Absorbing PML Boundary Layers for Wave-Like Equations

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Abstract

We consider absorbing layers that are extensions of the Berenger PML [6]. These will be constructed both for time problems and for Helmholtz-like equations. We shall consider applications to electricity & magnetism and acoustics (with a mean flow) in both physical space and in the time Fourier space (Helmholtz equation). Numerical results are presented showing the efficacy of this condition for the time dependent Maxwell equations.

1 Introduction

In recent years a number of models have been suggested to prevent reflections from artificial boundaries that surround domains. Berenger [6] proposed a PML layer for use with the time dependent Maxwell equations. Later other models were proposed both in the physical time domain and in the Fourier time domain. We compare several of these models in both the physical domain and in Fourier space. We show how to rewrite these models as a second order elliptic partial differential equation. In free space this reduces to a Helmholtz (reduced wave) equation. Hence, this boundary condition can be used when one is numerically solving the Helmholtz equation.

One can then take this boundary layer and recast it in a form more appropriate for acoustics. A change of variables then allows one to also consider a mean flow for the acoustic equations. We conclude with a discussion of a finite element code for the approximation to the Helmholtz equation. This equation is formally self-adjoint but contains complex coefficients. It is equivalent to a symmetric elliptic system of two equations with real coefficients.

2 Well-Posedness

The Maxwell equations form a symmetric hyperbolic system. For such a system one can demonstrate that the solution is well-posed. Roughly this implies that we can bound the solution at any time t in terms of the initial data, boundary conditions and forcing functions even when lower order terms (i.e. non-differentiated terms that depend on E and H) are added to the system. We stress that the definition of well-posedness allows a growth in time even an exponential growth. However, this growth is independent of the given initial and boundary conditions and also any forcing functions or lower order terms.

Some of the absorbing layers we shall discuss can not be reduced to a symmetric form for the principal part (i.e. dropping lower oder terms) of the system. This implies that the system is no longer automatically well-posed and instead each case has to be individually analyzed. Abarbanel and Gottlieb [1] have indeed shown that two dimensional Berenger PML equations are not well-posed. This implies that lower order perurbations can lead to exponential growth in time that cannot be bounded a priori. In particular the growth depends on the frequency of the initial conditions. We then have the following theorem [17] p283.

Consider the first order system

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} B_i(x,t)\frac{\partial u}{\partial x_i} + C(x,t)u + F(x,t) \qquad x = (x_1, \ldots, x_n)$$

In the strip

$$0 \leq x_1 \leq 1 \qquad -\infty < x_2, \ldots, x_n < \infty \quad t > 0$$

with initial condition

$$u(x,0) = f(x)$$

where the system is symmetric i.e.

$$B_i^*(x,t) = B_i(x,t).$$

We consider boundary conditions sufficiently close to characteristic (see [17] for the technical details) then for any finite interval $0 \leq t \leq T$ there exists a constant K_T such that

$$||u(\cdot,t)|| + \int_0^t ||u(\cdot,\tau)||_\Gamma^2 d\tau \leq K_T \left\{ ||f||^2 + \int_0^t ||g(\cdot,\tau)||_\Gamma^2 + ||F(\cdot,\tau)||^2 d\tau \right\}$$

where $Gamma$ is the boundary and K_T does not depend on f, g or F but can be an exponential function of T. This theorem states that the L_2 norm of the solution in space-time plus the norm of the solution on the boundary is boundary by the initial data, f, the boundary data, g, and the forcing function F. The equation contains lower order terms $C(x,t)u$ but the bound does not depend on C. We again stress that K_T is independent of f and in particular can not depend on the frequency of the initial condition.

Nevertheless, Bourdonnaye [9] has shown that if we add the conditions that the field is divergence free and also that $\Delta E_{zy} = \frac{\partial^2 E_z}{\partial^2 y}$ then the equations are well posed. Furthermore, if these two conditions are true for the initial conditions then they remain true for all time.

3 PML Layers

The general feature of PML layers is the construction of a layer exterior to the domain of interest that will damp all the waves that enter from the interior domain. This layer will be represented by a system of equations which are not necessarily physically realizable. Their only purpose is to prevent reflections from the outer artificial boundary. There have been many attempts in the past at what was called "sponge layers". These were layers that added dissipative terms to damp the outgoing waves. These methods were only moderately successful since reflections were also created by gradients within the layer. Furthermore, the properties of these layers frequently demanded knowledge about the outgoing waves especially their frequency and direction of propagation. The uniqueness of the PML layer is that the dissipation properties are essentially independent of the angle of incidence and also the frequency.

For the complete description of the dissipative properties of these layers we refer the reader to [6, 7, 1]. Here, we shall merely bring some of the highlights of the technique. We assume that one has the Maxwell equations on one side (left side) of the interface and the PML on the other side (right side) of the interface. The present analysis is based on the behavior of plane waves. This is a reasonable assumption for Cartesian coordinates as any wave can be decomposed into plane waves. The analysis is usually done for a interface between two regions with constant properties. In practice this discontinuity in properties is too demanding for numerical methods. Thus, the "interface" is replaced by a layer in which the parameters of the PML region vary continuously. There is a judicial compromise between a thin layer which requires a rapid variation of the parameters and a thick layer which requires more grid points and hence more computer time and more storage. In practice layers of about 8 points seem to be most common.

We thus assume that for two dimensions the incoming wave has the functional form

$$E_x = E_0 \sin\theta e^{i\omega(t-\alpha x-\beta y)}$$
$$E_y = E_0 \cos\theta e^{i\omega(t-\alpha x-\beta y)}$$

and similarly for the magnetic field. In essence this implies that we Fourier transform both the Maxwell equations and the PML equations in time and space. One demands that the wave be continuous at the interface and then calculates the reflection and transmission coefficients. One then desires that the transmitted wave decay exponentially in the PML layer independent of the angle of incidence (i.e. α and β) and also independent of the frequency ω. Frequently, one can demonstrate this not for all angles but only for angles far from tangential incidence to the interface. Abarbanel and Gottlieb [1] also extend some of this analysis to finite depth layers with variable coefficient parameters and also to finite difference schemes.

In particular we note that two sets of PML layers with related Fourier transforms have related dissipation properties. We also see that this analysis applies only to Cartesian coordinates. For general coordinates the plane wave analysis must be replaced by an appropriate system of incoming waves.

4 Time Dependent CEM

We consider a computational domain as given in figure 1. In the interior we consider the two dimensional Maxwell equations for the TE mode in free space. We normalize the time and space variables so that $\epsilon_0 = 1$ and $\mu_0 = 1$. We denote the magnetic component H_z by simply H. Then the TE equations are given by:

$$\begin{aligned}\frac{\partial E_x}{\partial t} &= \frac{\partial H}{\partial y}\\ \frac{\partial E_y}{\partial t} &= -\frac{\partial H}{\partial x}\\ \frac{\partial H}{\partial t} &= \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}.\end{aligned} \tag{4.1}$$

For the absorbing layer we shall introduce two functions σ_x, σ_y. To simplify matters we shall assume that σ_x is a function only of x and σ_y is a function only of y. We shall only consider the right and top boundaries with the extension to the other boundaries straightforward. In the top layer BC $\sigma_x = 0$ while in the right layer AB $\sigma_y = 0$ while in the corner BB_1B_2 both are non-zero.

Berenger constructed his layer by splitting H into two fields H_x and H_y. We then have four unknowns which satisfy the following system:

$$\begin{aligned}
\frac{\partial E_x}{\partial t} + \sigma_y E_x &= \frac{\partial H}{\partial y} \qquad H = H_x + H_y \\
\frac{\partial E_y}{\partial t} + \sigma_x E_y &= -\frac{\partial H}{\partial x} \\
\frac{\partial H_x}{\partial t} + \sigma_x^* H_x &= -\frac{\partial E_y}{\partial x} \\
\frac{\partial H_y}{\partial t} + \sigma_y^* H_y &= \frac{\partial E_x}{\partial y}
\end{aligned} \tag{4.2}$$

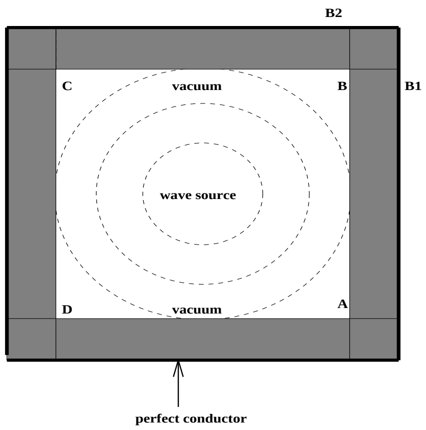

Figure 1: The PML technique

We Fourier transform in time and so replace time derivatives with multiplication by $i\omega$. We can then solve the equations to obtain (see also [19])

$$E_x = \frac{1}{i\omega + \sigma_y^*}\frac{\partial H}{\partial y} \tag{4.3a}$$

$$E_y = -\frac{1}{i\omega + \sigma_x^*}\frac{\partial H}{\partial x} \tag{4.3b}$$

$$H_x = -\frac{1}{i\omega + \sigma_x}\frac{\partial E_y}{\partial x} \tag{4.3c}$$

$$H_y = \frac{1}{i\omega + \sigma_y}\frac{\partial E_x}{\partial y} \tag{4.3d}$$

and hence

$$H = \frac{1}{i\omega + \sigma_y^*}\frac{\partial E_x}{\partial y} - \frac{1}{i\omega + \sigma_x^*}\frac{\partial E_y}{\partial x}. \tag{4.3e}$$

We assume that $\sigma_x = \sigma_x(x)$ and $\sigma_y = \sigma_y(y)$. Let $\sigma_x^* = \sigma_x$ and $\sigma_y^* = \sigma_y$. Define

$$s_x = 1 + \frac{\sigma_x}{i\omega} \quad , \quad s_y = 1 + \frac{\sigma_y}{i\omega}. \tag{4.4}$$

Then we can rewrite (4.3e) as

$$s_x s_y i\omega H = \frac{\partial(s_x E_x)}{\partial y} - \frac{\partial(s_y E_y)}{\partial x}.$$

We shall later show, (5.3), how to write this system for the time dependent equations without splitting the fields. Substituting (4.3a) and (4.3b) into (4.3e) we get

$$H = \frac{1}{i\omega + \sigma_x^*}\frac{\partial}{\partial x}\left(\frac{1}{i\omega + \sigma_x}\frac{\partial H}{\partial x}\right) + \frac{1}{i\omega + \sigma_y^*}\frac{\partial}{\partial y}\left(\frac{1}{i\omega + \sigma_y}\frac{\partial H}{\partial y}\right).$$

or

$$\frac{\partial}{\partial x}\left(\frac{s_y}{s_x}\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{s_x}{s_y}\frac{\partial H}{\partial y}\right) + \omega^2 s_x s_y H = 0. \tag{4.5}$$

Note that this equation is formally a self-adjoint elliptic equation. It is not self-adjoint in the usual sense because the coefficients are complex. However, we shall show that it can be symmetrized.

We note that the justification of this equation as a boundary condition, is slightly more difficult than for the time dependent case. For the time dependent Maxwell equations we have the principle of casuality. Hence, if we can show that the waves that reach the PML layer from the interior do not reflect back into the interior we are done. However, (4.5) is an elliptic equation. Hence, the layer exerts a global effect on the solution. For the interior Helmholtz equation there are incoming and outcoming solutions. The Sommerfeld radiation condition is designed to allow only the outgoing solutions to exist. Hence, it is necessary to show that (4.5) is some sense imposes the Sommerfeld condition. We shall currently only sketch the proof in one dimension.

In one dimension the Helmholtz equation reduced to the ordinary differential equation

$$\frac{d^2 H}{dx^2} + \omega^2 H = 0 \tag{4.6}$$

This has two fundamental solutions i.e. $e^{i\omega x}$ and $e^{-i\omega x}$. We wish that only the second wave exist for sufficiently large x. Imposing the Sommerfeld condition, in the far field, $\frac{dH}{dx} + i\omega H = 0$ accomplishes this. We reduce (4.5) to one dimension. In addition for simplicity we assume that σ_x is a positive constant. We then have a layer in the far field in which

$$\frac{d^2H}{dx^2} + K^2 H = 0 \quad , \quad K = \omega - i\sigma_x. \tag{4.7}$$

This has two fundamental solution e^{iKx} and e^{-iKx} or $e^{i\omega x}e^{\sigma_x x}$ and $e^{-i\omega x}e^{-\sigma_x x}$. In a semi-infinite layer $e^{\sigma_x x}$ is excluded. Hence, the only wave permitted is $e^{-i\omega x}e^{-\sigma_x x}$. Hence, this layer allows only the right moving wave to exist. Allowing σ_x to depend on x does not substantially change the proof (see [2] for a similar argument in a different context).

We can also write this equation as a set of two equations with real coefficients. Let $H = H_R + iH_I$. Then

$$\frac{\partial}{\partial x}\left(\frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\frac{\partial H_R}{\partial x} - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\frac{\partial H_I}{\partial x}\right) +$$

$$\frac{\partial}{\partial y}\left(\frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\frac{\partial H_R}{\partial y} + \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\frac{\partial H_I}{\partial y}\right) + (\omega^2-\sigma_x\sigma_y)H_R + \omega(\sigma_x+\sigma_y)H_I = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\frac{\partial H_I}{\partial x} + \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\frac{\partial H_R}{\partial y}\right) +$$

$$\frac{\partial}{\partial y}\left(\frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\frac{\partial H_I}{\partial y} - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\frac{\partial H_R}{\partial y}\right) + (\omega^2-\sigma_x\sigma_y)H_I - \omega(\sigma_x+\sigma_y)H_R = 0$$

We express the principal part of this in matrix form as

$$\begin{pmatrix} \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\partial_x^2 + \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\partial_y^2 & -\left[\frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\partial_x^2 - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\partial_y^2\right] \\ \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\partial_x^2 - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\partial_y^2 & \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\partial_x^2 + \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\partial_y^2 \end{pmatrix} \begin{pmatrix} H_R \\ H_I \end{pmatrix}$$

This matrix is not symmetric. However we now rewrite this matrix in terms of $(H_R, -H_I)$ and then we get

$$\begin{pmatrix} \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\partial_x^2 + \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\partial_y^2 & \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\partial_x^2 - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\partial_y^2 \\ \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_x^2}\partial_x^2 - \frac{\omega(\sigma_x-\sigma_y)}{\omega^2+\sigma_y^2}\partial_y^2 & -\left[\frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_x^2}\partial_x^2 + \frac{\omega^2+\sigma_x\sigma_y}{\omega^2+\sigma_y^2}\partial_y^2\right] \end{pmatrix} \begin{pmatrix} H_R \\ -H_I \end{pmatrix}$$

which is symmetric. The determinant of the matrix is negative definite and so the system is elliptic.

We next consider a Lorentz material model [40] for an absorbing layer.

$$\begin{aligned} \frac{\partial E_x}{\partial t} + \sigma_y E_x &= \frac{\partial H}{\partial y} - J_x \\ \frac{\partial E_y}{\partial t} + \sigma_x E_y &= -\frac{\partial H}{\partial x} - J_y \\ \frac{\partial H}{\partial t} + (\sigma_x + \sigma_y)H &= \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - K \\ \frac{\partial J_x}{\partial t} &= -\sigma_x \frac{\partial H}{\partial y} \\ \frac{\partial J_y}{\partial t} &= \sigma_y \frac{\partial H}{\partial x} \\ \frac{\partial K}{\partial t} &= \sigma_x \sigma_y H \end{aligned} \tag{4.8}$$

Abarbanel and Gottlieb [2] have noted that if we define $P_x = J_x + \sigma_x E_x$ and $P_y = J_y + \sigma_y E_y$ then the equations for P_x, P_y, K are ordinary differential equations rather than partial differential equations. This saves some computer time and also proves the stabilty of the modified equations. Berenger's PML equations however can have exponential growth under perturbations [1]. Since, we will show that both models yield the same equation for H the instability of the *PML* equations occurs only in H_x and H_y (see [1]). However, only the total field H affects the interior solution and so this instability does not usually appear in computations. Again Fourier transforming in time and eliminating J_x, J_y, K we obtain

$$\begin{aligned} i\omega E_x &= \frac{s_x}{s_y}\frac{\partial H}{\partial y} \\ i\omega E_y &= -\frac{s_y}{s_x}\frac{\partial H}{\partial x} \\ i\omega s_x s_y H &= \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}. \end{aligned} \tag{4.9}$$

s_x, s_y as in (4.4). This is the same as the absorbing layer given in [38] and in a TE formulation [12, 28]. Combining these equations we get

$$\frac{\partial}{\partial x}\left(\frac{s_y}{s_x}\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{s_x}{s_y}\frac{\partial H}{\partial y}\right)+\omega^2 s_x s_y H=0$$

which is identical to (4.5). Thus, both sets of absorbing layers formulations give identical equations, in Fourier space, for the magnetic component H_z. However E_x, E_y differ. By the uniqueness of the Fourier transform we conclude that the solution H_z of both the Berenger PML and the Lorentz model are identical.

We finally consider a third set of equations which is an extension of those suggested by Abarbanel and Gottlieb [2].

$$\begin{aligned}
\frac{\partial E_x}{\partial t}+2\sigma_y E_x+\sigma_y P_y &= \frac{\partial H}{\partial y}\\
\frac{\partial E_y}{\partial t}+2\sigma_x E_y+\sigma_x P_x &= -\frac{\partial H}{\partial x}\\
\frac{\partial H}{\partial t}-\sigma_x' Q_x-\sigma_y' Q_y &= \frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
\frac{\partial P_x}{\partial t} &= \sigma_x E_y & \qquad \frac{\partial P_y}{\partial t} &= \sigma_y E_x\\
\frac{\partial Q_x}{\partial t} &= -\sigma_x Q_x - E_y & \qquad \frac{\partial Q_y}{\partial t} &= -\sigma_y Q_y - E_x
\end{aligned}$$

Fourier transforming in time we get

$$i\omega s_y^2 E_x = \frac{\partial H}{\partial y}\tag{4.11a}$$

$$i\omega s_x^2 E_y = -\frac{\partial H}{\partial x}\tag{4.11b}$$

$$i\omega H = \frac{\partial E_x}{\partial y}-\frac{\partial E_x}{\partial x}-\frac{\sigma_x'}{i\omega s_x}E_y+\frac{\sigma_y'}{i\omega s_y}E_x$$

This leads to the Helmholtz equation

$$\frac{\partial}{\partial x}\left(\frac{1}{s_y^2}\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{s_x^2}\frac{\partial H}{\partial y}\right)++\frac{\sigma_x'}{i\omega s_x^3}\frac{\partial H}{\partial x}+\frac{\sigma_y'}{i\omega s_y^3}\frac{\partial H}{\partial y}+\omega^2 H=0\tag{4.12}$$

Multiplying through by s_x, s_y we get

$$s_x\frac{\partial}{\partial x}\left(\frac{1}{s_y}\frac{\partial H}{\partial x}\right)+s_y\frac{\partial}{\partial y}\left(\frac{1}{s_x}\frac{\partial H}{\partial y}\right)++\frac{\sigma_x' s_y}{i\omega s_x^2}\frac{\partial H}{\partial x}+\frac{\sigma_y' s_x}{i\omega s_y^2}\frac{\partial H}{\partial y}+s_x s_y\omega^2 H=0$$

If σ_x and σ_y are constant then

$$\frac{\partial}{\partial x}\left(\frac{s_x}{s_y}\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{s_y}{s_x}\frac{\partial H}{\partial y}\right)+\frac{\sigma_x' s_y}{(i\omega+\sigma_x)s_x}\frac{\partial H}{\partial x}+\frac{\sigma_y' s_x}{(i\omega+\sigma_y)s_y}\frac{\partial H}{\partial y}+s_x s_y \omega^2 H=0$$

Hence, if we consider constant coefficients σ_x and σ_y and setting $\sigma_x' = \sigma_y' = 0$ this reduces to the previous equation (4.5). However, for variable σ we get a different averaging of the σ's across cells. Furthermore, for $\sigma' \neq 0$ we get lower order terms.

5 Three Dimensions

The Berenger PML layer equations in three Cartesian dimensions are given by [7] (we set $\sigma_x^* = \sigma_x$ etc.)

$$\begin{aligned}
\frac{\partial E_{xy}}{\partial t} + \sigma_y E_{xy} &= \frac{\partial (H_{zx} + H_{zy})}{\partial y} \\
\frac{\partial E_{xz}}{\partial t} + \sigma_z E_{xz} &= -\frac{\partial (H_{yx} + H_{yx})}{\partial z} \\
\frac{\partial E_{yz}}{\partial t} + \sigma_z E_{yz} &= \frac{\partial (H_{xy} + H_{xz})}{\partial z} \\
\frac{\partial E_{yz}}{\partial t} + \sigma_x E_{yx} &= -\frac{\partial (H_{zx} + H_{zy})}{\partial x} \\
\frac{\partial E_{zx}}{\partial t} + \sigma_x E_{zx} &= \frac{\partial (H_{yz} + H_{yx})}{\partial x} \\
\frac{\partial E_{zy}}{\partial t} + \sigma_y E_{zy} &= -\frac{\partial (H_{xy} + H_{xz})}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial H_{xy}}{\partial t} + \sigma_y H_{xy} &= -\frac{\partial (E_{zx} + E_{zy})}{\partial y} \\
\frac{\partial H_{xz}}{\partial t} + \sigma_z H_{xz} &= \frac{\partial (E_{yx} + E_{yx})}{\partial z} \\
\frac{\partial H_{yz}}{\partial t} + \sigma_z H_{yz} &= -\frac{\partial (E_{xy} + E_{xz})}{\partial z} \\
\frac{\partial H_{yz}}{\partial t} + \sigma_x H_{yx} &= \frac{\partial (E_{zx} + E_{zy})}{\partial x} \\
\frac{\partial H_{zx}}{\partial t} + \sigma_x H_{zx} &= -\frac{\partial (E_{yz} + E_{yx})}{\partial x} \\
\frac{\partial H_{zy}}{\partial t} + \sigma_y H_{zy} &= \frac{\partial (E_{xy} + E_{xz})}{\partial y}
\end{aligned}
$$

Note, that if only $\sigma_x \neq 0$ we still require 10 equations and only the split fields E_x and H_x can be eliminated. Fourier transforming in time and eliminating the split fields and assuming σ_x is a function only of x etc. we get

$$s_y s_z i\omega E_x = \frac{\partial (s_z H_z)}{\partial y} - \frac{\partial (s_y H_y)}{\partial z} \tag{5.2a}$$

$$s_x s_z i\omega E_y = \frac{\partial (s_x H_x)}{\partial z} - \frac{\partial (s_z H_z)}{\partial x} \tag{5.2b}$$

$$s_x s_y i\omega E_z = \frac{\partial (s_y H_y)}{\partial x} - \frac{\partial (s_x H_x)}{\partial y} \tag{5.2c}$$

$$s_y s_z i\omega H_x = -\frac{\partial (s_z E_z)}{\partial y} + \frac{\partial (s_y E_y)}{\partial z} \tag{5.2d}$$

$$s_x s_z i\omega H_y = -\frac{\partial (s_x E_x)}{\partial z} + \frac{\partial (s_z E_z)}{\partial x} \tag{5.2e}$$

$$s_x s_y i\omega H_z = -\frac{\partial (s_y E_y)}{\partial x} + \frac{\partial (s_x E_x)}{\partial y} \tag{5.2f}$$

Let $\frac{dJ_j}{dt} = E_j$ and $\frac{dK_j}{dt} = H_j$, $j = 1 \ldots 3$. Transforming back to the physical time domain we have

$$\begin{aligned}
\frac{\partial E_x}{\partial t} + (\sigma_y + \sigma_z)E_x + \sigma_y\sigma_z P_x &= \frac{\partial(H_z + \sigma_z Q_z)}{\partial y} - \frac{\partial(H_y + \sigma_y Q_y)}{\partial z} \\
\frac{\partial E_y}{\partial t} + (\sigma_x + \sigma_z)E_y + \sigma_x\sigma_z P_y &= \frac{\partial(H_x + \sigma_x Q_x)}{\partial z} - \frac{\partial(H_z + \sigma_z Q_z)}{\partial x} \\
\frac{\partial E_z}{\partial t} + (\sigma_x + \sigma_y)E_z + \sigma_x\sigma_y P_z &= \frac{\partial(H_y + \sigma_y Q_y)}{\partial x} - \frac{\partial(H_x + \sigma_x Q_x)}{\partial y} \\
\frac{\partial H_x}{\partial t} + (\sigma_y + \sigma_z)H_x + \sigma_y\sigma_z Q_x &= -\frac{\partial(E_z + \sigma_z P_z)}{\partial y} + \frac{\partial(E_y + \sigma_y P_y)}{\partial z} \\
\frac{\partial H_y}{\partial t} + (\sigma_x + \sigma_z)H_y + \sigma_x\sigma_z Q_y &= -\frac{\partial(E_x + \sigma_x P_x)}{\partial z} + \frac{\partial(E_z + \sigma_z P_z)}{\partial x} \\
\frac{\partial H_z}{\partial t} + (\sigma_x + \sigma_y)H_z + \sigma_x\sigma_y Q_z &= -\frac{\partial(E_y + \sigma_y P_y)}{\partial x} + \frac{\partial(E_x + \sigma_x P_x)}{\partial y}
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\frac{\partial P_x}{\partial t} &= E_x & \frac{\partial P_y}{\partial t} &= E_y & \frac{\partial P_z}{\partial t} &= E_z \\
\frac{\partial Q_x}{\partial t} &= H_x & \frac{\partial Q_y}{\partial t} &= H_y & \frac{\partial Q_z}{\partial t} &= H_z
\end{aligned}$$

For a staggered mesh each J_i, K_i is located at the corresponding E_i, H_i location. We thus have an alternative to the Berenger PML layer. [12] and [33] have slightly different forms of these equations. This alternative has the identical Fourier transform for the physical variables and hence the same solution for the physical variables as the original Berenger layer. This formulation adds six ordinary equations to the Maxwell equations but does not split any of the physical fields.

We next rewrite this in a slightly different way by introducing

$$\begin{aligned}
\hat{E}_x &= E_x + \sigma_x P_x & \hat{E}_y &= E_y + \sigma_y P_y & \hat{E}_z &= E_z + \sigma_z P_z \\
\hat{H}_x &= H_x + \sigma_x Q_x & \hat{H}_y &= H_y + \sigma_y Q_y & \hat{H}_z &= H_z + \sigma_z Q_z
\end{aligned}$$

Then (5.3) becomes

$$\begin{aligned}
\frac{\partial \hat{E}_x}{\partial t} + (\sigma_y + \sigma_z - \sigma_x)\hat{E}_x + (\sigma_x - \sigma_y)(\sigma_x - \sigma_z)P_x &= \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z} \\
\frac{\partial \hat{E}_y}{\partial t} + (\sigma_x + \sigma_z - \sigma_y)\hat{E}_y + (\sigma_y - \sigma_x)(\sigma_y - \sigma_z)P_y &= \frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x} \\
\frac{\partial \hat{E}_z}{\partial t} + (\sigma_x + \sigma_y - \sigma_z)\hat{E}_z + (\sigma_z - \sigma_x)(\sigma_z - \sigma_y)P_z &= \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} \\
\frac{\partial \hat{H}_x}{\partial t} + (\sigma_y + \sigma_z - \sigma_x)\hat{H}_x + (\sigma_x - \sigma_y)(\sigma_x - \sigma_z)Q_x &= -\frac{\partial \hat{E}_z}{\partial y} + \frac{\partial \hat{E}_y}{\partial z} \\
\frac{\partial \hat{H}_y}{\partial t} + (\sigma_x + \sigma_z - \sigma_y)\hat{H}_y + (\sigma_y - \sigma_x)(\sigma_y - \sigma_z)Q_y &= -\frac{\partial \hat{E}_x}{\partial z} + \frac{\partial \hat{E}_z}{\partial x} \\
\frac{\partial \hat{H}_z}{\partial t} + (\sigma_x + \sigma_y - \sigma_z)\hat{H}_z + (\sigma_z - \sigma_x)(\sigma_z - \sigma_y)Q_z &= -\frac{\partial \hat{E}_y}{\partial x} + \frac{\partial \hat{E}_x}{\partial y}
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
\frac{\partial P_x}{\partial t} &= \hat{E}_x - \sigma_x P_x & \frac{\partial P_y}{\partial t} &= \hat{E}_y - \sigma_y P_y & \frac{\partial P_z}{\partial t} &= \hat{E}_z - \sigma_z P_z \\
\frac{\partial Q_x}{\partial t} &= \hat{H}_x - \sigma_x Q_x & \frac{\partial Q_y}{\partial t} &= \hat{H}_y - \sigma_y Q_y & \frac{\partial Q_z}{\partial t} &= \hat{H}_z - \sigma_z Q_z
\end{aligned}$$

We again stress that (5.4) is well-posed because it is a symmetric hyperbolic system (i.e. the Maxwell equations) plus lower order terms. Hence, it is stable to all lower order perturbations. On the other hand (5.1) can have linearly growing solutions for the split fields and perturbations can have exponentially growing modes that depend on the frequency of the initial data (see [1]). In a side layer in the x direction only σ_x is different than zero. Then only P_x and Q_x appear in (5.3). Hence, we need two extra ordinary differential equations in each side layer and six ODEs in a corner. Thus, this version of the PML requires less storage in three dimensions then the original model. If one defines the quantities $\hat{P}_i = P_i - \hat{E}_i$ and $\hat{Q}_i = Q_i - \hat{H}_i$ the one gets the uniaxial layer. See [12, 13, 14, 26, 38, 39].

We briefly consider how to approximate this system by a Yee-type scheme. We locate the new variables P_i at the same location as $\hat{E}_i$ and similarly for Q_i and $\hat{H}_i$. We next write explicit the finite difference scheme for $\hat{E}_x$ and P_x.

$$\begin{aligned}
\frac{\hat{E}_x^{n+1} - \hat{E}_x^n}{\Delta t} + 0.5(\sigma_y + \sigma_z - \sigma_x)(\hat{E}_x^{n+1} + \hat{E}_x^n) + 0.5(\sigma_x - \sigma_y)(\sigma_x - \sigma_z)(\hat{P}_x^{n+1} + \hat{P}_x^n) \\
= \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z}
\end{aligned}$$

$$\frac{P_x^{n+1} - P_x^n}{\Delta t} = 0.5(\hat{E}_x^{n+1} + \hat{E}_x^n) - 0.5\sigma_x * (P_x^{n+1} + P_x^n)$$

where the spatial derivatives are approximated by some difference formula. This gives us two equations for two unknowns. We solve this for $\hat{E}_x^{n+1}$ and P_x^{n+1}. Let

$$A = 1 + 0.5\Delta t(\sigma_y + \sigma_z - \sigma_x) \qquad B = 1 - 0.5\Delta t(\sigma_y + \sigma_z - \sigma_x)$$
$$C = -0.5\Delta t \qquad D = 1 + 0.5\sigma_x \Delta t$$

then the discriminant is given by $G = (1 + 0.5\sigma_y \Delta t)(1 + 0.5\sigma_z \Delta t)$.
Define

$$F_1 = 1 - 0.5\Delta t(\sigma_y + \sigma_z - \sigma_x)\hat{E}_x^n - 0.5(\sigma_x - \sigma_y)(\sigma_x - \sigma_z)\hat{P}_x^n + \frac{\partial \hat{H}_z}{\partial y} - \frac{\partial \hat{H}_y}{\partial z}$$
$$F_2 = 0.5\Delta t(\hat{E}_x^n - \sigma_x P_x^n).$$

Then we have

$$\hat{E}^{n+1} = \frac{DF_1 - BF_2}{G} \qquad P^{n+1} = \frac{AF_2 - CF_1}{G}$$

Similar formulas hold for the other variables.

We can also write equations (5.2a-5.2f) in vector form as

$$\begin{aligned} s_x s_y s_z i\omega\left(\frac{E_x}{s_x}, \frac{E_y}{s_y}, \frac{E_z}{s_z}\right) &= curl(s_x H_x, s_y H_y, s_z H_z) \\ s_x s_y s_z i\omega\left(\frac{H_x}{s_x}, \frac{H_y}{s_y}, \frac{H_z}{s_z}\right) &= -curl(s_x E_x, s_y E_y, s_z E_z). \end{aligned} \tag{5.5}$$

Let S, R be the 3×3 matrices $S = diag(s_x, s_y, s_z)$ and $R = (s_y s_z, s_x s_z, s_x s_y)$. Then (5.5) is equivalent to

$$\begin{aligned} i\omega R\vec{E} &= curl(S\vec{H}) \\ i\omega R\vec{H} &= -curl(S\vec{E}). \end{aligned} \tag{5.6}$$

We substitute the first three equations (5.2a-5.2c) in the last three equations (5.2d-5.2f). We then clear fractions to obtain

$$\begin{aligned} -s_y s_x \omega^2 H_x &= -\frac{\partial}{\partial y}\left(\frac{s_z}{s_x}\frac{\partial H_y}{\partial x} - \frac{s_z}{s_y}\frac{\partial H_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{s_y}{s_z}\frac{\partial H_x}{\partial z} - \frac{s_y}{s_x}\frac{\partial H_z}{\partial x}\right) \\ -s_x s_z \omega^2 H_y &= -\frac{\partial}{\partial z}\left(\frac{s_x}{s_y}\frac{\partial H_z}{\partial y} - \frac{s_x}{s_z}\frac{\partial H_y}{\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{s_z}{s_x}\frac{\partial H_y}{\partial x} - \frac{s_z}{s_y}\frac{\partial H_x}{\partial y}\right) \\ -s_x s_y \omega^2 H_z &= -\frac{\partial}{\partial x}\left(\frac{s_y}{s_z}\frac{\partial H_x}{\partial z} - \frac{s_y}{s_x}\frac{\partial H_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{s_x}{s_y}\frac{\partial H_z}{\partial y} - \frac{s_x}{s_z}\frac{\partial H_y}{\partial z}\right) \end{aligned}$$

In vector form we have

$$-\omega^2 R\vec{H} = -curl\left(SR^{-1}curl(S\vec{H})\right). \tag{5.7}$$

For the Lorentz material in three dimensions we introduce two diagonal 3×3 matrices D_1, D_2 in Fourier space. We then replace (4.9) (see also [28, 41]) by

$$\begin{aligned}
i\omega D_1\vec{E} &= curl(\vec{H})\\
i\omega D_2\vec{H} &= -curl(\vec{E}).
\end{aligned} \tag{5.8}$$

With this assumption we can no longer duplicate the system (5.7) as was done in two dimensions. However, the two sets are equivalent when we introduce new variables as in done in (5.11) (see [12, 35]). Substituting the first equation into the second we get

$$-curl\left(D_1^{-1}curl(\vec{H})\right) + \omega^2 D_2\vec{H} = 0. \tag{5.9}$$

This is a three dimensional extension of (4.5). Let $D_1 = D_2 = diag(d_1, d_2, d_3)$. Denote the x component of the curl by $[curl]_x$ then we rewrite (5.8) in Cartesian components as

$$\begin{array}{lll}
d_1 i\omega\vec{E}_x = [curl(\vec{H})]_x & d_2 i\omega\vec{E}_y = [curl(\vec{H})]_y & d_3 i\omega\vec{E}_z = [curl(\vec{H})]_z\\
d_1 i\omega\vec{H}_x = -[curl(\vec{E})]_x & d_2 i\omega\vec{H}_y = -[curl(\vec{E})]_y & d_3 i\omega\vec{H}_z = -[curl(\vec{E})]_z
\end{array}$$

Set $d_i = \frac{s_j s_k}{s_i}$ as (i,j,k) cycle from 1 to 3. We recover the two dimensional case when $s_z = 1$. But $i\omega s_x s_y = i\omega + \sigma_x + \sigma_y + \frac{\sigma_x\sigma_y}{i\omega}$. Hence, we clear fractions and transform back to the physical time domain and get

$$
\begin{aligned}
&\frac{\partial E_x}{\partial t} + (\sigma_y + \sigma_z)E_x + F_x = [curl(\vec{H})]_x - J_x \\
&\frac{\partial F_x}{\partial t} = \sigma_y \sigma_z E_x \qquad\qquad \frac{\partial J_x}{\partial t} = -\sigma_x [curl(\vec{H})]_x \\
&\frac{\partial E_y}{\partial t} + (\sigma_x + \sigma_z)E_y + F_y = [curl(\vec{H})]_y - J_y \\
&\frac{\partial F_y}{\partial t} = \sigma_x \sigma_z E_y \qquad\qquad \frac{\partial J_y}{\partial t} = -\sigma_y [curl(\vec{H})]_y \\
&\frac{\partial E_z}{\partial t} + (\sigma_x + \sigma_y)E_z + F_z = [curl(\vec{H})]_z - J_z \\
&\frac{\partial F_z}{\partial t} = \sigma_x \sigma_y E_y \qquad\qquad \frac{\partial J_z}{\partial t} = -\sigma_z [curl(\vec{H})]_z
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&\frac{\partial H_x}{\partial t} + (\sigma_y + \sigma_z)H_x + G_x = -[curl(\vec{E})]_x - K_x \\
&\frac{\partial G_x}{\partial t} = \sigma_y \sigma_z H_x \qquad\qquad \frac{\partial K_x}{\partial t} = \sigma_x [curl(\vec{E})]_x \\
&\frac{\partial H_y}{\partial t} + (\sigma_x + \sigma_z)H_y + G_y = -[curl(\vec{E})]_y - K_y \\
&\frac{\partial G_y}{\partial t} = \sigma_x \sigma_z H_y \qquad\qquad \frac{\partial K_y}{\partial t} = \sigma_y [curl(\vec{E})]_y \\
&\frac{\partial H_z}{\partial t} + (\sigma_x + \sigma_y)H_z + G_z = -[curl(\vec{E})]_z - K_z \\
&\frac{\partial G_z}{\partial t} = \sigma_x \sigma_y H_z \qquad\qquad \frac{\partial K_z}{\partial t} = \sigma_z [curl(\vec{E})]_z
\end{aligned}
$$

The functions F and G appear only within corners. We seem to have added 12 new functions. However, within a layer only a single σ is non-zero and so only 2 new functions are added (one J and one K). In a two dimensional corner 6 functions are needed and only in a three dimensional corner are all 12 needed.

Similar to the suggestion of Gottlieb and Abarbanel we rearrange these equation so that all the new equations are ordinary differential equations. This also has the benefit that we need only one new variable per equation rather than the two used above. Let

$$
\begin{aligned}
&\hat{P}_x = F_x + \sigma_x E_x + J_x \qquad \hat{P}_y = F_y + \sigma_y E_y + J_y \qquad \hat{P}_z = F_z + \sigma_z E_z + J_z \\
&\hat{Q}_x = G_x + \sigma_x H_x + K_x \qquad \hat{Q}_y = G_y + \sigma_y H_y + K_y \qquad \hat{Q}_z = G_z + \sigma_z H_z + K_z
\end{aligned}
\tag{5.11}
$$

Then (5.10) becomes

$$
\begin{aligned}
\frac{\partial E_x}{\partial t} + (\sigma_y + \sigma_z - \sigma_x)E_x + \hat{P}_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\
\frac{\partial E_y}{\partial t} + (\sigma_x + \sigma_z - \sigma_y)E_y + \hat{P}_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\
\frac{\partial E_z}{\partial t} + (\sigma_x + \sigma_y - \sigma_z)E_z + \hat{P}_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \\
\frac{\partial H_x}{\partial t} + (\sigma_y + \sigma_z - \sigma_x)H_x + \hat{Q}_x &= -\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} \\
\frac{\partial H_y}{\partial t} + (\sigma_x + \sigma_z - \sigma_y)H_y + \hat{Q}_y &= -\frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x} \\
\frac{\partial H_z}{\partial t} + (\sigma_x + \sigma_y - \sigma_z)H_z + \hat{Q}_z &= -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\frac{\partial \hat{P}_x}{\partial t} + \sigma_x \hat{P}_x &= (\sigma_x - \sigma_y)(\sigma_x - \sigma_z)E_x & \qquad \frac{\partial \hat{P}_y}{\partial t} + \sigma_y \hat{P}_y &= (\sigma_y - \sigma_x)(\sigma_y - \sigma_z)E_y \\
\frac{\partial \hat{P}_z}{\partial t} + \sigma_z \hat{P}_z &= (\sigma_z - \sigma_z)(\sigma_z - \sigma_y)E_z & \qquad \frac{\partial \hat{Q}_x}{\partial t} + \sigma_x \hat{Q}_x &= (\sigma_x - \sigma_y)(\sigma_x - \sigma_z)H_x \\
\frac{\partial \hat{Q}_y}{\partial t} + \sigma_y \hat{Q}_y &= (\sigma_y - \sigma_x)(\sigma_y - \sigma_z)H_y & \qquad \frac{\partial \hat{Q}_z}{\partial t} + \sigma_z \hat{Q}_z &= (\sigma_z - \sigma_z)(\sigma_z - \sigma_y)H_z
\end{aligned}
$$

This is the same system as derived by Ziolkowsky [41] except for the normalization of $\hat{P}_i, \hat{Q}_i$. Comparing this with (5.4) we see that the modified Berenger model, the uniaxial model and the Lorentz model are identical but with different artificial variables (see also [12, 35]). Hence, their properties are similar. In particular the uniaxial and Lorentz models are well posed mathematically and stable under lower order perturbations. The Berenger model is only weakly well-posed because of the split equations. However, this weak instability is rarely seen in practice and so all behave in a similar fashion. The differences are mainly due to changes in the numerical implementation of these conditions.

At the back end of the PML one usually specificies boundary conditions for a perfectly conducting media. One can improve this by using characteristic boundary conditions (for a nonstaggered mesh) or a accurate nonrflecting boundary condition. Under most circumstances the reflection off the back of the PML layer is less than the truncation error of the scheme. Hence, one would expect that the boundary conditions at the end of the layer are not very important [2, 23].

6 Polar Coordinates

Having reformulated the PML layer in vector form we can now find the formulation for cylinderical coordinates. We assume that the domain is periodic in θ and so there are no layers in the θ direction. So $s_r = s_r(r)$, $s_\theta = 1$ and $s_z = s_z(z)$. We first write the general expression for the curl in cylinderical coordinates for any vector A.

$$curl(s_r A_r, A_\theta, s_z A_z) = \left(\frac{s_z}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(s_r \frac{\partial A_r}{\partial z} - s_z \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - s_r \frac{\partial A_r}{\partial \theta}\right)\hat{k}$$

We can express the time dependent version of (5.6) in cylinderical component form as

$$\begin{aligned}
\frac{\partial E_r}{\partial t} + \sigma_z E_r &= \frac{1}{r}\frac{\partial (H_z + \sigma_z K_z)}{\partial \theta} - \frac{\partial H_\theta}{\partial z} \\
\frac{\partial E_\theta}{\partial t} + (\sigma_r + \sigma_z)E_\theta + \sigma_r \sigma_z J_\theta &= \frac{\partial (H_r + \sigma_r K_r)}{\partial z} - \frac{\partial (H_z + \sigma_z K_z)}{\partial r} \\
\frac{\partial E_z}{\partial t} + \sigma_r E_z &= \frac{1}{r}\left(\frac{\partial r H_\theta}{\partial r} - \frac{\partial (H_r + \sigma_r K_r)}{\partial \theta}\right) \\
\frac{\partial H_r}{\partial t} + \sigma_z H_r &= -\frac{1}{r}\frac{\partial (E_z + \sigma_z J_z)}{\partial \theta} + \frac{\partial E_\theta}{\partial z} \\
\frac{\partial H_\theta}{\partial t} + (\sigma_r + \sigma_z)H_\theta + \sigma_x \sigma_z K_\theta &= -\frac{\partial (E_r + \sigma_r J_r)}{\partial z} + \frac{\partial (E_z + \sigma_z J_z)}{\partial r} \\
\frac{\partial H_z}{\partial t} + \sigma_r H_z &= -\frac{1}{r}\left(\frac{\partial (r E_\theta)}{\partial r} - \frac{\partial (E_r + \sigma_r J_r)}{\partial \theta}\right)
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
\frac{\partial J_r}{\partial t} &= E_r & \frac{\partial J_\theta}{\partial t} &= E_\theta & \frac{\partial J_z}{\partial t} &= E_z \\
\frac{\partial K_r}{\partial t} &= H_r & \frac{\partial K_\theta}{\partial t} &= H_\theta & \frac{\partial K_z}{\partial t} &= H_z
\end{aligned}$$

In two dimensional polar coordinates this simplifies to (TE mode)

$$\begin{aligned}
\frac{\partial E_r}{\partial t} &= \frac{1}{r}\frac{\partial H_z}{\partial \theta} \\
\frac{\partial E_\theta}{\partial t} + \sigma_r E_\theta &= -\frac{\partial H_z}{\partial r} \\
\frac{\partial H_z}{\partial t} + \sigma_r H_z &= -\frac{1}{r}\left(\frac{\partial (r E_\theta)}{\partial r} - \frac{\partial (E_r + \sigma_r J_r)}{\partial \theta}\right) \\
\frac{\partial J_r}{\partial t} &= E_r
\end{aligned}$$

This system has the same Fourier transform as the split field systems of Navarro et.al. [20] and Rappaport [25]. The derivation of the PML properties depended on the use of plane waves reaching the interface. For polar coordinates we replace this analysis with cylinderical waves. The metrics now contain variable coefficients which complicate the theory. One finds that there are significant reflections from this PML layer.

There have been several other extensions of the PML algorithm to cylinderical and general curvilinear coordinates, see [26, 30, 31, 18, 22]. Here we present the the development of Yang et.al. [36] for an extension of the PML layer to polar coordinates. This has the advantage that it is constructed in order to be a well posed system of equations.

$$
\begin{aligned}
\frac{\partial E_r}{\partial t} + \frac{\sigma_r}{r} E_r &= \frac{1}{r}\frac{(\partial H_{zr} + H_{z\theta)}}{\partial \theta} \\
\frac{\partial E_\theta}{\partial t} + \sigma_r' E_\theta &= -\frac{\partial (H_{zr} + H_{z\theta})}{\partial r} \\
\frac{\partial H_{zr}}{\partial t} + \frac{\sigma_r}{r} H_{zr} &= \frac{1}{r}\frac{\partial E_r}{\partial \theta} - \frac{E_\theta}{r} \\
\frac{\partial H_{z\theta}}{\partial t} + \sigma_r' H_{z\theta} &= -\frac{\partial E_\theta}{\partial r}
\end{aligned} \tag{6.2}
$$

σ_r' represents the derivative of σ_r with respect to r. If we translate their system to the unsplit version we get

$$
\begin{aligned}
\frac{\partial E_r}{\partial t} + \frac{\sigma_r}{r} E_r &= \frac{1}{r}\frac{\partial H_z}{\partial \theta} \\
\frac{\partial E_\theta}{\partial t} + \sigma_r' E_\theta &= -\frac{\partial H_z}{\partial r} \\
\frac{\partial H_z}{\partial t} + (\frac{\sigma_r}{r} + \sigma_r') H_z + \frac{\sigma_r \sigma_r'}{r} Q &= -\frac{1}{r}\left(\frac{\partial (r E_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right) + (\frac{\sigma_r'}{r} P_r - \frac{\sigma_r}{r} P_\theta) \\
\frac{\partial Q}{\partial t} = H_z \qquad \frac{\partial P_r}{\partial t} &= \frac{\partial E_r}{\partial \theta} - E_\theta \qquad \frac{\partial P_\theta}{\partial t} = E_\theta
\end{aligned} \tag{6.3}
$$

σ' is the derivative of σ with respect to r. In contrast to the Cartesian case one of the additional equations is a partial differential equation instead of just ordinary differential equations. This requires three extra functions which makes (6.3) more costly than the split version of Yang et.al (6.2). A different approach for general curvilinear coordinates is given in [11].

We also present the PML layer, in polar coordinates suggested by Collino and Monk [11].

$$\begin{aligned}
\frac{\partial E_r}{\partial t} + \overline{\sigma} E_r &= \frac{1}{r}\frac{\partial H_z}{\partial \theta} \\
\frac{\partial E_\theta}{\partial t} + \sigma E_\theta &= -\frac{\partial H_z}{\partial r} \\
\frac{\partial H_z}{\partial t} + \overline{\sigma} H_z + (\sigma - \overline{\sigma})Q &= -\frac{1}{r}\left(\frac{\partial (rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right) \\
\frac{\partial Q}{\partial t} + \sigma Q &= -\frac{\partial E_\theta}{\partial r}
\end{aligned}$$

where $\sigma = (r\overline{\sigma})_r$.

7 Acoustics

The linearized two dimensional acoustic equations without a mean flow are given by

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial t} &= -\rho_0 c_0 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})
\end{aligned} \tag{7.1}$$

p is a perturbation of the pressure and u,v are components of the perturbed velocity. This is equivalent to the *TE* equations (4.1) under the transformation $t \to c_0 t$ and

$$E_x \to -v \quad E_y \to u \quad H \to \frac{1}{\rho_0 c_0} p$$

Using either the Berenger or Ziolkowski layers and Fourier transforming in time we can translate the Helmholtz-like equation (4.5) into acoustic notation as

$$\frac{\partial}{\partial x}\left(\frac{s_x}{s_y}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{s_y}{s_x}\frac{\partial p}{\partial y}\right) + k^2 s_x s_y p = 0 \tag{7.2}$$

where $s_x = 1 + \frac{\sigma_x}{ik}$, $s_y = 1 + \frac{\sigma_y}{ik}$. This reduces to the Helmholtz equation when $\sigma_x = \sigma_y = 0$.

The two dimensional version of the new Berenger PML in the x side layer under this transformation becomes

$$
\begin{aligned}
\frac{\partial u}{\partial t} + c_0\sigma_x u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial t} + c_0\sigma_x p &= -\rho_0 c_0^2(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \rho_0 c_0^2 \sigma_x \frac{\partial Q}{\partial y} \\
\frac{\partial Q}{\partial t} &= c_0 v
\end{aligned} \tag{7.3}
$$

We next consider the acoustic equations with a constant mean flow. We choose the x direction as the direction of this mean flow. $(U_0, 0)$. The acoustic equations then become

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U_0\frac{\partial u}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + U_0\frac{\partial v}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial t} + U_0\frac{\partial p}{\partial x} &= -\rho_0 c_0^2(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}).
\end{aligned} \tag{7.4}
$$

This yieldss the convected wave equation for p

$$p_{tt} + 2U_0 p_{xt} - (c_0^2 - U_0^2)p_{xx} - c_0^2 p_{yy} = 0 \tag{7.5}$$

Define $M = \frac{U_0}{c_0}$. We change variables to reduce the convective wave equation to a standard wave equation. We choose the change of variables suggested in [4]:

$$x = (1 - M^2)^{1/2}\xi \quad , \quad t = (1 - M^2)^{-1/2}(\tau - \frac{M}{c_0}\xi) \tag{7.6}$$

and so

$$w_\tau = (1 - M^2)^{-1/2} w_t \quad , \quad w_\xi = -\frac{M}{c_0}(1 - M^2)^{-1/2} w_t + (1 - M^2)^{1/2} w_x$$

$$w_t = (1 - M^2)^{1/2} w_\tau \quad , \quad w_x = \frac{M}{c_0}(1 - M^2)^{-1} w_\tau + (1 - M^2)^{-1/2} w_\xi$$

This reduces (7.5) to the wave equation

$$p_{\tau\tau} = c_0^2(p_{\xi\xi} + p_{yy}).$$

We now use (7.3) as a boundary condition in (ξ, y, τ) variables and then use (7.6) to transform to (x, y, t) variables. The results is

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{M}{\rho_0 c_0}\frac{\partial p}{\partial t} + (1-M^2)^{1/2} c_0 \sigma_x u &= -\frac{1-M^2}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} &= -\frac{(1-M^2)^{1/2}}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial t} + \rho_0 c_0 M \frac{\partial u}{\partial t} + (1-M^2)^{1/2} c_0 \sigma_x p &= \\
-\rho_0 c_0^2 \left[(1-M^2)\frac{\partial u}{\partial x} + (1-M^2)^{1/2} (\frac{\partial v}{\partial y} + \frac{\partial J}{\partial y} \right] \\
\frac{\partial J}{\partial t} &= c_0 (1-M^2)^{1/2} v
\end{aligned}
$$

This boundary condition is different than that of Hu [16] in two respects. First it is based on (5.3) rather than the split field formulation used by Hu. Second the extension to include the mean field is done in a completely different manner. In this treatment we have used the transformation on the entire system. Another possibility is to split the system into its convective and acoustic parts [27]. One then applies the above method only to the acoustic portion of the equations. One then specifies the incoming data and absorb the outgoing variables. The convective parts are scalar equations and sothe numerical reflection from an outgoing wave with a nonreflecting boundary condition is minimal.

A different approach is that of Abarbanel and Gottlieb described above. This is a formal mathematical procedure without any physical justification. Nondimensionalizing (7.4) one gets

$$
\begin{aligned}
\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + M\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} &= 0 \\
\frac{\partial p}{\partial t} + M\frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{aligned}
\tag{7.7}
$$

Where M is the Mach number of the mean flow. We are only considering the case that the mean flow is along the x axis, i.e. we consider flow at zero degrees angle of attack. We now choose a different transformation than above to simplify the equations, i.e.

$$\xi = x \qquad \eta = \sqrt{1-M^2}y = \gamma y \qquad \tau = Mx + \gamma^2 t \tag{7.8}$$

The system (7.7) now becomes

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{M}{\gamma}\frac{\partial v}{\partial \eta} + \frac{\partial p}{\partial \xi} &= 0 \\ \frac{\partial v}{\partial \tau} + M\frac{\partial v}{\partial \xi} + \gamma\frac{\partial p}{\partial \eta} &= 0 \\ \frac{\partial p}{\partial \tau} + \frac{\partial u}{\partial \xi} + \frac{1}{\gamma}\frac{\partial v}{\partial \eta} &= 0. \end{aligned} \tag{7.9}$$

Abarbanel and Gottlieb et.al. [3] construct the following system to be used in all absorbing layers, inflow, outflow and and in the direction normal to the flow.

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{M}{\gamma}\frac{\partial v}{\partial \eta} + \frac{\partial p}{\partial \xi} &= -\sigma_x u \\ \frac{\partial v}{\partial \tau} + M\frac{\partial v}{\partial \xi} + \gamma\frac{\partial p}{\partial \eta} &= -\sigma_x v + 2\sigma_x Q + \sigma_x^2 \Psi + \sigma_x' MP - 2\sigma_y v + \sigma_y^2 R + 2\sigma_y \nu + \sigma_y' \Gamma \\ \frac{\partial p}{\partial \tau} + \frac{\partial u}{\partial \xi} + \frac{1}{\gamma}\frac{\partial v}{\partial \eta} &= -\sigma_x p \\ \frac{\partial Q}{\partial \tau} &= -\gamma\frac{\partial p}{\partial \eta} \\ \frac{\partial P}{\partial \tau} &= v - \sigma_x P \\ \frac{\partial \Psi}{\partial \tau} &= Q \\ \frac{\partial \nu}{\partial \tau} &= -M\frac{\partial v}{\partial \xi} \\ \frac{\partial \Gamma}{\partial \tau} &= \gamma p - \sigma_y \Gamma \\ \frac{\partial R}{\partial \tau} &= M\nu - v \end{aligned}$$

Or in physical coordinates

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -\sigma_x(u + Mp) \\
&\frac{\partial v}{\partial t} + M\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = -\sigma_x v + 2\sigma_x Q + \sigma_x^2\Psi + \sigma_x' MP - 2\sigma_y v + \sigma_y^2 R + 2\sigma_y \nu + \sigma_y'\Gamma \\
&\frac{\partial p}{\partial t} + M\frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\sigma_x(Mu + p) \\
&\qquad\frac{\partial Q}{\partial t} + \gamma^2\frac{\partial p}{\partial y} = 0 \\
&\qquad\frac{\partial P}{\partial t} = \gamma^2(v - \sigma_x P) \\
&\qquad\frac{\partial \Psi}{\partial t} = \gamma^2 Q \\
&\frac{\partial \nu}{\partial t} + M^2\frac{\partial v}{\partial t} + \gamma^2 M\frac{\partial v}{\partial x} = 0 \\
&\qquad\frac{\partial \Gamma}{\partial t} = \gamma^2(\gamma p - \sigma_y\Gamma) \\
&\qquad\frac{\partial R}{\partial t} = \gamma^2(M\nu - v)
\end{aligned}
\tag{7.10}
$$

σ_x is a function of only x and σ_y is a function of only y. The prime signifies a derivative of σ with respect to its argument. After a fairly convoluted analysis the above system is found to represent solutions which are superpositions of plane waves in the free space computational domain and a superposition of decaying plane waves in the absorbing layers in all four directions and the corners. This set can be shown to be well-posed and nonreflecting for infinite absorbing layers. Note, that for $M = 0$ we do not recover any of the standard nonreflecting layers for the nonconvective acoustic equations (equivalent in two dimensions to the Maxwell equations). If we add and subtract the first and third equation we get characteristic equations in the x direction for $p \pm u$ and additional lower order terms. The second equation for v is already in characterstic form. Thus, we can combine the characteristic approach with the PML layer.

Several computations have been reported in extensions to acoustics (see [15] and [16]). These extensions seem to work well for linear acoustics with a mean flow but there is a tremendous degradation compared with CEM codes. For the Maxwell equations one typically finds that the PML layer can reduce reflections by five orders of magnitude. For acoustics standard techniques reduce the reflection from the outer boundary to about one percent of the energy reaching the outer surface. The PML layer of Hu reduces this by about another factor of two or three. Thus the reflection is still larger than 0.1 percent of the incoming signal. For the nonlinear acoustic equations (i.e. Euler equations) the reflections are considerably larger and there is still much work that needs to be done to improve the efficiency of the method.

8 Computations

We shall only consider the time dependent two dimensional Maxwell equations for our numerical computations. There have been several works that have also applied the PML layer to the Helmholtz equation and to linear acoustics [16, 15]. For the Helmholtz equation there are three basic alternatives. Either a local boundary condition based on asymptotic equations [4, 5], a global integral condition such as the DtN condition or else infinite element methods. Both the DtN and infinite elements work best when the outer boundary is a circle or ellipse. On the other hand the PML method works best in Cartesian coordinates. Hence, to some extent these different approaches are complementary.

We recapitulate the boundary conditions that have been presented for the Maxwell Equations in Cartesian coordinates. The original suggestion of Berenger involved the splitting of some of the fields into non-physical subcomponents. The two dimensional version is presented in (4.2) and the three dimensional version in (5.1). We then Fourier transformed the Berenger system and solved explicitly for the physical components. Transforming back to physical space yielded the system (5.3). This system could be rewritten so that the leading order terms are the original Maxwell equations and the additional equations are all ordinary differential equations. This is presented by system (5.4). A different approach was suggested by Ziolkowsky who built a model based on a physical Lorentz system. The two dimensional version is presented in (4.8) and our three dimensional extension by (5.12). These have already been rewritten so that they consist of the Maxwell equations plus lower order terms and additional ordinary differential equations. Finally, Abrbanel and Gottlieb started from the mathematical viewpoint that we wish to describe the most general system which can be expressed as the Maxwell equations plus lower order terms and additional ordinary differential equations. They then chose coefficients of the lower order terms to maximize the absorption properties of the PML layer. A two dimensional extension of their system is presented in (4.10).

We stress that all the Berenger-type layers and the Lorentz model layers satisfy the identical equations for the Fourier transform of the physical variables. Nevertheless, the time behavior of these systems is not identical because of the different ways of intoducing the nonphysical variables. In particular Abarbanel and Gottlieb have shown that the original split field decomposition of Berenger can lead to an ill-posed system. In practice this ill-posed and consequent exponential growth only is observed in practice in extreme situations. All the systems that can be written as the Maxwell equations plus lower order terms and additional ordinary differential equations are automatically well-posed as they form a symmetric hyperbolic system. The system intoduced by Abarbanel and Gottlieb is inherently different than the other systems.

Some of the systems e.g. (5.1) and (5.4) differ only with respect to the way they are written. Thus, in (5.4) we rewrite the previous system so that only ordinary differential equations are added. Nevertheless, on the numerical level these may lead to different discretizations. In particular if a staggered grid is used coupled with a Yee-like scheme the different formulations lead to a different placement of the artificial variables.

We shall now compare some of the previous formulations for some simple two dimensional configurations based on the Maxwell equations. We consider a monochromatic isotropic point source of wavelength 0.25, that is switched on at $t = 0$. The domain is $0 \leq x, y \leq 0.4$. For Yee's scheme we choose $h = \frac{1}{40}$, $\Delta t = \frac{2h}{3} = \frac{1}{60}$, for Ty(4,4): $h = \frac{1}{40}$, $\Delta t = \frac{h}{4} = \frac{1}{100}$ while for Ty(2,4) $h = \frac{1}{40}$, $\Delta t = \frac{h}{18} = \frac{1}{720}$. The point source is modeled by adding a term representing a current $I_z(t) = 0.01 \sin(8\pi t)\epsilon(t)$ at $(x, y) = (\frac{1}{10}, \frac{1}{10})$ where $\epsilon(t)$ denotes the Heaviside unit-step function. For the pulse under consideration, the radiated field is the solution of

$$\partial_x^2 E_z + \partial_y^2 E_z - \partial_t^2 E_z = Z\partial_t I_z(t)\delta(r - r_s) \tag{8.1}$$

The solution consists of rotationally symmetric outgoing waves. The exact solution is

$$E_z(r,t) = -\frac{Z}{2\pi}\int_0^\infty \frac{\partial_t \boldsymbol{I_z(t - \sqrt{|(r - r_s|^2 + \xi^2)}}}{\sqrt{|(r - r_s|^2 + \xi^2}} d\xi$$

We use three different methods to solve the two dimensional Maxwell Equations. All the methods use the same staggered placement of the elecric and magnetic components. The first method is the Yee scheme. This is a central difference in space and Leapfrog is time all based on the staggered mesh. It is second order accurate in space and time. The second method (Ty(2,4)) adds an implicit operator to the calculation of all space derivatives. This implicit operator is constructed so that the method is fourth order accurate in space. Since the same staggered in time Leapfrom method is still used the scheme is second order accurate in time. Hence, small time steps are used to improve the time accuracy relative to the space accuracy. The final method (called Ty(4,4)) uses the same implicit staggered space scheme as the second method. Now, however, the leapfrog scheme is replaced by a fourth order accurate Runge-kutta method. Hence, the scheme is now uniformly fourth order accurate (see [32] for more details).

The leapfrog scheme in time for the two dimensional version of (5.4) is given by

$$
\begin{aligned}
&\frac{E_z^{n+1}-E_z^n}{\Delta t}+\frac{\sigma_x+\sigma_y}{2}(E_z^{n+1}+E_z^n)+\frac{\sigma_x\sigma_y}{2}(P_z^{n+1}+P_z^n)=\delta_x H_y^{n+1/2}-\delta_y H_x^{n+1/2}\\
&\frac{H_x^{n+3/2}-H_x^{n+1/2}}{\Delta t}+\frac{\sigma_y-\sigma_x}{2}(H_x^{n+3/2}+H_x^{n+1/2})-\frac{\sigma_x(\sigma_x-\sigma_y)}{2}(Q_x^{n+3/2}+Q_x^{n+1/2})=-\delta_y E_z^n\\
&\frac{H_y^{n+3/2}-H_y^{n+1/2}}{\Delta t}+\frac{\sigma_x-\sigma_y}{2}(H_y^{n+3/2}+H_y^{n+1/2})-\frac{\sigma_x(\sigma_x-\sigma_y)}{2}(Q_y^{n+3/2}+Q_y^{n+1/2})=\delta_x E_z^n
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
\frac{P_z^{n+1}-P_z^n}{\Delta t}&=\frac{1}{2}(E_z^{n+1}+E_z^n)\\
\frac{Q_x^{n+3/2}-Q_x^{n+1/2}}{\Delta t}&=\frac{1}{2}(H_x-\sigma_x Q_x)^{n+3/2}+\frac{1}{2}(H_x-\sigma_x Q_x)^{n+1/2}\\
\frac{Q_y^{n+3/2}-Q_y^{n+1/2}}{\Delta t}&=\frac{1}{2}(H_y-\sigma_y Q_x)^{n+3/2}+\frac{1}{2}(H_x-\sigma_y Q_x)^{n+1/2}
\end{aligned}
$$

Where E_z, P_z are at (i,j), H_x, Q_x are at $(i, j+1/2)$ and H_y, Q_y are at $(i+1/2, j)$. each pair is also at the same time level. This requires the trivial solution of a 2×2 system for each pair. Alternatively we could stagger Q and H in time. In this case we can explicitly advance each variable in the correct order but the analysis is more difficult. Here

Yee:

$$
\begin{aligned}
\delta_x U_{i,j}&=U_{i+1/2,j}-U_{i-1/2,j}\\
\delta_y U_{i,j}&=U_{i,j+1/2}-U_{i,j+1/2}
\end{aligned}
\tag{8.3}
$$

U is evaluated at the appropriate time and space location.

For the fourth order implicit scheme we replace the approximation of the space derivative (8.3) by: **Ty(2,4):**

$$
\delta_y U:\ \frac{U_{i,j+1}-U_{i,j}}{\Delta y}=
\tag{8.4}
$$

$$
\left(\frac{11}{12}\frac{\partial U}{\partial y}\right)_{i,j+1/2}+\frac{1}{24}\left[\left(\frac{\partial U}{\partial y}\right)_{i,j+3/2}+\left(\frac{\partial U}{\partial y}\right)_{i,j-1/2}\right].
$$

Finally for the Ty(4,4) we replace the leapfrog in time by a fourth order Runge-Kutta method. We simplify the classical scheme and save storage since the problems are linear in time.

We condcuted a series of numerical experiments comparing the PML schemes described above. The parameter σ decreases quadratically from 0 to σ_m. We plotted the error as a function of time for different values of σ_m. This enabled us to find computational the optimal σ_m. We found that it depends both on the numerical scheme and on the PML model being used. Furthermore, some PML models are relatively insensitive to the value of σ_m while others are more sensitive. In addition we compared the error levels that can be achieved for various schemes coupled with the different PML models. The best error level achieved for each case is summarized in Table 1. We do not display the graphs here due to space restrictions.

From these tests cases it seems that the Berenger and uniaxial layers are consistently the models that yield the lowest errors. In addition they seem to be the most insensitive to the value of σ_m. Nevertheless, Abarbanel and Gottlieb [1] have shown that Berenger's method is only weakly well-posed and so might lead to difficulties. The tests runs conducted here and similar results in other works have shown that in general these possible problems with the Berenger PML do no arise. However, when we used an extremely thick PML layer we observe 2 that the Berenger PML caused the calculation to become unstable while the other methods did not have this problem.

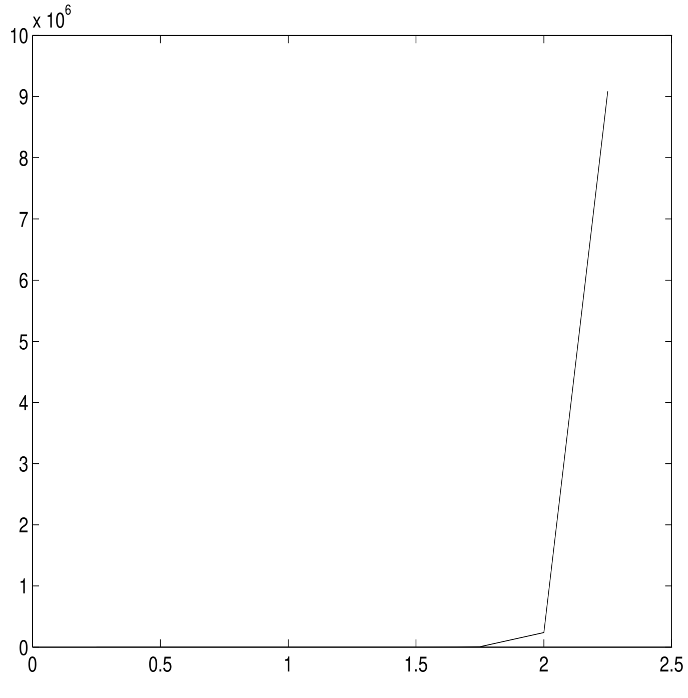

Figure 2: Yee & Berenger PML as a function of cell thickness

scheme	cells	Berenger	Turkel	Ziolkowsky	Abarbanel	Uniaxial
Ty(2,4)	12	1.1715×10^{-4} $\sigma_m = 40-80$	2.1431×10^{-4} $\sigma_m = 20-140$	2.138×10^{-4} $\sigma_m = 20-160$	2.382×10^{-4} $\sigma_m = 20-50$	2.148×10^{-4} $\sigma_m = 40-700$
Ty(4,4)	12	1.1715×10^{-4} $\sigma_m = 40-80$	1.7561×10^{-4} $\sigma_m = 30-140$	1.9×10^{-4} $\sigma_m = 20-25$	2.2294×10^{-4} $\sigma_m = 20-40$	1.757×10^{-4} $\sigma_m = 40-400$
Yee	12	1.763×10^{-3} $\sigma_m = 30-160$	1.77×10^{-3} $\sigma_m = 20$	1.785×10^{-3} $\sigma_m = 20-25$	1.754×10^{-3} $\sigma_m = 20-50$	1.76×10^{-3} $\sigma_m = 30-700$
Ty(2,4)	8	2.1756×10^{-4} $\sigma_m = 40-160$	2.1553×10^{-4} $\sigma_m = 20-140$	2.12×10^{-4} $\sigma_m = 40-120$	4.1×10^{-4} $\sigma_m = 20-30$	2.1757×10^{-4} $\sigma_m = 40-700$
Ty(4,4)	8	1.2261×10^{-4} $\sigma_m = 80-160$	1.7724×10^{-4} $\sigma_m = 30-140$	2.36×10^{-4} $\sigma_m = 20$	3.8857×10^{-4} $\sigma_m = 10-30$	1.7802×10^{-4} $\sigma_m = 40 - 400$
Yee	8	1.716×10^{-3} $\sigma_m = 10-160$	1.7165×10^{-3} $\sigma_m = 10-20$	1.705×10^{-3} $\sigma_m = 10-25$	1.68×10^{-3} $\sigma_m = 10-20$	1.704×10^{-3} $\sigma_m = 10-700$
Ty(2,4)	4	2.1435×10^{-4} $\sigma_m = 80-160$	2.0778×10^{-4} $\sigma_m = 60-140$	1.9601×10^{-4} $\sigma_m = 40-120$	2.2×10^{-3} $\sigma_m = 20-30$	2.1437×10^{-4} $\sigma_m = 80-400$
Ty(4,4)	4	1.4214×10^{-4} $\sigma_m = 120-200$	1.4824×10^{-4} $\sigma_m = 30-140$	1.356×10^{-3} $\sigma_m = 30$	2.22×10^{-3} $\sigma_m = 20-30$	1.842×10^{-4} $\sigma_m = 80-400$
Yee	4	1.625×10^{-3} $\sigma_m = 40-160$	1.625×10^{-3} $\sigma_m = 30-40$	1.785×10^{-3} $\sigma_m = 30-38$	1.998×10^{-3} $\sigma_m = 30-40$	1.62×10^{-3} $\sigma_m = 20-700$

Table 1: L_2 errors for pulse using various difference schemes and PML models. σ_m is maximum value of PML parameter σ

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