j$ define the correction or residual obtained from the implicit smoothing procedure. Then,

$$(1 - \beta \nabla \Delta)\Delta \overline{w}_j = \Delta w_j, \tag{7.2.4}$$

and the Fourier symbol of (7.2.3) is replaced by

$$z = -\mathrm{i} \frac{N \sin\theta}{\left(1 + 4\beta \sin^2 \frac{\theta}{2}\right)}. \tag{7.2.5}$$

A sufficient condition for stability is as follows

$$\max\ |z| \le N^* \qquad \text{for all } \theta, \tag{7.2.6}$$

and N^* is the Courant number of the unsmoothed scheme. Solving for $\sin\theta$ and $\cos\theta$ corresponding to the maximum of $|z|$ we obtain

$$\sin\theta = \frac{\sqrt{1 + 4\beta}}{(1 + 2\beta)}, \qquad \cos\theta = \frac{2\beta}{(1 + 2\beta)}. \tag{7.2.7}$$

Using (7.2.7) and the sufficient condition of (7.2.6) the smoothing coefficient is determined by

$$\beta \geq \frac{1}{4}\left[\left(\frac{N}{N^*}\right)^2 - 1\right] \quad \text{or} \quad \beta \geq \frac{1}{4}\left[\left(\frac{\Delta t}{\Delta t^*}\right)^2 - 1\right], \tag{7.2.8}$$

where Δt^* is the time step of the unsmoothed scheme. In subsequent discussion, this β will be referred to as the one-dimensional (1-D) smoothing coefficient and will be designated by $\beta_{1-\mathrm{D}}$.

Suppose we instead consider the diffusion equation

$$\frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2}.$$

If once again the spatial derivative is approximated with a central difference and a Fourier transform is taken of the resulting semidiscrete equation, the Fourier symbol of the product of Δt and the difference operator is given by

$$z = -N_D \sin^2 \frac{\theta}{2},$$

where the diffusion number $N_D = 4\Delta t \mu / \Delta x^2$. If we apply the residual smoothing operator of (7.2.4), then

$$z = \frac{-N_D \sin^2 \frac{\theta}{2}}{(1 + 4\beta_D \sin^2 \frac{\theta}{2})}. \tag{7.2.9}$$

As discussed previously, a sufficient stability condition for a R-K scheme is

$$\max |z| \leq N_D^* \qquad \text{for all } \theta,$$

where N_D^* is the diffusion number of the unsmoothed scheme. In the same manner as for the convection equation, we can show that

$$\beta_D \geq \frac{1}{4}\left[\frac{N_D}{N_D^*} - 1\right] = \frac{1}{4}\left[\frac{\Delta t_D}{\Delta t_D^*} - 1\right]. \tag{7.2.10}$$

For the scalar diffusion equation the smoothing coefficient β_D is proportional to $\mu \Delta t / (\Delta x)^2$. As will be shown this type of β can be combined with the type of β given by (7.2.8) to yield a formulation suitable for a convection-diffusion equation.

Now we examine some properties of the implicit residual smoothing. If the residual smoothing is applied on each stage of a R-K scheme, the stability function given by (6.1.8) still applies, with the z of the original (basic) scheme modified as in (7.2.5) or (7.2.9). This stability behavior leads to the following theorem:

Theorem 7.1 *Let $\hat{\mathcal{L}}$ be the Fourier symbol of any discrete spatial operator for the convection-diffusion equation. Let (6.1.8) be the stability polynomial for an explicit m-stage R-K scheme. Apply implicit residual smoothing, as in (7.2.4), after every stage of the R-K scheme.*

If the original scheme is unconditionally unstable, then the smoothed scheme is also unconditionally unstable. If the original scheme is conditionally stable, then the smoothed scheme can be made unconditionally stable by choosing $\beta \Delta t$ sufficiently large.

Proof. Let z and z_s be the symbols of the original and smoothed schemes, respectively. Then,

$$z_s = \frac{z}{(1 + 4\beta \sin^2 \frac{\theta}{2})}. \tag{7.2.11}$$

Define $\mathbf{r}$ to be the position vector corresponding to z. Thus, $\mathbf{r}$ emanates from the origin of the complex domain and has magnitude $|z|$. Let $\mathbf{r_s}$ be the position vector associated with z_s. If the original R-K scheme is stable, then $\mathbf{r}$ does not terminate outside the stability region $\mathcal{S}$ determined by the von Neumann condition $|g(z)| \leq 1$, where $g(z)$ is the stability polynomial. If the original scheme is unstable, then there is no $\Delta t > 0$ that is small enough to allow $\mathbf{r}$ to be in $\mathcal{S}$. Since the denominator of (7.2.11) merely acts as a scaling factor of $\mathbf{r}$, the residual smoothing cannot stabilize the unstable original scheme.

Suppose β is proportional to Δt, as in (7.2.8) or (7.2.10). Then, $\mathbf{r_s}$ does not terminate outside the boundary of $\mathcal{S}$ for any value of Δt, since *beta* can always be made sufficiently large. Moreover, the scheme is unconditionally stable.

Remark 7.2 *Even though the explicit R-K scheme can be made unconditionally stable with the implicit residual smoothing, there is a practical limit on the time step when solving the hyperbolic problem and taking $\beta \propto \Delta t^2$, as in (7.2.8). That is, if Δt is too large convergence slows down.*

Lemma 7.3 *Apply an explicit m-stage R-K scheme with implicit residual smoothing to the scalar equation*

$$\frac{\partial w}{\partial t} + \varepsilon^{(4)} \Delta x^3 \frac{\partial^4 w}{\partial x^4} = 0, \tag{7.2.12}$$

where $\varepsilon^{(4)}$ is a constant coefficient. Assume that the residual smoothing coefficient β is proportional to N^2. Then the symbol of the smoothed scheme vanishes as $\Delta t \to \infty$, and the stability polynomial $g(z) \to 1$.

Proof. The symbol for the difference operator of (7.2.12) when implicit residual smoothing is applied is given by

$$z_s = \frac{-16\varepsilon^{(4)} N \sin^4 \frac{\theta}{2}}{(1 + 4\beta \sin^2 \frac{\theta}{2})}.$$

Using $\beta \sim N^2$ and taking Δt to be large, then

$$z_s \sim -\frac{4}{N} \varepsilon^{(4)} \sin^2 \frac{\theta}{2}.$$

Therefore, $z_s \to 0$ as $\Delta t \to \infty$. From (6.1.8) it follows immediately that $g(z) \to 1$ as $z_s \to 0$.

Remark 7.4 *As evident from Lemma 7.3, the limit on the extension of stability with the implicit smoothing and (7.2.8) is caused by the requirement to have a certain background dissipation (i.e., high-frequency damping). If $\beta \sim N$, as for the parabolic problem, then the symbol z_s does not vanish.*

The use of constant coefficients, as determined by (7.2.8), in the implicit treatment has proven to be satisfactory (extending the Courant number by a factor of two to three) even for the highly stretched meshes of viscous flow computations [95], provided additional support such as enthalpy damping [32] is introduced. However, the use of enthalpy damping, which assumes constant total enthalpy throughout the flow field, precludes the solution of problems with heat transfer effects. By using variable coefficients β_ξ and β_η, which account for the variation in mesh cell aspect ratio, residual smoothing can be applied without the support of enthalpy damping.

7.2.2 Variable Coefficients

By considering the Alternating Direction Implicit (ADI) scheme and the implicit scheme of Lerat [54], one can see a functional dependence of variable smoothing coefficients on the characteristic speeds λ_ξ and λ_η, as defined in Section 4.2. In Appendix B we show this dependence. Then, with a 2-D stability analysis similar to the 1-D analysis of the previous subsection, we obtain variable smoothing coefficients

$$\begin{aligned}\beta_\xi &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}}\right)^2-1\right],0\right\},\\ \beta_\eta &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}^{-1}}\right)^2-1\right],0\right\},\end{aligned}\tag{7.2.13}$$

where again $r_{\eta\xi}=\lambda_\eta/\lambda_x i$. In the limiting cases, we have

$$\beta_\xi \to \beta_{1-\mathrm{D}}, \qquad \beta_\eta \to 0 \quad \text{as} \quad r_{\eta\xi}\to 0$$

and

$$\beta_\xi \to 0, \qquad \beta_\eta \to \beta_{1-\mathrm{D}}, \quad \text{as} \quad r_{\eta\xi}\to \infty$$

There is a problem with the smoothing coefficients of (7.2.13). In the typical case of $N/N^*=2$, the smoothing coefficients vanish when $r_{\eta\xi}=1$, making the scheme unstable. In [67] Martinelli eliminates this difficulty by modifying the residual smoothing coefficients of (7.2.13) as follows:

$$\begin{aligned}\beta_\xi &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{\phi(r_{\eta\xi})}{1+r_{\eta\xi}}\right)^2-1\right],0\right\},\\ \beta_\eta &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{\phi(r_{\eta\xi}^{-1})}{1+r_{\eta\xi}^{-1}}\right)^2-1\right],0\right\},\end{aligned}\tag{7.2.14}$$

where

$$\phi(r)=1+r^\zeta,\tag{7.2.15}$$

and the exponent ζ is taken to be $\frac{2}{3}$. Other values for the exponent ζ, such as $\frac{1}{2}$, have been used ([77]). We note that on most machines calculating a square root ($\zeta=\frac{1}{2}$) is considerably faster than calculating an exponent. In tests of the full three-dimensional viscous multi-block code *TLNS3D* choosing the square root for the residual smoothing and artificial dissipation instead of the exponent saved about 5% of the entire running time. The function ϕ is the same one used in scaling the artificial dissipation coefficients. The introduction of this function seems appropriate due to the direct relationship between residual smoothing and artificial dissipation. For example, a desired high-frequency damping behavior of the scheme can be maintained when the dissipation is increased by increasing the residual smoothing.

The variable smoothing coefficients β_ξ and β_η of (7.2.14) cannot be uniquely determined from a sufficient condition for stability, as the constant coefficient β was in (7.2.8). In [123] Wigton and Swanson use an additional constraint to derive the parameters of (7.2.14). For the sake of completeness we repeat that short derivation here.

Consider the sufficient condition for stability

$$\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}}\,\frac{1}{\sqrt{1+4\beta_\xi}}+\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}^{-1}}\,\frac{1}{\sqrt{1+4\beta_\eta}}\leq 1, \tag{7.2.16}$$

as derived in Appendix B. Let the Courant numbers for the two coordinate directions ξ and η be given by

$$\begin{aligned} N_\xi &= \frac{\Delta t_{\text{act}}}{\Delta t_\xi} &= N\frac{\lambda_\xi}{(\lambda_\xi+\lambda_\eta)} &= \frac{N}{1+r_{\eta\xi}}, \\ N_\eta &= \frac{\Delta t_{\text{act}}}{\Delta t_\eta} &= N\frac{\lambda_\eta}{(\lambda_\xi+\lambda_\eta)} &= \frac{N}{1+r_{\eta\xi}^{-1}}, \end{aligned} \tag{7.2.17}$$

where Δt_{act} is the 2-D allowable time step for convection, and Δt_ξ, Δt_η are the corresponding 1-D time steps. If we substitute for the Courant Number N in (7.2.16), according to (7.2.17), we obtain

$$\frac{N_\xi}{N^*}\,\frac{1}{\sqrt{1+4\beta_\xi}}+\frac{N_\eta}{N^*}\,\frac{1}{\sqrt{1+4\beta_\eta}}\leq 1. \tag{7.2.18}$$

As suggested by Lemma 7.3, one wants β_ξ and β_η to be as small as possible and still maintain stability. With this objective in mind, $\beta_\xi+\beta_\eta$ is minimized subject to equality in (7.2.18). This is accomplished by applying the method of Lagrange multipliers and considering the function

$$F(\beta_\xi,\beta_\eta)=\beta_\xi+\beta_\eta+\omega\left(\frac{N_\xi}{N^*}\,\frac{1}{\sqrt{1+4\beta_\xi}}+\frac{N_\eta}{N^*}\,\frac{1}{\sqrt{1+4\beta_\eta}}\right).$$

After equating the partial derivatives of $F(\beta_\xi,\beta_\eta)$ with respect to β_ξ and β_η to zero, we find

$$N_\xi\,\frac{1}{(1+4\beta_\xi)^{3/2}}=N_\eta\,\frac{1}{(1+4\beta_\eta)^{3/2}}. \tag{7.2.19}$$

If we now solve for $\sqrt{1+4\beta_\eta}$ and substitute the resulting expression into (7.2.16), we get that

$$\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}}\,\frac{1}{\sqrt{1+4\beta_\xi}}\left[1+\frac{1+r_{\eta\xi}}{1+r_{\eta\xi}^{-1}}\left(\frac{N_\eta}{N_\xi}\right)^{-1/3}\right]\leq 1, \tag{7.2.20}$$

With the equality of (7.2.20) holding and using (7.2.17), one obtains the smoothing coefficients of (7.2.14). As pointed out in [31], the function ϕ arises without any consideration of numerical dissipation terms. It is important to recognize that the role of the ϕ function is just to connect the values of a β corresponding to low-aspect-ratio and high-aspect-ratio cells.

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Figure 9a: Variation of implicit residual smoothing coefficient β_ξ with mesh cell aspect ratio $r_{\eta\xi}$ for two forms of variable coefficients.

The variation of β_ξ from (7.2.14) with $r_{\eta\xi}$ is shown in Figure 9a as a solid line. For this curve the ratio of Courant numbers N/N^* is assumed to be two. One observes that $\beta_\xi \to \beta_{1-\mathrm{D}}$ for $r_{\eta\xi} \to 0$ and $\beta_\xi \to 0$ for $r_{\eta\xi} \to \infty$. In the case of $r_{\eta\xi} = 1$, $\beta_\xi = \beta_\eta = \beta_{1-\mathrm{D}}$ (a value of 0.75 when $N/N^* = 2$). Based upon numerical calculations for inviscid flows using typical inviscid meshes, smoothing coefficients β_ξ and β_η that are constant with a value of about 0.4 result in rapid convergence. Values for $\beta_\xi \geq 0.75$ and/or $\beta_\eta \geq 0.75$ can cause a significant slowdown in convergence. In order to provide improved smoothing coefficients when $r_{\eta\xi} \approx 1$ the formulas of (7.2.14) can be replaced with

$$\begin{aligned} \beta_\xi &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{1}{1+\psi\, r_{\eta\xi}}\right)^2 - 1\right], 0\right\}, \\ \beta_\eta &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\,\frac{1}{1+\psi\, r_{\eta\xi}^{-1}}\right)^2 - 1\right], 0\right\}, \end{aligned} \tag{7.2.21}$$

where ψ is a parameter to be specified. Here we have simply removed the connection function ϕ by introducing ψ. The curve representing β_ξ when $N/N^* = 2$ and $\psi = 0.25$ is given in Figure 9a. For the case of $r_{\eta\xi} = 1$, β_ξ is 0.39.

In order to investigate stability using the smoothing coefficients of (7.2.21), consider the sufficient stability condition

$$\frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}}\,\frac{\sin\theta_\xi}{\chi_\xi\chi_\eta} + \frac{N}{N^*}\,\frac{1}{1+r_{\eta\xi}^{-1}}\,\frac{\sin\theta_\eta}{\chi_\xi\chi_\eta} \le 1, \tag{7.2.22}$$

for all θ_ξ and θ_η. In (7.2.22)

$$\begin{aligned}\chi_\xi &= 1 + 2\beta_\xi\left(1-\cos\theta_\xi\right),\\ \chi_\eta &= 1 + 2\beta_\eta\left(1-\cos\theta_\eta\right),\end{aligned} \tag{7.2.23}$$

and $r_{\eta\xi} = \lambda_\eta/\lambda_\xi$. This condition comes from the 2-D stability analysis that is given in Appendix B (see (B.18)). If $r_{\eta\xi} \ll 1$, the condition of (7.2.22) reduces to approximately

$$\frac{N}{N^*}\,\frac{\sin\theta_\xi}{\chi_\xi} \le 1 \qquad \text{for all } \theta_\xi. \tag{7.2.24}$$

Using (7.2.7) we obtain

$$\frac{N}{N^*}\,\frac{1}{\sqrt{1+4\beta_\xi}} \le 1. \tag{7.2.25}$$

Substituting for β_ξ according to (7.2.21), the inequality (7.2.25) is satisfied. Now, if $r_{\eta\xi} \gg 1$, the condition of (7.2.22) reduces to approximately

$$\frac{N}{N^*}\,\frac{\sin\theta_\eta}{\chi_\eta} \le 1 \qquad \text{for all } \theta_\eta. \tag{7.2.26}$$

¿From (7.2.7) and the definition of β_η in (7.2.21), the inequality (7.2.26) is satisfied. Consider the case of $r_{\eta\xi} = 1$. If we assume $\theta_\xi = \theta_\eta$ and $\beta_\xi = \beta_\eta$, then (7.2.22) becomes

$$\tilde{F} \le 1 \qquad \text{for all } \theta_\xi, \tag{7.2.27}$$

where

$$\tilde{F} = \frac{N}{N^*}\,\frac{\sin\theta_\xi}{{\chi_\xi}^2}.$$

It can be shown that

$$\tilde{F}_{max} \le \frac{N}{N^*}\left[\frac{\left(1+8\beta_\xi\right)^5}{\left(1+4\beta_\xi\right)\left(1+10\beta_\xi\right)^4}\right]^{\frac{1}{2}}\,\frac{1}{\sqrt{1+4\beta_\xi}}.$$

Then,

$$\tilde{F}_{max} \le 0.9\frac{N}{N^*}\,\frac{1}{\sqrt{1+4\beta_\xi}}. \tag{7.2.28}$$

Substituting for β_ξ and taking $\psi \approx 0.11$, the condition of (7.2.27) is satisfied. From numerical experiments this estimate for ψ seems to be conservative. A value for ψ of 0.125 has worked well for both central and upwind schemes. Moreover, for central schemes $\psi = 0.25$ also works without causing stability problems.

To evaluate the variable smoothing coefficients of (7.2.14) and (7.2.21), calculations were performed for a case of inviscid transonic flow over an RAE 2822 airfoil. A typical inviscid mesh with 224×32 cells was used. Figure 9b shows the convergence histories corresponding to the formulations of Martinelli (7.2.14) and Swanson (7.2.21). Convergence is measured by the logarithm of the root-mean-square of the residual of the continuity equation. For each computation the basic explicit scheme of (6.1.3) and a multigrid method, which will be described in Section 7.3, were employed. The average rate of reduction of the residual was defined by

$$R_f = (\text{rate})^N R_i,$$

where R is the residual for the continuity equation, the subscripts f and i mean final and initial values, respectively, and N denotes the number of multigrid cycles. With the coefficients of (7.2.14) the average rate of residual reduction is 0.889, while with the coefficients of (7.2.21) it is 0.789. As one might expect, the two formulations exhibit only small differences in convergence behavior in the case of turbulent flow calculations, since the high-aspect-ratio cells of the mesh usually defined to resolve the boundary layer determine the convergence rate.

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Figure 9b: Residual histories for two forms of variable coefficients for implicit residual smoothing (RAE 2822 airfoil, $M_\infty = 0.75$, $\alpha = 3°$, 224 × 32 grid).

7.2.3 Removal of Diffusion Limit

The diffusion restriction on the time step (see (6.3.1) can be a significant factor in viscous regions of a flow field, causing excessive restrictions on the allowable time step Δt. We now show how one can utilize the diffusion-based smoothing coefficient (7.2.10) to construct a new smoothing parameter that allows the removal of this restriction.

Suppose we consider the thin-layer form of the 2-D Navier-Stokes equations, then one could simply use the smoothing coefficient of (7.2.21) in the streamwise-like (ξ) direction. One would anticipate that a possible formulation for the normal η direction would depend on a diffusion type β near the surface and a convection type β when the viscous effects are no longer important. Consider the dependency in (7.2.10) on the ratio of the actual Δt to the Δt of the basic explicit scheme. Since we want to remove the diffusion limit on the time step, the actual time step must be independent of diffusion effects. Thus, we set $N_D = 0$ in (6.3.2), giving

$$\Delta t = \Delta t_{\text{act}} = \frac{N\,\Omega}{\lambda_\xi + \lambda_\eta}, \tag{7.2.29}$$

where Ω is the area of the mesh cell being considered. In the part of the boundary layer where diffusion effects dominate, we define the time step of the unsmoothed scheme (Δt^*) by

$$\Delta t^* = (\Delta t_D)_\eta = \frac{N_D\,\Omega}{(\lambda_D)_\eta}, \tag{7.2.30}$$

where

$$(\lambda_D)_\eta = \frac{\sqrt{\gamma} M}{Re}\frac{\gamma\mu}{\rho Pr}\Omega^{-1}(x_\xi^2 + y_\xi^2).$$

Then, we define

$$(\beta_D)_\eta = \frac{1}{4}\left[\frac{\Delta t_{\text{act}}}{(\Delta t_D)_\eta} - 1\right]. \tag{7.2.31}$$

If $(\Delta t_D)_\eta = \Delta t_{\text{act}}$, $\beta_D = 0$. This means that the full parabolic stability limit is being used for the physical diffusion terms. Since the numerical dissipation of the scheme is not included in the analysis, we replace (7.2.31) by

$$(\beta_D)_\eta = \frac{1}{4}\left[C_1\frac{\Delta t_{\text{act}}}{(\Delta t_D)_\eta} - 1\right], \tag{7.2.32}$$

where C_1 is a constant. In so doing we can also take into account any possible influence on the stability due to a single evaluation of the physical viscous terms in the multistage time-stepping scheme. Using (7.2.29) and (7.2.30) to replace Δt_{act} and $(\Delta t_D)_\eta$, respectively, equation (7.2.32) can be rewritten as

$$(\beta_D)_\eta = \frac{1}{4}\left[C_1\frac{N}{N_D}\frac{(\lambda_D)_\eta}{\lambda_\xi + \lambda_\eta} - 1\right], \tag{7.2.33}$$

which we can approximate simply as

$$(\beta_D)_\eta = \frac{1}{4}\bar{C}_1\frac{(\lambda_D)_\eta}{\lambda_\xi + \lambda_\eta}. \tag{7.2.34}$$

It does not appear to matter whether one uses the form of (7.2.33) or that of (7.2.34), provided the constant is defined properly. Both forms successfully remove the diffusion restriction on the time step. Here we have elected to use the simpler form of (7.2.34), which has also been considered by Radespiel and Kroll [80]. From numerical experiments we found that a satisfactory value for $\bar{C}_1$ is 5.

For the full Navier-Stokes equations (including all viscous terms), one needs to also define a coefficient β_D for the streamwise-like direction (ξ). We use the form of $(\beta_D)_\eta$ in (7.2.34) and define $(\beta_D)_\xi$ as

$$(\beta_D)_\xi = \frac{1}{4}\bar{C}_1 \frac{(\lambda_D)_\xi}{\lambda_\xi + \lambda_\eta}, \tag{7.2.35}$$

where

$$(\lambda_D)_\xi = \frac{\sqrt{\gamma}M}{Re}\frac{\gamma\mu}{\rho Pr}\Omega^{-1}(x_\eta^2 + y_\eta^2).$$

The variable coefficient of (7.2.34) cannot generally be used by itself. For example, in an airfoil flow, $(\beta_D)_\eta$ goes to zero too fast at the leading edge, resulting in a zero value in the inviscid region. We have overcome this difficulty by calculating β_η as

$$\beta_\eta = \max[(\beta_D)_\eta, (\beta_C)_\eta],$$

where $(\beta_C)_\eta$ is defined by (7.2.21). In a similar manner we redefine $(\beta_D)_\xi$ in (7.2.35).

According to the theory presented in [96] one need evaluate the residual smoothing only on the even steps of a Runge-Kutta time scheme. In practice one frequently evaluates the residual smoothing every stage. This is more expensive but produces a more robust algorithm.

7.2.4 Extensions to Residual Smoothing

The one-dimensional residual that we considered until now was expressed as

$$(1 - \beta\nabla\Delta)\bar{\mathcal{R}}_i = \mathcal{R}_i.$$

This requires the inversion of a tridiagonal matrix which is accomplished by the standard *LU* decomposition. This requires 2N operations for the forward sweep and another 2N operations for the backward sweep. If the same parameters are used for each equation in the system, then the same forward parameters can be used for each equation. Hence, for m equations we require $2(m+1)$ operations. Zhu, Lacor and Hirsch [126] noted that we can replace this by an explicit *LU* form

$$(1 + \beta_f\nabla)(1 - \beta_b\Delta)\bar{\mathcal{R}}_i = \mathcal{R}_i.$$

This can be solved as

$$\begin{aligned} -\beta_f\widetilde{\mathcal{R}}_{i-1} + (1+\beta_f)\widetilde{\mathcal{R}}_i &= \mathcal{R}_i, \\ (1+\beta_b)\bar{\mathcal{R}}_i - \beta_b\bar{\mathcal{R}}_{i+1} &= \widetilde{\mathcal{R}}_i \end{aligned}$$

The two formulations are identical if $\beta_b = \beta_f$ and $\beta = \beta_f(1 + \beta_f)$. If β is given, then $\beta_f = \sqrt{1 + 4\beta}$. Let σ^* be the Courant number of the original scheme, and let σ be the Courant number of the modified scheme. Then for central differences the scheme is stable if

$$\frac{\sigma}{\sigma^*} \le 1 + \beta_f + \beta_b.$$

If β is given arbitrarily, then we require a square root to calculate β_f. However, as indicated in the previous section, we usually choose β based on the Courant number σ we wish (i.e., $\beta_f + \beta_b = (\sigma/\sigma^*) - 1$). Thus, we have a nice formula for the sum, but the central formula $\beta_b = \beta_f$ requires a square root.

Zhu et al. allow $\beta_b \ne \beta_f$ in order to change the stability properties. Similar work was done by Blazek [12]. Thus, for example, choosing $\beta_b = 0$ or $\beta_f = 0$ allows for an upwind residual smoothing (see also [11]). For a system of equations this requires a characteristic decomposition, and it is not clear if the gains achieved justify the additional expense. A simpler strategy is to choose the parameters to depend on the sign of the local velocity or Mach number ([126]) similar to the *AUSM* scheme. Thus, one possibility, similar to that suggested by Zhu et al., is

$$\begin{aligned}
\beta_f &= \overline{\beta}\min\left[\max(\frac{1+M}{2}, 0), 1\right], \\
\beta_b &= \overline{\beta}\min\left[\max(\frac{1-M}{2}, 0), 1\right],
\end{aligned}$$

where M is a signed Mach number in the coordinate direction. Then the residual smoothing is central only at $M = 0$. The parameter $\overline{\beta}$ is chosen to enlarge the time step and can be a function of the aspect ratio as described above. Then, β_f is zero for supersonic flow to the left, and β_b is zero for supersonic flow to the right. Since $\beta_f + \beta_b = \overline{\beta}$, the time step depends only on $\overline{\beta}$. Thus, we can choose $\overline{\beta} = (\sigma/\sigma^*) - 1$.

A different extension is to modify the right hand side of (7.2.1) in addition to modifying the left hand side. For example, we modify the explicit portion as well as the implicit portion (see [21] and [22]). Then,

$$(1 - \beta_\xi \nabla_\xi \Delta_\xi)(1 - \beta_\eta \nabla_\eta \Delta_\eta)\bar{\mathcal{R}}_{i,j}^{(m)} = (1 - \delta_\xi \nabla_\xi \Delta_\xi - \delta_\eta \nabla_\eta \Delta_\eta + \gamma \nabla_\xi \Delta_\xi \nabla_\eta \Delta_\eta)\mathcal{R}_{i,j}^{(m)}.$$

One can the choose $\delta_\xi, \delta_\eta, \gamma$ to improve the properties of the symbol, especially at higher frequencies. Let $\alpha = \sigma/\sigma^*$. The suggestion of Enander et al. for stretched grids is

$$\begin{aligned}
\beta_\xi &= \max\left\{\frac{1}{4}\left[\left(\alpha\frac{\phi(r_{\eta\xi})}{1 + r_{\eta\xi}}\right)^2 - 1\right], 0\right\}, \\
\beta_\eta &= \max\left\{\frac{1}{4}\left[\left(\alpha\frac{\phi(r_{\eta\xi}^{-1})}{1 + r_{\eta\xi}^{-1}}\right)^2 - 1\right], 0\right\}, \\
\delta_\xi &= \frac{\left(\frac{1+4\beta_\xi}{\alpha} - 1\right)}{4}, \qquad (7.2.36)
\end{aligned}$$

$$\begin{aligned}
\delta_\eta &= \frac{\left(\frac{1+4\beta_\eta}{\alpha} - 1\right)}{4}, \\
\gamma &= \frac{\left(\frac{\left(1+4\beta_\xi\right)(1+\beta_\eta)}{\alpha} - 4(\delta_\xi + \delta_\eta) - 1\right)}{16}.
\end{aligned}$$

For inviscid subsonic flow about a bump in a channel Enander and Karlsson [22] found a speed up of about 35% in CPU time. They obtained a smaller increase in efficiency for the transonic case. However, the explicit portion adds considerably to the cost of the residual smoothing, and it must be demonstrated that the improvements justify the cost for more difficult problems including viscous effects. One should be able to combine both of these approaches, choose β_f and β_b as functions of the Mach number, and also add a right-hand-side operator acting on the residual with parameters that again will be functions of the Mach number.

The analysis of these extensions is based on a single hyperbolic equation. It is also necessary to include viscous effects, as we have done for the standard implicit residual method. When we discuss preconditioners, we shall also discuss an extension of residual smoothing that applies directly to the Euler equations rather than to a scalar equation. Additional analysis of residual smoothing for equalizing the wave speeds is presented in Part II.

7.3 Multigrid Method

The concept of multigrid acceleration of an iterative scheme was first suggested by Fedorenko [24]. The fundamental ideas of this approach that are currently used in many applications are principally due to [13]. Although most of the theory developed for the multigrid method is for elliptic problems, a number of effective multigrid solvers (i.e., [35, 37, 71]) have been constructed for the Euler equations of gas dynamics, which are hyperbolic. Transonic as well as subsonic flows have been computed with these solvers. Some multigrid methods (i.e., [97, 68, 77, 119]) have also been devised for the numerical solution of the compressible Navier-Stokes equations. In the initial part of this section, the basic theory of the multigrid process is briefly reviewed. Then, the operators used in the present method are defined. This section is concluded with a discussion of the various elements of the multigrid technique of this work.

7.3.1 Basic Concepts

In the simplest sense the multigrid method involves applying a sequence of grids to solve a discrete problem. More specifically, a faster rate of development of the solution on a fine grid is achieved by approximating the fine-grid problem on successively coarser grids in the sequence. With suitable coarse-grid approximations of the fine-grid problem, the low-frequency error components on the fine grid appear as high-frequency error components on the coarser grids. The low-frequency components on the fine grid, where the discrete solution is desired, are precisely the error components that dramatically slow down the convergence of single-grid schemes. Thus, with a good high-frequency damping scheme, an effective multigrid process (i.e., much more rapid removal of low-frequency errors than a single-grid scheme) can be constructed. As will become evident, the driving scheme for the multigrid process is not only important in providing smoothing on each grid but also in removing high-frequency errors resulting from interpolation of corrections for the fine-grid approximation.

There are two additional advantages derived from displacing part of the effort in solving a set of discrete equations to coarse grids. One advantage is that the larger mesh spacing permits larger time steps. This means that information is propagated rapidly in the domain of interest. Moreover, for explicit time-stepping schemes, such as the multistage schemes described previously, the increased time step is especially important since the allowable time step depends on the speed of sound. This acoustic dependence is even more critical for viscous flows. Another benefit of the coarse grids is that they require less computational work. For example, in two dimensions, the computational effort needed is decreased roughly by a factor of four on successively coarser meshes. So, clearly the objective in the multigrid process is to spend much more time on the coarse grids than on the fine grid.

The basic ideas of the multigrid process are revealed by considering the continuum problem

$$\mathcal{L}W(x,y) = S(x,y), \qquad \Lambda W(x,y) = \Phi(x,y),$$

where the first equation is associated with the domain Ω, and the second equation is associated with its boundary $\partial\Omega$. The symbols $\mathcal{L}$ and Λ are general nonlinear differential operators, and both S and Φ are source terms. Let $G_0, G_1, ..., G_N$ be a set of grids, where G_N is the finest grid and each successively coarser grid $G_k (k \leq N-1)$ is generated by eliminating every other mesh line in each coordinate direction of the next finer mesh. The discrete problem on G_N is as follows:

$$\begin{aligned} \mathcal{L}_N W_N(x,y) &= S_N(x,y) \quad \text{for } x,y \in G_N, \\ \Lambda_N W_N(x,y) &= \Phi_N \quad \text{for } x,y \in \partial G_N, \end{aligned} \tag{7.3.1}$$

and W_N is the exact discrete solution. If $w_N(x,y)$ is an approximate discrete solution, equation (7.3.1) can be written as

$$\mathcal{L}_N w_N = S_N + R_N, \qquad \Lambda_N w_N = \Phi_N + (R_B)_N, \tag{7.3.2}$$

where R_N and $(R_B)_N$ are residual functions. Subtracting (7.3.2) from (7.3.1) the residual equations for the G_N problem are obtained. That is,

$$\mathcal{L}_N W_N - \mathcal{L}_N w_N = -R_N, \qquad \Lambda_N W_N - \Lambda_N w_N = -(R_B)_N.$$

These equations can be adequately approximated on G_{N-1} if the residual functions and corrections $(W_N - w_N)$ are smooth. The smoothing is accomplished by performing an iteration with an effective high-frequency damping scheme. The approximations of the residual equations on the coarser grid G_{N-1} are

$$\begin{aligned} \mathcal{L}_{N-1} W_{N-1} - \mathcal{L}(I_N^{N-1} w_N) &= -I_N^{N-1} R_N, \\ \Lambda_{N-1} W_{N-1} - \Lambda(I_N^{N-1} w_N) &= -I_N^{N-1} (R_B)_N, \end{aligned} \tag{7.3.3}$$

where I_N^{N-1} is a restriction operator. Note that if $R_N = 0$ then $W_{N-1} = I_N^{N-1} w_N$ and so once a steady state is reached on the fine grid all corrections on the coarse grid are zero. Furthermore, for a linear problem we can combine the two terms on the left hand side of (7.3.3) and obtain the error equation $\mathcal{L}(\text{error}) = -I_N^{N-1} R_N$. In general, the operator I_l^m is used to indicate restriction when $l > m$ and prolongation when $l < m$. Thus, a restriction operator transfers information from a fine grid to a coarse grid, and a prolongation operator (i.e., interpolating polynomial) transfers information from a coarse grid to a fine grid. The equations of (7.3.3) can be rewritten as

$$\mathcal{L}_{N-1} w_{N-1} = \bar{S}_{N-1}, \qquad \Lambda_{N-1} w_{N-1} = (\bar{S}_B)_{N-1},$$

where

$$\bar{S}_{N-1} = R_{N-1} + F_{N-1}, \qquad (\bar{S}_B)_{N-1} = (R_B)_{N-1} + (F_B)_{N-1},$$

and

$$\begin{aligned} F_{N-1} &= I_N^{N-1}(-R_N) + \mathcal{L}_{N-1}(I_N^{N-1} w_N), \\ (F_B)_{N-1} &= I_N^{N-1}(-R_B)_N) + \Lambda_{N-1}(I_N^{N-1} w_N) \end{aligned} \tag{7.3.4}$$

Thus, the discrete problem on G_{N-1} has the same form as that on G_N, except the forcing functions of (7.3.4) are added to the residual functions. An improvement to the approximate solution w_N can be obtained by adding a coarse-grid correction. The fine-grid solution is then given by

$$w_N \leftarrow w_N + I_{N-1}^N (w_{N-1} - I_N^{N-1} w_N),$$

where the correction $(w_{N-1} - I_N^{N-1} w_N)$ is an approximation to the smoothed function $W_N - w_N$.

In practice the smoother we choose to solve (7.3.3) is a multistage Runge-Kutta of the type discussed in Section 6. Thus, we add a time derivative to the steady state equations and march with several iterations of the multistage method. Usually one complete Runge-Kutta cycle is performed on the finest mesh and two or three on coarser meshes.

Instead of immediately passing a correction from G_{N-1} to G_N, the solution w_{N-1} and residual R_{N-1} can be restricted to the grid G_{N-2}. Then, iteration sweeps can be performed to obtain a smooth approximation of the correction function $W_{N-2} - w_{N-2}$. If this correction is passed to G_{N-1}, iterations performed, and the correction of G_{N-1} is transferred to G_N, a multigrid cycle of three grids is completed. This cycle is called a V cycle. There are other fixed cycle strategies (i.e., W cycle), as well as variable ones, which depend on a prescribed residual level or a certain slowdown in smoothing rate before changing to a coarser grid problem (see [13]). For each coarse grid G_k the full current approximation w_k and the initial (basic) approximation w_{k+1} (the approximation on grid G_{k+1} that is transferred to grid G_k) are stored. The approximation w_k is the sum of the G_k correction and the basic approximation. Brandt refers to a scheme that stores the full current approximation rather than just the correction as the Full Approximation Storage (FAS) scheme.

7.3.2 Transfer Operators

The inter-grid transfer operators employed in the present multigrid method are the same as those introduced by Jameson [35], assuming that the unknowns are stored at the center of a mesh cell. The restriction operator for the residual is defined by

$$I_{k+1}^{k} R_{k+1} = \frac{1}{\Omega_k} \sum_{l=1}^{4} \, (\Omega_{k+1} R_{k+1})_l, \tag{7.3.5}$$

where the residual function R_{k+1} is expressed in the usual way as

$$R_{k+1} = \frac{1}{\Omega_{k+1}} \mathcal{L}_{k+1} w_{k+1},$$

with $\mathcal{L}_{k+1}$ and Ω_{k+1} denoting the spatial discretization operator and the cell area, respectively, on grid G_{k+1}. Thus, the modified residuals $\mathcal{L}_{k+1} w_{k+1}$ of the four fine-grid cells corresponding to a coarse-grid cell are summed. In this manner, the residual transfer operation is conservative. To transfer the solution from G_{k+1} to G_k the following volume-weighted operator is used:

$$I_{k+1}^{k} w_{k+1} = \frac{\sum_{l=1}^{4} (\Omega_{k+1} \, w_{k+1})_l}{\sum_{l=1}^{4} (\Omega_{k+1})_l}. \tag{7.3.6}$$

Again the summations are over the four fine-grid cells, and the operator conserves mass, momentum, and energy. The prolongation of corrections from G_k to G_{k+1} is accomplished with bilinear interpolation.

In elliptic multigrid methods the residual restriction operator is frequently defined as the adjoint of the correction prolongation operator. This means that one operator is the transpose of the other (see Appendix 7.3.3 for discussion of the adjoint property). Such a relationship can prove convenient in analyzing multigrid schemes [28]. In typical multigrid methods using a cell-vertex, finite-volume formulation for spatial discretization [35],[98], the restriction operator is defined with full weighting [28] and bilinear interpolation is used for the prolongation operator. For full weighting the restriction operator I_{k+1}^{k} is defined by

$$I_{k+1}^{k} (R_{i,j})_{k+1} = 4 \mu_x^2 \mu_y^2 (R_{i,j})_{k+1},$$

where μ is a standard averaging operator, and thus

$$\mu_x R_{i,j} = \frac{1}{2} (R_{i+1/2,j} + R_{i-1/2,j}),$$

and

$$\mu_x^2 R_{i,j} = \frac{1}{4} (R_{i+1,j} + 2R_{i,j} + R_{i-1,j}).$$

These operators are adjoint on a uniformly spaced mesh. The operators of the present scheme do not have the adjoint property. Suppose we replace the bilinear interpolation operator with a piecewise constant prolongation operator. This new prolongation operator transfers the same correction to all fine-grid cells that comprise a coarse-grid cell. Using the inner product definition of functions one can easily show (see Appendix 7.3.3) that this operator is the adjoint of the restriction operator of (7.3.5). However, this type of prolongation is not considered an appropriate choice. That is, if the Navier-Stokes equations are solved, it does not allow the following requirement for the inter-grid transfer operators to be satisfied. The sum of the order m_p of the prolongation operator and the order m_r of the restriction operator needs to exceed the order $2m$ of the differential operator being considered [28]. With the piecewise constant prolongation, $m_p + m_r = 2$. In the case of the bilinear interpolation, $m_p + m_r = 3 > 2m = 2$. Then, the requirement is satisfied. It should be mentioned that it is quite common to choose a restriction operator that is not the adjoint of the prolongation operator.

7.3.3 Adjoint Transfer Operators

In the restriction to coarser meshes we use a volume-weighted average, as given in (7.3.6). One justification for this is based on conservation. Thus, to get the density on a coarse cell, we sum the mass of the fine cells to get the mass of the coarse cell containing the fine cells. Since the mass is equal to the density times the volume, this is equivalent to volume weighting the density in the prolongation. However, this limits the idea of volume weighting to conservation laws. In Section 4.10 we consider the use of other variables besides the conservation variables. However, in all cases we use the volume weighting. Here, we present another justification for volume weighting by analyzing the adjoint operator.

In constructing a multigrid method one must choose appropriate inter-grid transfer (i.e., restriction and prolongation) operators. Often these operators are selected so that they are adjoint operators. Such a choice provides convenience in the analysis of the multigrid scheme (i.e., two-level multigrid analysis). In this section we consider the natural choice for the residual function restriction operator when a cell-centered, finite-volume scheme is used for discretization. Moreover, the restriction process involves simply summing the fine-grid residuals for the fine-grid cells that comprise the coarse-grid cell. We show that a piecewise constant prolongation operator is the adjoint operator.

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Figure 10: Cells of two grid levels.

Consider a one-dimensional domain $\Omega = \{x \in \Re : \ 0 \leq x \leq L\}$. Define a fine grid G_f and a coarse grid G_c that cover the domain Ω such that $G_c \subset G_f$. Generate G_c by eliminating every other mesh point of G_f, which are delineated by crossed lines in Figure 10. Let the mesh interval $(\Delta x_j)_f = \left(x_{j+1/2} - x_{j-1/2}\right)_f$ of G_f be constant. Define $h = h_f = (\Delta x_j)_f$. Then the coarse-grid mesh interval is $h_c = 2h$. Let R be the residual function, and let $(v)_h$

be a correction to the fine-grid solution. Assume that the unknowns are stored at the center of a mesh cell. The restriction operator for the residual is defined by

$$I_h^{2h} R_h = \frac{1}{h_c} \sum_{l=1}^{2} \ (h_f R_f)_l,$$

Suppose that the prolongation operator I_{2h}^h for the coarse-grid correction simply transfers the same correction to the fine-grid cells that determine the coarse-grid cell. The operators I_h^{2h} and I_{2h}^h are adjoint operators if

$$(R_h \, , \, I_{2h}^h v_{2h}) = ((I_{2h}^h)^* R_h \, , \, v_{2h}) = (I_h^{2h} R_h \, , \, v_{2h}), \tag{7.3.7}$$

where $(\cdot \, , \, \cdot)$ denotes an inner product and the asterisk indicates transpose. In order to show that these operators satisfy (7.3.7), we consider the inner productive definition for functions. Let the index k denote a coarse-grid cell, and let the indices j and $j-1$ represent the corresponding fine-grid cells. Then,

$$(I_h^{2h} R_h \, , \, v_{2h})_{2h} = \sum_k \left(\frac{h_j}{h_k} \, (R_j)_h + \frac{h_{j-1}}{h_k} \, (R_{j-1})_h \right) (v_k)_{2h} \, h_k,$$

and

$$\begin{aligned} (R_h \, , \, I_{2h}^h v_{2h})_h \quad &= \quad \sum_{j \text{ even}} \ (R_j)_h \, (v_k)_{2h} \, h_j + \sum_{j \text{ even}} \ (R_{j-1})_h \, (v_k)_{2h} \, h_{j-1} \\ &= \quad \sum_{j \text{ even}} \ [(R_j)_h \, h_j + (R_{j-1})_h \, h_{j-1}] \, (v_k)_{2h}. \end{aligned}$$

Thus, we have shown that these operators are adjoint operators. Note that if the piecewise constant prolongation operator is replaced by a linear interpolation operator, the operators are not adjoint.

If one repeats the above argument for nonuniform cells one can show the adjoint of linear prolongation is volume weighted restriction. Thus, if one assumes that the use of adjoint operators for restriction and prolongation is reasonable then one should use a volume weighted restriction. We stress that formal proofs of the useful of adjoint operators exist only for finite element type methods for elliptic equations.

7.3.4 Elements of Present Method

It was shown above that a forcing function is required to properly define a coarse-grid problem for the multigrid method. After initialization of the coarse-grid solution, the forcing term P_k is constructed as

$$P_k = I_{k+1}^k \bar{R}_{k+1} - R_k(I_{k+1}^k w_{k+1}),$$

where $\bar{R}_{k+1}$ is the sum of the residual R_{k+1} and forcing function P_{k+1}, and $0 < k < N$. If $k = N-1$, then $\bar{R}_N = R_N$. In the case of the multistage time-stepping scheme, the $(q+1)$st stage becomes

$$w_k^{(q+1)} = w_k^{(0)} - \alpha_{q+1} \Delta t_k \bar{R}_k,$$

where

$$\begin{aligned}\bar{R}_k &= R_k(w_k^{(q)}) + P_k^{(0)},\\ R_k(w_k^{(q)}) &= \frac{1}{\Omega_k}\left[\mathcal{L}_k^C w_k^{(q)} + \mathcal{L}_k^D w_k^{(0)} - AD^{(q)}\right],\end{aligned}$$

and the superscripts C and D mean that the discrete operators are associated with the convection and physical viscous terms, respectively. The quantity AD represents the appropriate artificial dissipation terms for a given stage. The residuals on G_k are smoothed with a Runge-Kutta scheme. Information is transferred from one grid to another with a fixed cycle strategy.

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Figure 11a: Four-level cycle.

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Figure 11b: Five-level cycle (inside rectangle means four-level cycle without fine grid).

We have considered both V-type and W-type cycles. The structure of these W-type cycles with four and five levels is shown in Figure 11a and Figure 11b. The comparable V-type cycles consist of the first and last legs of these cycles. Although the W-type cycle becomes complex as the number of grids increases, it has a recursive definition. Thus, it is essentially as easy to program as the V-type cycle. The work of these cycles are as follows:

$$\begin{aligned} \text{Work}_{\text{ MG}} &< \frac{4}{3}\ \text{Work}_{\text{ FINE}} \quad \text{for V} - \text{type cycle,} \\ \text{Work}_{\text{ MG}} &< 2\ \text{Work}_{\text{ FINE}} \quad \text{for W} - \text{type cycle,} \end{aligned}$$

the subscript MG indicates multigrid cycle, and the subscript FINE refers to one time step on the finest mesh. The work associated with grid transfer operations has been neglected. At a given grid level additional Runge-Kutta (R-K) steps can be performed in both cycles. In particular, the application of two R-K steps on G_{N-1} and three R-K steps on all successively coarser grids has proven to be an effective strategy. Multiple coarse grid time steps reduce the number of cycles to reach a prescribed level of convergence (i.e., engineering accuracy, which means three to four orders of magnitude of reduction in the residual). However, the computational time required to realize this level is about the same with or without the additional steps. The principal advantage of these multiple iterations is the improved smoothing of residuals, which is especially important for difficult nonlinear flow problems.

Without additional coarse grid sweeps, the W-type cycle generally requires about the same amount of computer time for convergence (engineering accuracy) as the V-type cycle. The advantage of the W-type cycle is that it provides improved robustness. Therefore, for the applications of this paper we elected to use a W-type cycle.

In solving the Navier-Stokes equations the viscous terms are computed on each mesh in the multigrid process rather than just on the finest mesh (i.e., as in the convective coarse-grid correction scheme of Johnson [47]). Computing on each mesh was found to provide improved convergence behavior for low Reynolds number (i.e., $O(1000)$) flow cases. For turbulent flows the viscosity associated with Reynolds stresses is evaluated only on the finest grid and then determined on each successively coarser grid by a simple averaging of surrounding finer grid values. This is done in order to obtain a consistent estimate of the eddy viscosity on coarse meshes when an algebraic turbulence model is being applied. The artificial dissipation model for the finest grid is replaced on coarser grids with a simple constant coefficient second-difference dissipation model. On each grid the boundary conditions are updated at every Runge-Kutta stage.

In describing the multigrid method (Section 7.3.1) we indicated that on coarse grids one constructs approximations to the residual equations at the boundary points (see (7.3.3)). In so doing the solution at the boundary points on a coarse grid is driven by the residuals for the boundary points on the next finer grid. However, for the present cell-centered, finite-volume scheme, such a treatment for the boundary points is not convenient from a computational point of view (i.e., the boundary points do not lie on the boundaries themselves but are located in auxiliary cells outside of the domain). Instead, we apply the fine-grid boundary conditions discussed in Section 5 on the coarse grids. This means that fine-grid accuracy is not maintained at coarse-grid boundary points. Therefore, when transferring coarse-grid corrections to a finer grid, we use only the changes in the solution at the boundary points due to Runge-Kutta time stepping. Although such a procedure can possibly affect the asymptotic convergence rate of the multigrid method, it does not change the fine-grid boundary values if the method converges.

The robustness of the multigrid method is enhanced significantly by smoothing the corrections for the fine grid solution. That is,

$$W^{(n+1)} = W^{(n)} + \Delta W_{\text{tot}},$$

where

$$\Delta W_{\text{tot}} = \Delta W_f + \Delta W_c.$$

The quantity ΔW_f is the solution correction from the finest grid, and ΔW_c is the resultant solution correction from the coarse grids. This smoothing reduces the high-frequency oscillations introduced by the bilinear interpolation of the coarse-mesh corrections; and hence, it allows convergence of the scheme for a broader range of artificial dissipation coefficients. The factored scheme described for implicit residual averaging with constant coefficients ($\epsilon_\xi = \epsilon_\eta = 0.1$) is used for the smoothing.

In order to provide an improved initial solution on the finest grid in the multigrid procedure, the Full Multigrid (FMG) method is employed. With the FMG method the solution is initialized on a coarser grid of the basic sequence of grids and iterated for a prescribed number of cycles using the FAS scheme. The solution is then interpolated to the next finer grid. The process is repeated until the finest grid is reached. In the current work, three refinement levels are used for a standard mesh density (e.g., 320 × 64 cells). The first and second levels include three and four grids, respectively, and 50 cycles are performed on each. There are five grids in the final level.

7.3.5 Hyperbolic Equations

In the standard elliptic multigrid theory (e.g., [13]) we decompose the error into low-frequency and high-frequency components. We assume that a smoother is being used that rapidly reduces the high-frequency components in the error. Thus, after a few sweeps of the smoother, only low-frequency components remain in the error, and these are not effectively reduced by the smoother. We then transfer the error to a coarser grid where parts of the low-frequency error are now high-frequency error. We can therefore again apply the smoother to reduce these high-frequency components and repeat the process on an even coarser grid. The corrections are then all combined on the finest grid. The object of this procedure is to reduce the amplification matrix uniformly below one. This implies that all components of the error are decreased independent of the mesh size. For the high-frequency components of the error this is accomplished by the smoother (in our case Runge-Kutta) and for the low-frequency components by the multi-level process.

For the wave equation we have conservation of energy. A physical problem approaches a steady state only because all disturbances are propagated outside the domain of interest. A wave equation can never achieve a steady state in the entire space. Hence, on the differential equation level the norm of the symbol is identically one. Any consistent scheme must propagate the low frequencies with minimal attenuation of the error. An "ideal" algorithm would propagate all the errors out of the domain within a few work units and would have an amplification matrix with norm one. Including the viscous terms of the Navier-Stokes equations introduces only a minimal dissipation outside of the boundary layer for the high Reynolds numbers that occur in aerodynamic problems. On the numerical level we assume that the low-frequency information (which corresponds to the PDE) is propagated outside the domain of integration. This places a strong emphasis on the far-field boundary conditions in the sense that they should not reflect significant error back into the domain. Hence, we cannot measure the efficiency of multigrid for hyperbolic problems with the same tools used for elliptic and parabolic equations. Instead we expect the coarser grids to enhance the propagation of waves so they exit the domain of integration. Hence, we wish the smoother to be stable under a large CFL condition, for the low-frequency error. The high-frequency information is not consistent with the physics and so needs to be damped out. This is especially important with a multigrid process where we do not wish the high-frequency errors to become aliased on the coarser grids to low-frequency components. In summary, the low-frequency error is convected outside the domain of interest. This is enhanced by the multigrid process where we transfer the residual to the next coarser grid and choose a time step that is twice as large. After transferring the error corrections back to the finest grid we have an effective time step that is equal to the sum of the time steps on all the coarser grids. The higher frequency errors are damped effectively by the smoother and so are not interpolated to coarser grids.

We now compare the elliptic and hyperbolic problems on the fine and coarse grids of the multigrid process. In both cases we expect the smoother to reduce the high-frequency components. This is done not only to remove part of the error but also to prevent aliasing on the coarser grids. For the elliptic problem the smoother accomplishes very little on the very long wave length components of the error. However, for the hyperbolic problem the smoother also propagates the long wave length components outside the domain. Hence, we wish a smoother that simultaneously reduces the high frequencies while maintaining a reasonable time step to propagate the low frequencies. This makes it harder to construct good smoothers for hyperbolic operators than for elliptic ones. As an example, the Jacobi relaxation for the Laplace equation is equivalent to using forward Euler for the heat equation. It is well known that this is not a good smoother for multigrid. Introducing an underrelaxation of Jacobi scheme is equivalent to choosing a smaller time step for the forward Euler scheme. For some choices of this time step we achieve good smoothing rates. The fact that a smaller time step is chosen is irrelevant to the convergence rate of the multigrid method. However, for a hyperbolic problem choosing a smoother that is very efficient at removing high-frequency errors and that has a very small time step would not lead to fast convergence rates.

After solving the problem on one grid we then transfer the residuals to the next coarser grid and solve for the error. For an elliptic problem this is done in order to convert low-frequency components on the finer mesh to higher frequency components on the coarser mesh, and thus, to remove them by the smoother. For the hyperbolic problem this is done in order to choose a time step that is twice as large on the coarse mesh as on the fine mesh, and thus, to propagate the low-frequency errors twice as fast. These errors will then reach the outer boundary and (hopefully) be absorbed at the outer boundary.

With respect to far-field boundary conditions, a global treatment can be very effective for the Euler and Navier-Stokes equations when coupled with multigrid [110]. This analysis demonstrates why multigrid is less effective on highly stretched grids where it is difficult to find a good smoother in both directions. This is complicated in hyperbolic problems where the time step in the various directions may differ by orders of magnitude. Using an explicit smoother requires choosing the minimum time step of the various directions. Thus, while in one direction the error may propagate one cell each time step on the finest mesh, it may travel only a tiny fraction of a cell in another direction. Using multigrid to increase the propagation speed is still inefficient. Use of an implicit method allows more of a choice in the time step but still does not solve the basic problem that the time step that is appropriate in one direction is not appropriate in the other direction. Nevertheless, the use of implicit-explicit residual smoothing may alleviate some of the difficulties. Preconditioning methods attempt to equalize the speeds of the different waves. Hence, the time step chosen is appropriate to all the waves rather than just one wave. So one would expect that preconditioning is even more effective when coupled with multigrid then when used by itself. Semicoarsening offers some hope, but it is not clear that it justifies the cost of choosing all the possible variations especially in three dimensions.

The above discussion on hyperbolic multigrid has been based on the theoretical work of Löstedt and Gustafsson [27], [58] and [59].

In Part II we present a way of decomposing the Euler equations into "elliptic" and "hyperbolic" portions. For the steady state two canonical forms appear. They are the wave operator, or elliptic form of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and the convective operator, or hyperbolic form of

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0.$$

We wish to convert the elliptic operator into a heat equation while converting the hyperbolic operator into a time-dependent convection equation.

8 Turbulence Modeling

The numerical solution of the instantaneous Navier-Stokes equations for turbulent flows requires computing power well beyond what is currently available [85]. In order to make turbulent flow problems tractable using existing computers, we must solve a time-averaged form of the equations. If the appropriate expansions in terms of Favre variables [84] are substituted for the flow variables in (3.1), and the resulting equations are time averaged, the mass-averaged form of the Navier-Stokes equations is obtained. These equations have the same form as their laminar flow counterparts, except that the stress tensor is augmented by the Reynolds stress tensor, the heat flux vector is augmented by the heat flux terms associated with turbulence, and additional mean energy dissipation terms appear (in many cases these terms can be neglected). Closure for this system of time-averaged equations is realized by using the eddy-viscosity hypothesis, which says that the Reynolds stress and heat flux terms are related to mean flow field gradients. Moreover, the effective viscosity is obtained by simply adding the turbulent viscosity to the molecular one. The Reynolds heat flux terms are approximated using the constant Prandtl number assumption. Thus, the effective nondimensional transport coefficients for diffusion and heat conduction are

$$\begin{aligned} \mu &= \mu_l + \mu_t, \\ k &= k_l + k_t = \left(\frac{\mu}{Pr}\right)_l + \left(\frac{\mu}{Pr}\right)_t, \end{aligned} \tag{8.1}$$

where the subscripts l and t refer to laminar and turbulent values. The laminar and turbulent Prandtl numbers are taken to be 0.72 and 0.9, respectively.

¿From the point of view of aerodynamic computations, the primary requirement for an eddy-viscosity model is to provide a sufficiently good representation of turbulence so as to allow accurate predictions of mean flow field characteristics. The desire to utilize such capability on a routine basis creates the need for turbulence models to also have a high degree of numerical compatibility. That is, these models must demonstrate a favorable interaction with numerical schemes, and thus they must not prevent reliable and efficient calculations. In this section the two turbulent viscosity models used herein are presented. Specific modifications of the originally published forms of the models to improve physical modeling and/or numerical compatibility are discussed.

The basic turbulence model considered is the now widely used algebraic model of Baldwin and Lomax [8]. With this two-layer model the nondimensional turbulent viscosity is defined as

$$\mu_t = \min\ \left[(\mu_t)_i\ ,\ (\mu_t)_o\right], \tag{8.2}$$

where the subscripts i and o denote inner and outer values. The viscosity in each layer is proportional to the product of a length scale and a velocity scale. In the inner part of the boundary layer,

$$(\mu_t)_i = \overline{Re}\ \rho\ L^2\ \Omega,$$

where $\overline{Re} = Re/(\sqrt{\gamma}M)$, Ω is the magnitude of the vorticity vector, and the length scale L is given by

$$L = K\ D\ d \tag{8.3}$$

with $K = 0.4$ (von Karman's constant), D representing the van Driest damping factor, and d denoting the distance from the wall. The damping factor D is defined as

$$D = 1 - \exp\left(\frac{-d^+}{A^+}\right), \qquad d^+ = d\ \frac{\sqrt{\overline{Re}\ \rho_w\ (\tau_l)_{\max}}}{\mu_w}, \tag{8.4}$$

where $A^+ = 26$. In the definition of the law-of-the-wall coordinate d^+, the original shear stress at the wall has been replaced with the maximum laminar value. This substitution prevents the eddy viscosity from vanishing when the shear stress goes to zero at a separation point. So it not only eliminates a nonphysical behavior of the turbulence but also generally removes numerical difficulties. In the outer part of the boundary layer, the turbulent viscosity becomes

$$(\mu_t)_o = \overline{Re}\ C_{\text{Clau}}\ C_{\text{cp}}\ \rho\ F_{\text{wake}}\ F_{\text{Kleb}}(d), \tag{8.5}$$

where $C_{\text{Clau}} = 0.0168$ (Clauser's constant), the additional constant $C_{\text{cp}} = 1.6$, and

$$F_{\text{wake}} = \min\left(d_{\max}\ F_{\max}\ ,\ C_{\text{wake}}\ d_{\max}\ \frac{{U_{\text{dif}}}^2}{F_{\max}}\right) \tag{8.6}$$

with $F_{\max}$ being the maximum value of the function

$$F(d) = d\,\Omega\,D \tag{8.7}$$

across the layer, and $d_{\max}$ is the value of d at which $F_{\max}$ occurs. The quantity $U_{\rm dif}$ is the difference between the magnitudes of the maximum and minimum velocity vectors that occurs across the layer. The function $F_{\rm kleb}(d)$ represents the Klebanoff intermittency factor and is defined by

$$F_{\rm Kleb}(d) = \left[1 + 5.5\left(C_{\rm Kleb}\frac{d}{d_{\max}}\right)^6\right]^{-1}. \tag{8.8}$$

In [8] Baldwin and Lomax defined the constant $C_{\rm wake}$ to be 0.25. This value was found generally to be unsatisfactory in transonic airfoil flows, where it caused oscillatory movement of a shock wave. A remedy for this problem is to set $C_{\rm wake} = 1.0$.

The Baldwin-Lomax (B-L) model just described is also used for wake regions. For wake flows the van Driest damping factor is set to unity. The model can also be used to represent transition to turbulence. In general, however, the specification of a transition location according to experiment is preferred.

In the implementation of the B-L model, care must be exercised in the determination of the maximum of the function $F(d)$, especially for complex flows. Multiple peaks can occur in this function, for example, in the vicinity of separation. The second peak is chosen in this case. Due to the rapid evolution of the numerical solution with the multigrid method, the turbulent viscosity is updated every multigrid cycle. Less frequent evaluation can cause either a slowdown or a stall in convergence.

The B-L turbulence model represents a balance of production and dissipation of turbulence. When the boundary layer on a solid surface is subjected to a sufficiently strong adverse pressure gradient to cause flow separation, such a balance breaks down. The inner part of the boundary layer responds immediately to the adverse pressure gradient, but the outer part experiences a delayed reaction. This behavior creates a disequilibrium of the two regions. If the size of the separated flow region is large enough to alter the surface pressure distribution, then the history effects cannot be neglected in the turbulence model. In general, both the convection and diffusion of turbulence need to be modeled in order to accurately predict the turbulent stresses. Johnson and King [46] proposed a model to account for nonequilibrium effects. They used the two-layer, algebraic model of Cebeci and Smith [17] as a foundation for their model. In principle, one could choose any equilibrium model, i.e., Baldwin-Lomax model. The basic idea of their model is to find an appropriate nonequilibrium factor to modify the variation of the equilibrium outer eddy viscosity. The nonequilibrium factor is determined in such a way that a transport equation for the maximum shear stress in the boundary layer is satisfied. In [78] the implementation of a modified version of this nonequilibrium model is presented. A similar modified form is given in [2]. The elements of these forms of the Johnson-King (J-K) model are described in the remainder of this section.

In [78] a turbulence reference quantity is defined as

$$G_m = \left(\Omega\frac{\mu_t}{\rho}\right)_m, \tag{8.9}$$

where the index m denotes the maximum of G across the shear layer. This quantity is then used to replace the maximum turbulent shear stress divided by density (i.e., the correlation of fluctuating velocity components given by $(-\overline{u'v'})_m$ appearing in the original J-K model [46]. The advantage of using G_m is that it is invariant with respect to coordinate systems. In the formulation of [46], the turbulent viscosity is constructed as an exponential blending of the inner and outer viscosities. That is,

$$\mu_t = (\mu_t)_o \left[1 - \exp\left(\frac{(\mu_t)_i}{(\mu_t)_o}\right)\right]. \tag{8.10}$$

The inner viscosity is given by

$$(\mu_t)_i = Re\ \rho\ D^2\ K\ d\ \sqrt{Re\ g_m} \tag{8.11}$$

with

$$D = 1 - \exp\left[\frac{-d}{A^+ \mu_w} \sqrt{\rho_w\ \max(\rho_w g_m\ ,\ \tau_w)}\right]. \tag{8.12}$$

An appropriate value of A^+ is 17 (rather than the equilibrium value of 26) according to [3]. The original model of Johnson and King requires the determination of the edge of the boundary layer, since the foundation model was that of Cebeci and Smith. This requirement is removed in [78],[2] by using the B-L model. Moreover, the outer turbulent viscosity is expressed as

$$(\mu_t)_o = \sigma\ \overline{Re}\ C_{\mathrm{Clau}}\ C_{\mathrm{cp}}\ \rho\ F_{\mathrm{wake}}\ F_{\mathrm{Kleb}}(d), \tag{8.13}$$

where σ is the previously mentioned nonequilibrium factor, and the other quantities are defined the same as for the B-L model.

Assuming that G is proportional to the turbulent kinetic energy and introducing a time derivative, the ordinary differential equation in [46] governing the trajectory of the maximum shear stress is replaced with

$$\frac{\partial G_m}{\partial t} + u_m \frac{\partial G_m}{\partial x} + v_m \frac{\partial G_m}{\partial y} + a_1 \frac{G_m}{L_m}\left({G_m}^{1/2} - (G_{\mathrm{eq}})_m^{1/2}\right) + G_m^{3/2} \mathcal{D}_m = 0, \tag{8.14}$$

where u_m and v_m are the Cartesian velocity components at the location of G_m. The quantity $(G_{\mathrm{eq}})_m$ is the equilibrium value of G at the location of G_m, and the length scale L_m is defined as

$$L_m = 0.4 d_m \quad \text{for} \quad \frac{\mathrm{d_m}}{\delta} \leq 0.225, \tag{8.15}$$

$$L_m = 0.09\delta \quad \text{for} \quad \frac{\mathrm{d_m}}{\delta} > 0.225, \tag{8.16}$$

with δ being the boundary-layer thickness. An estimate of δ, which is given in [91], is $1.9\ d_{\max}$. In the original J-K model, the diffusion term $\mathcal{D}_m$ is defined as

$$\mathcal{D}_m = \frac{a_2 F(\sigma)}{\delta\left(0.7 - \left(\frac{d_m}{\delta}\right)\right)}, \tag{8.17}$$

where

$$F(\sigma) = |\sigma^{1/2} - 1|. \tag{8.18}$$

The constants a_1 and a_2 are taken to be 0.25 and 0.5. With the $F(\sigma)$ given by (8.18), there is a singular-like (nonphysical) behavior of the diffusion term at $\sigma = 1$. In [78] $F(\sigma)$ is expressed as

$$F(\sigma) = \max\Big(0\ ,\ \sigma^{1/2} - 1\Big)\,, \tag{8.19}$$

and thus, it has a smooth behavior at $\sigma = 1$. This $F(\sigma)$ makes the diffusion term zero in regions of reverse flow (where $\sigma < 1$). It can be demonstrated that the use of (8.19) produces some deterioration in the accuracy of the predicted shock position in comparison with experiment for a transonic shock-induced separated airfoil flow with strong nonequilibrium effects.

If ${g_m}^{-2}$ is substituted for G_m in (8.14), the resulting linear equation is given by

$$\frac{\partial g_m}{\partial t} + u_m \frac{\partial g_m}{\partial x} + v_m \frac{\partial g_m}{\partial y} + \mathcal{S}_m - \frac{1}{2}\left(\mathcal{D}_m + \frac{a_1}{L_m}\right) = 0, \tag{8.20}$$

where the source term is

$$\mathcal{S}_m = \frac{a_1}{2L_m}\,(g_{\mathrm{eq}})_m^{-1}\,g_m.$$

Equation (8.14) is strictly valid along the curve determined by the maximum shear stress. However, to facilitate the numerical solution of the equation, it is solved along the solid surface of interest. As pointed out in [78], the misalignment between these surfaces creates errors in the convection terms. In order to reduce these errors, the velocity components u_m and v_m are replaced with their projections onto the wall boundary. The spatial discretization of the modified equation is accomplished by applying the finite-volume technique to the layer of mesh cells adjacent to the solid surface. A fourth-difference dissipation term is appended to this semidiscrete equation. The same five-stage Runge-Kutta scheme described in Section 6 in conjunction with local time stepping is used to numerically integrate the equations in time. With implicit treatment of the linear source term in (8.20), a Courant number of about three can be used. The computation of g_m, and thus the turbulent viscosity, must be adequately converged in order to allow convergence of the fluid dynamic system of equations. The turbulence model is applied once every time step in the solution of the Navier-Stokes equations, and Runge-Kutta time steps are performed for each update of the turbulence field.

Once the distribution of g_m is known, a new variation for the nonequilibrium factor σ is calculated with the following relation

$$\sigma^{n+1} = \sigma^n \left[\frac{g_m}{\left(\Omega\frac{\mu_t}{\rho}\right)_m}\right], \tag{8.21}$$

where σ^n is σ at time level n, introduced by Abid [3].

An alternative technique to solve for g_m, which is equal to ${(-\overline{u'v'})_m}^{-1/2}$, is a space marching procedure, as considered in [46]. In applying this procedure we eliminate the time-derivative of (8.20), transform to arbitrary curvilinear coordinates (ξ, η), and assume g_m is independent of the normal coordinate η (i.e., a ξ curve coincides with the transport path of g_m). Then, we obtain

$$U_m \frac{\partial g_m}{\partial \xi} + \mathcal{S}_m = \frac{1}{2}\left(\mathcal{D}_m + \frac{a_1}{L_m}\right), \tag{8.22}$$

with $U_m = \xi_x u_m + \xi_y v_m$. A discrete equation for g_m can be written as

$$\{(g_{\text{eq}})_m^{-1}\left[A_1 + (g_{\text{eq}})_m\right]\}_i \, (g_m)_{i+1} = (A_1 + g_m)_i + (A_1 \frac{L_m}{a_1}\mathcal{D}_m)_i, \tag{8.23}$$

where the index i means evaluated at the previous ξ location,

$$A_1 = \frac{a_1 \Delta s_\xi}{2 L_m U_m},$$

with Δs_ξ representing distance along the integration path, and U_m is defined here to be the magnitude of the velocity vector at the actual location where g_m occurs. Again, for convenience, the integration path is taken to be coincident with the geometry being considered. In general, we have found this particular approach to be somewhat more robust; and thus, we use it for all Johnson-King computations. The original argument in favor of the time-dependent technique concerned simplicity in extending to three dimensions. However, as indicated in [69] the steady equation for g_m can be solved fairly easily with first-order upwind differencing and point Gauss-Seidel relaxation. For additional discussion concerning implementation and various forms of the Johnson-King model, see [86].

9 Numerical Results

In the numerical applications presented here we assess the shock capturing capabilities of *CUSP* and its convergence behavior in combination with multistage schemes and multigrid. The first case is similar to the application published in [43]. Inviscid transonic flow over NACA 0012 airfoil is shown in Figure 12. The shock waves on the upper and lower surfaces are captured with a single interior point. The differences between the accurate dissipation coefficients, (4.9.14) and (4.9.16), on the one hand and the simplified coefficients, (4.9.18) on the other are small. Careful examination reveals that the exact formulation is somewhat more accurate on the upper surface where the shock is stronger. These inviscid solutions were obtained with a five-stage time stepping scheme, 4-level multigrid and sawtooth-type cycling. They converged at a rate of about 0.90 per multigrid cycle.

The second case is the hypersonic 2D flow over a blunt wedge. Figure 13 displays the second-order accurate solutions obtained for viscous and inviscid flow by using identical meshes of 64x48 cells. Physical diffusion is so large that the shock profile is significantly smeared in the viscous result. For inviscid flow, on the other hand, we obtain perfect capturing with a single interior point in the shock structure by using the formulation (4.9.14), (4.9.16). Detailed comparisons of the hypersonic wedge flow solutions yielded by *CUSP* and *AUSM* have been presented in [79]. It was found that the shock capturing capabilities of both schemes are essentially equal. A comparison of shock profiles for the exact and the simplified coefficients is given in Figure 14. Here, we have chosen the first-order scheme in order to address the pure shock capturing capability of *CUSP* without interference of the limiter. The simplified dissipation coefficients, (4.9.18), produce strong oscillations at the shock, even though there is substantial physical diffusion present. Hence it is concluded that an accurate implementation of dissipation coefficients is a must for hypersonic flows with strong shocks.

Transonic flow over the RAE 2822 airfoil is the next test case. The coordinate grid, flow solution and convergence histories are displayed in Figure 15. Five levels of multigrid have been used for this case and 70 multigrid cycles were executed on two coarser meshes in order to obtain an initial solution for the fine grid with 320x64 cells. The results indicate that convergence is somewhat improved by using the 2D formulation for the dissipation coefficient νc, (4.9.16). Note that convergence with $\zeta = 0$ was possible for the transonic case but not for the hypersonic case presented above.

Comparisons of computation times indicate that *CUSP* needs about 50-60% more computer time the basic scalar dissipation of section 4.2. Moreover, because of the lower inherent dissipation, computations with the *CUSP* scheme converge somewhat slower for transonic flows than those with simple scalar dissipation. The major advantage of the *CUSP* approach is that it is more accurate and more robust than scalar viscosity. Our preliminary tests indicate that the accuracy of *CUSP* is intermediate in between scalar and matrix -valued dissipation for transonic flows. For hypersonic flows it seems to be more robust than the matrix viscosity even though it is not *TVD*.

Since *CUSP* is implemented through artificial dissipative terms, these do not have to be reevaluated at each stage of the Runge-Kutta method. In particular, it can be evaluated only at the first, third, and fifth stages of a five stage method, just as we typically do for various forms of numerical dissipation.

-bb-error = =

Figure 12: Influence of HCUSP dissipation coefficients on transonic flow over NACA 0012 airfoil.

-bb-error = =

Figure 13: Viscous and inviscid hypersonic flow over 2-D wedge (second-order result).

-bb-error = =

Figure 14: Influence of HCUSP dissipation coefficients on hypersonic flow over 2-D wedge.

-bb-error = =

Figure 15: Computation of 2-D turbulent flow over RAE 2822 airfoil (c_l = 0.847 , c_{d_p} = 0.0121, c_{d_f} = 0.00570).

10 Concluding Remarks

In this report we have defined and discussed the elements of a class of explicit multistage time-stepping schemes with centered spatial differencing and multigrid. We have introduced additional understanding for a number of components of these schemes by their analysis. Through this approach, the benefit of a local mode analysis in evaluating boundary-point difference stencils for the numerical dissipation has been demonstrated. The stability of the multistage schemes has been examined. Hyperbolic and parabolic operator splitting has been applied to determine a reasonable time-step estimate for solving the compressible Navier-Stokes equations. The basic properties of the implicit process of residual smoothing for extending stability have been given. Two forms of variable coefficients for the residual smoothing procedure have been considered. The formulas introduced by the authors have been shown to perform much better than those of [68] for typical meshes used to compute inviscid airfoil flow solutions. With these formulas, a new set of variable coefficients has been constructed that can eliminate the general requirement to include a diffusion limit in the time-step estimate. The implicit residual smoothing has also been used as the basis for one of the several techniques that have been included to enhance the robustness of the basic multigrid method.

Numerical dissipation plays a very important role in fluid-dynamic computations. It affects both solution accuracy and convergence in the case of stationary solutions. Consequently, considerable attention in this report has been focussed on numerical dissipation. For schemes involving a blending of second-difference and fourth-difference dissipation, such as the class of schemes described in this report, a sufficient condition for the *TVD* property has been given. Flux limiters, including the family of limiters introduced in the *SLIP* scheme of Jameson [40], have been discussed. In order to improve accuracy with the *SLIP* scheme, a limiter function from [40] has been modified to allow reduced background dissipation. This limiter can also be used with the *CUSP* scheme of Jameson [40], which allows capturing of shock waves with a single interior point. The *CUSP* scheme has been described in this report.

For turbulence closure, we have considered both the equilibrium model of Baldwin and Lomax and the nonequilibrium model of Johnson and King. The implementation of these models, including two alternatives for the J-K model, has been described in detail. Some modifications to the original formulations of the models have been made. These changes were made in order to improve numerical compatibility of the models (i.e., make it easier to converge numerical algorithm with the model), and in the case of the J-K model, to simplify implementation and to improve prediction capability.

A Equations for Boundary Points

Consider the elements of the solution vector

$$\mathbf{W}' = [\, c \quad u \quad v \quad s \,]^T$$

as dependent variables, where c is the speed of sound, and s is the entropy. If we write the Euler equations relative to the rotated Cartesian coordinate system (x_t, x_n), then

$$\frac{\partial \mathbf{W}'}{\partial t} + A'' \frac{\partial \mathbf{W}'}{\partial x_t} + B'' \frac{\partial \mathbf{W}'}{\partial x_n} = 0, \tag{A.1}$$

where

$$\begin{aligned} A'' &= A' \cos\theta + B' \sin\theta, \\ B'' &= -A' \sin\theta + B' \cos\theta, \end{aligned}$$

and

$$A' = \begin{bmatrix} u & c(\gamma-1)/2 & 0 & 0 \\ 2c(\gamma-1)^{-1} & u & 0 & -c^2(\gamma-1)^{-1} \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix},$$

$$B' = \begin{bmatrix} v & 0 & c(\gamma-1)/2 & 0 \\ 0 & v & 0 & 0 \\ 2c(\gamma-1)^{-1} & 0 & v & -c^2(\gamma-1)^{-1} \\ 0 & 0 & 0 & v \end{bmatrix}.$$

In (A.1) θ is the angle the rotated coordinate system makes with the unrotated system. Suppose now we desire to have the Riemann invariants of one-dimensional gas dynamics as dependent variables. This can be done by first assuming that the flow is locally homentropic and by redefining the matrices A' and B' as the reduced matrices

$$A' = \begin{bmatrix} u & c(\gamma-1)/2 & 0 \\ 2c(\gamma-1)^{-1} & u & 0 \\ 0 & 0 & u \end{bmatrix}, \quad B' = \begin{bmatrix} v & 0 & c(\gamma-1)/2 \\ 0 & v & 0 \\ 2c(\gamma-1)^{-1} & 0 & v \end{bmatrix}. \tag{A.2}$$

Then, with the matrix

$$Q^{-1} = \begin{bmatrix} 0 & \cos\theta & -\sin\theta \\ 1/\sqrt{2} & -(\gamma-1)\sin\theta/(2\sqrt{2}) & (\gamma-1)\cos\theta/(2\sqrt{2}) \\ 1/\sqrt{2} & (\gamma-1)\sin\theta/(2\sqrt{2}) & -(\gamma-1)\cos\theta/(2\sqrt{2}) \end{bmatrix}.$$

the reduced form of the solution vector $\mathbf{W}'$ can be transformed to a new one that is a function of the Riemann invariants. In addition, the similarity transformation with Q^{-1} and

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ \cos\theta & -\sqrt{2}\sin/(\gamma-1) & -\sqrt{2}\sin\theta/(\gamma-1) \\ \sin\theta & -\sqrt{2}\cos/(\gamma-1) & -\sqrt{2}\cos\theta/(\gamma-1) \end{bmatrix}.$$

can be used to diagonalize the reduced form of the matrix B''. Thus, if (A.1) with A' and B' defined by (A.2) is premultiplied by Q^{-1}, we obtain

$$Q^{-1}\frac{\partial \mathbf{W}'}{\partial t} + \left(Q^{-1}A''Q\right)Q^{-1}\frac{\partial \mathbf{W}'}{\partial x_t} + \left(Q^{-1}B''Q\right)Q^{-1}\frac{\partial \mathbf{W}'}{\partial x_n} = 0. \tag{A.3}$$

If Q^{-1} is considered to be locally constant, and the variation of $\mathbf{W}'$ in the tangential direction is taken to be negligible, then (A.3) becomes

$$\frac{\partial \tilde{\mathbf{W}}}{\partial t} + \Lambda_{B''}\frac{\partial \tilde{\mathbf{W}}}{\partial x_n} = 0, \tag{A.4}$$

where $\Lambda_{B''}$ is a diagonal matrix of the eigenvalues $(q_n, q_n + c, q_n - c)$ of B'', and $\tilde{\mathbf{W}}$ is the vector of characteristic variables defined by

$$\tilde{\mathbf{W}} = \begin{bmatrix} q_t & \frac{1}{\sqrt{2}}\frac{\gamma-1}{2}R^{+} & \frac{1}{\sqrt{2}}\frac{\gamma-1}{2}R^{-} \end{bmatrix}^T$$

with

$$R^{+} = q_n + \frac{2c}{\gamma - 1}, \qquad R^{-} = q_n - \frac{2c}{\gamma - 1},$$
$$q_t = u\ \cos\theta + v\ \sin\theta, \qquad q_n = -u\ \sin\theta + v\ \cos\theta.$$

B Development of Residual Smoothing Coefficients

In order to obtain insight into an appropriate form for variable smoothing coefficients, consider first the approximate factorization scheme

$$(I + \frac{\Delta t}{\Omega}\mu_\xi \delta_\xi \tilde{A})\;(I + \frac{\Delta t}{\Omega}\mu_\eta \delta_\eta \tilde{B})\;\Delta\widetilde{\mathbf{W}}_{i,j} = -\mathcal{R}_{i,j}, \tag{B.1}$$

where $\Delta\widetilde{\mathbf{W}}_{i,j}$ is the product of the solution vector for the Euler equations and the volume Ω, as determined by a transformation Jacobian, $\mathcal{R}_{i,j}$ is the residual vector for the system, and (ξ, η) are arbitrary curvilinear coordinates. The operators μ and δ are standard averaging and central difference operators, respectively. Thus,

$$\mu_\xi \widetilde{\mathbf{W}}_{i,j} = \frac{1}{2}\left(\widetilde{\mathbf{W}}_{i+1/2,j} + \widetilde{\mathbf{W}}_{i-1/2,j}\right), \qquad \delta_\xi \widetilde{\mathbf{W}}_{i,j} = \left(\widetilde{\mathbf{W}}_{i+1/2,j} - \widetilde{\mathbf{W}}_{i-1/2,j}\right).$$

The transformed flux Jacobian matrices are defined as

$$\tilde{A} = \xi_x A + \xi_y B, \qquad \tilde{B} = \eta_x A + \eta_y B,. \tag{B.2}$$

The spectral radii of these matrices are as follows

$$\sigma_{\tilde{A}} = \sigma(\tilde{A}) = \frac{\lambda_\xi}{\Omega}, \qquad \sigma_{\tilde{B}} = \sigma(\tilde{B}) = \frac{\lambda_\eta}{\Omega}, \tag{B.3}$$

where λ_ξ and λ_η are the characteristic speeds defined in (4.2.5). If the matrices $\tilde{A}$ and $\tilde{B}$ are approximated as

$$\tilde{A} = \sigma_{\tilde{A}} I, \qquad \tilde{B} = \sigma_{\tilde{B}} I,$$

then (B.1) can be replaced with

$$(I + \frac{\Delta t}{\Omega}\sigma_{\tilde{A}}\mu_\xi \delta_\xi)\;(I + \frac{\Delta t}{\Omega}\sigma_{\tilde{B}}\mu_\eta \delta_\eta)\;\Delta\widetilde{\mathbf{W}}_{i,j} = -\mathcal{R}_{i,j}, \tag{B.4}$$

when the scalings are taken to be locally constant. Define

$$\beta_\xi = \frac{\Delta t}{\Omega}\sigma_{\tilde{A}}, \qquad \beta_\eta = \frac{\Delta t}{\Omega}\sigma_{\tilde{B}} \tag{B.5}$$

as the implicit smoothing coefficients for the ξ and η directions, respectively. Using (B.3) and taking

$$\Delta t = \frac{\Omega}{(\lambda_\xi + \lambda_\eta)}, \tag{B.6}$$

the smoothing coefficients of (B.5) become

$$\beta_\xi = \frac{\lambda_\xi}{(\lambda_\xi + \lambda_\eta)}, \qquad \beta_\eta = \frac{\lambda_\eta}{(\lambda_\xi + \lambda_\eta)}. \tag{B.7}$$

Now we investigate the case where parabolic implicit smoothing operators are used instead of hyperbolic ones. In particular, consider the implicit finite-volume method of Lerat

[54]. This scheme includes two stages. The first stage is a physical one in which the change of the solution vector of the Euler equations is evaluated using a Lax-Wendroff type scheme. In order to remove the time step limit of the explicit scheme, a mathematical stage is applied in the following integral form:

$$\iint_{\Omega_{i,j}} (\Delta\mathbf{W})^* d\Omega + \omega\frac{\Delta t^2}{2}\int_{\Gamma_{i+\frac{1}{2},j}\cup\Gamma_{i-\frac{1}{2},j}} \hat{\sigma}^2_{\tilde{A}}\left[\mathbf{n}\cdot\nabla(\Delta\mathbf{W})^*\right] d\Gamma = \iint_{\Omega_{i,j}} \Delta\overline{\mathbf{W}} d\Omega, \tag{B.8}$$

$$\iint_{\Omega_{i,j}} \Delta\mathbf{W} d\Omega + \omega\frac{\Delta t^2}{2}\int_{\Gamma_{i,j+\frac{1}{2}}\cup\Gamma_{i,j-\frac{1}{2}}} \hat{\sigma}^2_{\tilde{B}}\left[\mathbf{n}\cdot\nabla(\Delta\mathbf{W}\right] d\Gamma = \iint_{\Omega_{i,j}} (\Delta\mathbf{W})^* d\Omega, \tag{B.9}$$

where $\Delta\mathbf{W}$ is the change in the solution vector, the $(*)$ superscript indicates a provisional value, the overbar refers to a value from the explicit physical stage, the quantity $\Omega_{i,j}$ is a mesh cell volume, the vector $\mathbf{n}$ is a unit normal to the boundary curve Γ, and ω is a constant taken to be -1/2. In (B.8) and (B.9) the eigenvalues $\hat{\sigma}_{\tilde{A}}$ and $\hat{\sigma}_{\tilde{B}}$ are related to the spectral radii of (B.3) as

$$\hat{\sigma}_{\tilde{A}} = \frac{\sigma_{\tilde{A}}\,\Omega}{\sqrt{{x_\eta}^2 + {y_\eta}^2}}, \qquad \hat{\sigma}_{\tilde{B}} = \frac{\sigma_{\tilde{B}}\,\Omega}{\sqrt{{x_\xi}^2 + {y_\xi}^2}}.$$

Assuming that the quantities inside the integral signs associated with the boundary curves are locally constant and that the curvilinear coordinates ξ and η are orthogonal, the integral equations (B.8) and (B.9) can be approximated by

$$\begin{aligned}\Delta\mathbf{W}^*_{i,j} \quad &- \quad \frac{1}{4}\frac{\Delta t^2}{\Omega_{i,j}}\left[\left(\frac{\hat{\sigma}^2_{\tilde{A}}\Gamma^2}{\Omega}\right)_{i+\frac{1}{2},j}\left((\Delta\mathbf{W})^*_{i+1,j} - (\Delta\mathbf{W})^*_{i,j}\right)\right] \\ &+ \quad \frac{1}{4}\frac{\Delta t^2}{\Omega_{i,j}}\left[\left(\frac{\hat{\sigma}^2_{\tilde{A}}\Gamma^2}{\Omega}\right)_{i-\frac{1}{2},j}\left((\Delta\mathbf{W})^*_{i,j} - (\Delta\mathbf{W})^*_{i-1,j}\right)\right] = \overline{(\Delta\mathbf{W})}_{i,j},\end{aligned} \tag{B.10}$$

$$\begin{aligned}\Delta\mathbf{W}_{i,j} \quad &- \quad \frac{1}{4}\frac{\Delta t^2}{\Omega_{i,j}}\left[\left(\frac{\hat{\sigma}^2_{\tilde{B}}\Gamma^2}{\Omega}\right)_{i,j+\frac{1}{2}}\left((\Delta\mathbf{W})_{i,j+1} - (\Delta\mathbf{W})_{i,j}\right)\right] \\ &+ \quad \frac{1}{4}\frac{\Delta t^2}{\Omega_{i,j}}\left[\left(\frac{\hat{\sigma}^2_{\tilde{B}}\Gamma^2}{\Omega}\right)_{i,j-\frac{1}{2}}\left((\Delta\mathbf{W})_{i,j} - (\Delta\mathbf{W})_{i,j-1}\right)\right] = (\Delta\mathbf{W})^*{}_{i,j},\end{aligned} \tag{B.11}$$

where the unknowns are located at the cell centers. If the coefficients

$$\frac{\hat{\sigma}^2_{\tilde{A}}\Gamma^2}{\Omega}, \qquad \frac{\hat{\sigma}^2_{\tilde{B}}\Gamma^2}{\Omega},$$

which are evaluated at the cell faces, are taken to be locally constant, and the time step is defined as (B.6), then the smoothing parameters β_ξ and β_η depend upon

$$\left[\frac{\lambda_\xi}{(\lambda_\xi + \lambda_\eta)}\right]^2, \qquad \left[\frac{\lambda_\eta}{(\lambda_\xi + \lambda_\eta)}\right]^2, \tag{B.12}$$

respectively, which are the squares of the smoothing coefficients obtained for the ADI scheme.

The results from the 1-D stability analysis of Section 7.2.1 and the understanding of the functional dependence of the smoothing coefficients on λ_ξ and λ_η provide a foundation for developing β_ξ and β_η. In order to determine formulas for β_ξ and β_η we examine the two-dimensional (2-D) stability of a multistage time-stepping scheme with implicit residual smoothing. Consider the 2-D scalar hyperbolic wave equation

$$\frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = 0. \tag{B.13}$$

Using central difference approximations for the spatial derivatives, a semidiscrete form for (B.13) is written as

$$\Delta t\frac{dw}{dt} = -\frac{N_\xi}{2}\left(w^n_{i+1,j} - w^n_{i-1,j}\right) - \frac{N_\eta}{2}\left(w^n_{i,j+1} - w^n_{i,j-1}\right), \tag{B.14}$$

where the Courant numbers

$$\begin{aligned} N_\xi &= \lambda_\xi\frac{\Delta t}{\Omega} = (a\Delta y)\frac{\Delta t}{\Omega}, \\ N_\eta &= \lambda_\eta\frac{\Delta t}{\Omega} = (b\Delta x)\frac{\Delta t}{\Omega}. \end{aligned} \tag{B.15}$$

Taking the Fourier transform of (B.14) the following is obtained:

$$\Delta t\frac{d\hat{w}}{dt} = z\hat{w}^n,$$

where the Fourier symbol z is given by

$$z = -\mathrm{i}\left(N_\xi \sin\theta_\xi + N_\eta \sin\theta_\eta\right), \tag{B.16}$$

the caret indicates a transformed quantity, and θ_ξ, θ_η are the Fourier angles for the two coordinate directions. If implicit residual smoothing is applied, the Fourier symbol of (B.16) is replaced by

$$z = -\mathrm{i}\frac{\left(N_\xi \sin\theta_\xi + N_\eta \sin\theta_\eta\right)}{\chi_\xi\chi_\eta},$$

where

$$\begin{aligned} \chi_\xi &= 1 + 2\beta_\xi\left(1 - \cos\theta_\xi\right), \\ \chi_\eta &= 1 + 2\beta_\eta\left(1 - \cos\theta_\eta\right). \end{aligned} \tag{B.17}$$

A sufficient condition for stability can be written as

$$\max\ |z| \leq N^* \qquad \text{for all } \theta_\xi, \theta_\eta, \tag{B.18}$$

where N^* is the Courant number of the unsmoothed scheme. Let

$$\tilde{F} = |z| = N_\xi\frac{\sin\theta_\xi}{\chi_\xi\chi_\eta} + N_\eta\frac{\sin\theta_\eta}{\chi_\xi\chi_\eta}.$$

Clearly,

$$\tilde{F} \leq N_\xi \frac{\sin\theta_\xi}{\chi_\xi} + N_\eta \frac{\sin\theta_\eta}{\chi_\eta},$$

or

$$\tilde{F} \leq N_\xi f(\theta_\xi) + N_\eta g(\theta_\eta).$$

Certainly,

$$\tilde{F}_{\max} \leq N_\xi f_{\max} + N_\eta g_{\max}.$$

Then, a sufficient condition for stability is given by

$$N_\xi f_{\max} + N_\eta g_{\max} \leq N^*. \tag{B.19}$$

¿From (7.2.7) of Section 7.2.1 it follows that

$$f_{\max} = \frac{1}{\sqrt{1+4\beta_\xi}}, \qquad g_{\max} = \frac{1}{\sqrt{1+4\beta_\eta}}. \tag{B.20}$$

Substituting (B.19) into (B.20), we obtain

$$N_\xi \frac{1}{\sqrt{1+4\beta_\xi}} + N_\eta \frac{1}{\sqrt{1+4\beta_\eta}} \leq N^*. \tag{B.21}$$

But

$$\begin{aligned} N_\xi &= \frac{\Delta t_{\text{act}}}{\Delta t_\xi} = N\frac{\lambda_\xi}{(\lambda_\xi+\lambda_\eta)} = \frac{N}{1+r_{\eta\xi}} \\ N_\eta &= \frac{\Delta t_{\text{act}}}{\Delta t_\eta} = N\frac{\lambda_\eta}{(\lambda_\xi+\lambda_\eta)} = \frac{N}{1+r_{\eta\xi}^{-1}}, \end{aligned} \tag{B.22}$$

where $r_{\eta\xi}$ is the ratio of the modified characteristic speeds $(\lambda_\eta/\lambda_\xi)$ and is also proportional to mesh cell aspect ratio. Thus, (B.21) becomes

$$\frac{N}{N^*}\frac{1}{1+r_{\eta\xi}}\frac{1}{\sqrt{1+4\beta_\xi}} + \frac{N}{N^*}\frac{1}{1+r_{\eta\xi}^{-1}}\frac{1}{\sqrt{1+4\beta_\eta}} \leq 1. \tag{B.23}$$

In the cases of low-aspect-ratio cells ($r_{\eta\xi} \ll 1$) and high-aspect-ratio cells ($r_{\eta\xi} \gg 1$), equation (B.23) can be replaced by

$$\frac{N_\xi}{N^*}\frac{1}{\sqrt{1+4\beta_\xi}} \leq 1, \qquad \frac{N_\eta}{N^*}\frac{1}{\sqrt{1+4\beta_\eta}} \leq 1,$$

respectively. Thus, we can write

$$\begin{aligned} \beta_\xi &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\frac{1}{1+r_{\eta\xi}}\right)^2 - 1\right], 0\right\}, \\ \beta_\eta &= \max\left\{\frac{1}{4}\left[\left(\frac{N}{N^*}\frac{1}{1+r_{\eta\xi}^{-1}}\right)^2 - 1\right], 0\right\}. \end{aligned} \tag{B.24}$$

Note that these expressions are related to the smoothing coefficients of (B.12) for the implicit method of Lerat.

The formulas of (B.24) can also be obtained by substituting the appropriate time-step estimates into the 1-D smoothing coefficient of (7.2.8) in Section 7.2.1. That is, the time step of the smoothed scheme is defined as in (B.6) and that of the unsmoothed scheme is a 1-D time step.

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