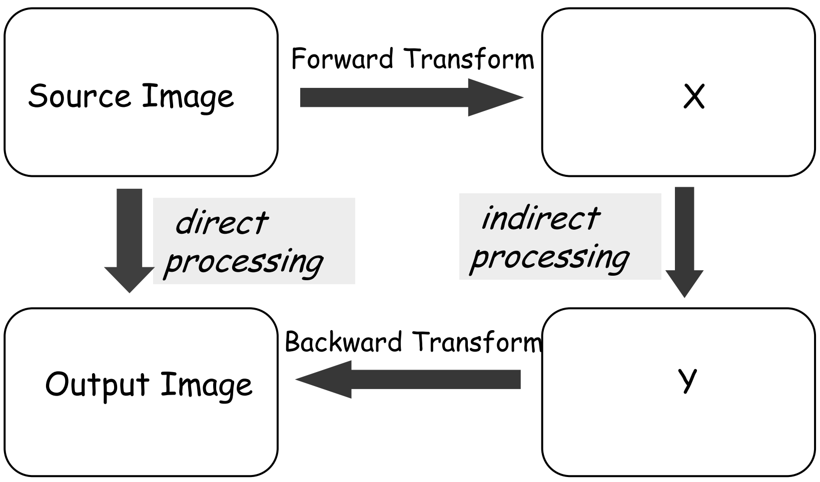
Image Transforms

- Fourier, FFT
- Wavelets
- Radon, Hough

A general note on image Transforms

It is sometimes easier to a problem in a different space



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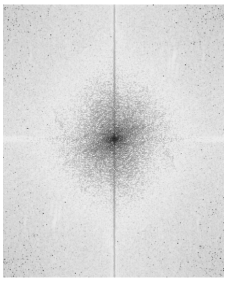
Jean Baptiste Joseph Fourier

Fourier | FFT of Fourier

Image domain



Frequency domain

$$f(x) = \sum_{n=-N}^{n=N} F_n e^{in\pi x/L}$$

Euler: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Every digital signal may be expanded in a sum of sine & cosine functions

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Simple Fourier expansions

(Demo: uiFourier)

$$Square = \frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \frac{\sin(9x)}{9} + \ldots$$

$$Sawtooth = \frac{\sin(x)}{1} + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \frac{\sin(4x)}{4} + \frac{\sin(5x)}{5} + \ldots$$

$$Triangular = \frac{\sin(x)}{1\cdot 1} + \frac{\sin(3x)}{3\cdot 3} + \frac{\sin(5x)}{5\cdot 5} + \frac{\sin(7x)}{7\cdot 7} + \frac{\sin(9x)}{9\cdot 9} + \ldots$$

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Fast Fourier Transform (FFT)

The function f(x) is therefore completely specified by the expansion coefficients F_n

F is usually computed by an efficient algorithm discovered by Tukey and Cooley in 1965 - the FFT algorithm

FFT Demo (DMfft)

A simple rgb image compression using fftn (multi dimensional fft)

```
% I = source image (rgb)
% mask = binary mask around the origin

FI = fftn(I);                    % fft image
MFI = fftshift(mask).*FI;        % masked fft image
RI = abs(ifftn(MFI));            % reconstruction
```

Fourier Transform

Infinite interval Case

$X(\omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-i\omega t}$

Inverse

$x(t)=\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$

Continuous Case - finite interval

$X(\omega) = \sum\limits_{n=-\infty}^{\infty} x_n e^{-in\omega}$

Inverse

$x_n=\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} X(\omega)e^{in\omega}d\omega$

Discrete Version (DFT)

$\widehat{X}_\upsilon=\sum\limits_{n=0}^{N-1} x_n e^{-\frac{2\pi in\upsilon}{N}} \qquad \upsilon = 0, 1, ...N-1$

Inverse

$x_n=\frac{1}{N} \sum\limits_{\upsilon=0}^{N-1} \widehat{X}_\upsilon e^{\frac{2\pi in\upsilon}{N}} \qquad n = 0, 1, ...N-1$

Note: placement of constants is arbritray. Only product of the constants in each direction counts.

Properties

Separability

$$F_{\mu\nu} = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{nm} e^{-\frac{2\pi i (n\mu + m\upsilon)}{N}} \qquad \upsilon, \nu = 0, 1, ..., N-1$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} F_{n\nu} e^{-\frac{2\pi i n\mu}{N}}$$

where

$$F_{n\nu} = \frac{1}{N} \sum_{m=0}^{N-1} F_{nm} e^{-\frac{2\pi i m\nu}{N}}$$

Translation

$$F_{\mu-\mu_0} \leftrightarrow f_n e^{\frac{2\pi i n\mu_0}{N}}$$

$$F_\mu e^{-\frac{2\pi i n\mu_0}{N}} \leftrightarrow f_{n-n_0}$$

Convolution

Continuous

$f * g(x) = \int f(y)g(x-y)dy$

Discrete

$f * g(x) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)g(x-n)$

Fourier

$f(x,y) * g(x,y) \leftrightarrow F(u,v)\cdot G(u,v)$

Correlation

Continuous

$f \circ g(x) = \int f^*(y)g(x+y)dy = f^*(y) * g(-y)$

Discrete

$f \circ g(x) = \frac{1}{N}\sum_{n=0}^{N-1} f^*(n)g(x+n)$

Fourier

$f(x,y) \circ g(x,y) \leftrightarrow F^*(u,v)\cdot G(u,v)$
$f^*(x,y)g(x,y) \leftrightarrow F(u,v) \circ G(u,v)$

Point Spread Function (PSF)

Response of a filter to a point image.
Let $P_{mn} = 1$ if $m = n = 0$ and 0 otherwise. Then

$$P'_{mn} = \sum_{i=-r}^{r}\sum_{j=-r}^{r} H_{ij}P_{m-i,n-j} = H_{mn}$$

Since convolution is linear when we know the effect of a filter on a point image we know it on any image.

Theorem: A linear shift-invariant operator must be a convolution operator in space.

Gaussian convolution

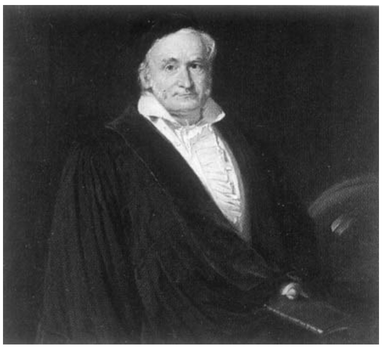
The solution of this equation (D = 1) is given by the ***Convolution Integral:***

$$u(\vec{x},t)=\begin{cases} f(\vec{x}) & (t=0) \\ \left(G_{\sqrt{2t}} \otimes f\right)(\vec{x}) & (\mathrm{t}>0) \end{cases}$$

where

$$G_\sigma(\vec{x})=\frac{1}{2\pi\sigma^2}\cdot\exp\left(-\frac{|\vec{x}|^2}{2\sigma^2}\right) \qquad f:\mathbf{R}^2\rightarrow\mathbf{R}$$

Johann Carl Friedrich Gauss



Born: 30 April 1777 in Brunswick, Duchy of Brunswick (now Germany)
Died: 23 Feb 1855 in G ttingen, Hanover (now Germany)

Heisenberg Uncertainty Principle (band limited in physical space and Fourier space)

Normalize f(x) so that $\int_{-\infty}^{+\infty} |f(x)|^2 dx = 1$ and $\lim_{x\to\infty} \sqrt{x} f(x) = 0$
Let F(k) be the Fourier transform of f(x).
Then by Parseval's theorem $\int_{-\infty}^{+\infty} |F(k)|^2 dk = 1$

$$|\int_{-\infty}^{\infty} x(f(x)f'(x)dx| \leq \sqrt{\int_{-\infty}^{\infty} |xf(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |f'(x)|^2 dx} \qquad \text{by the Schwarz inequal}$$

$$= \sqrt{\int_{-\infty}^{\infty} |xf(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |ikF(k)|^2 dk} \qquad \text{by Parseval's equality}$$

$$\int_{-\infty}^{\infty} x(f(x)f'(x)dx = \frac{1}{2}x\,[f(x)]^2\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}\,[f(x)]^2$$

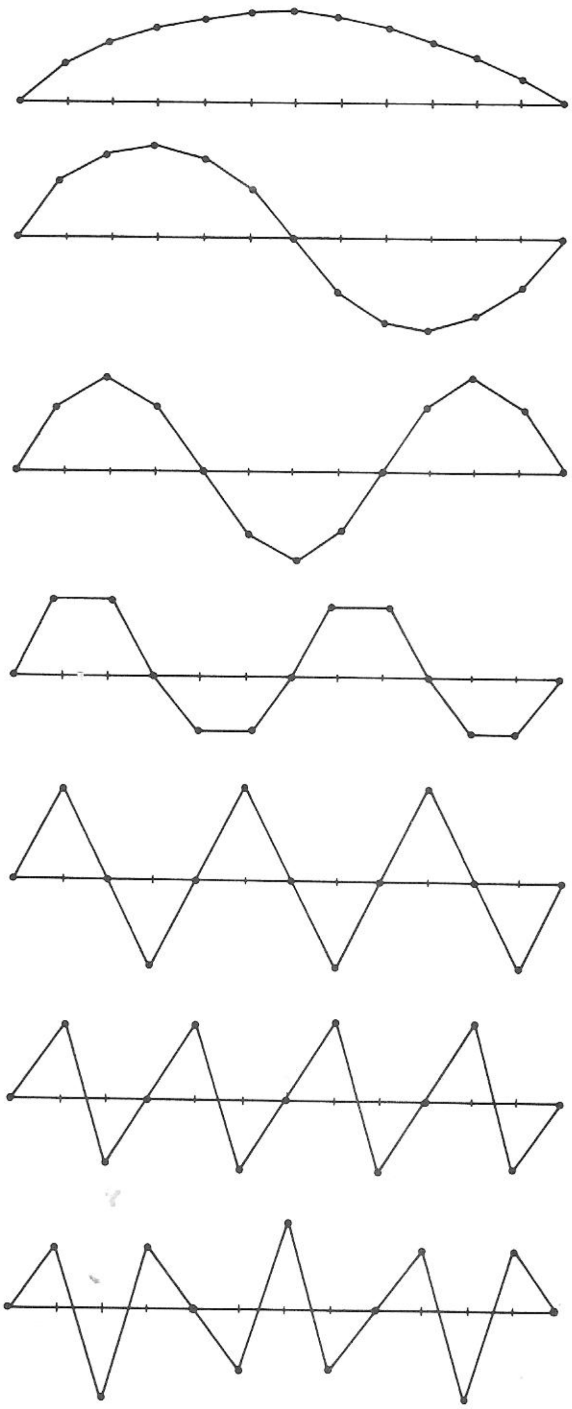
$$= 0 - \frac{1}{2} = -\frac{1}{2} \qquad \text{by integration by parts}$$

Hence

$$\sqrt{\int_{-\infty}^{\infty} |xf(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |ikF(k)|^2 dk} \geq \frac{1}{2}$$

So we can't make both integrals small. If we concentrate the physical function we spread the Fourier transform and vice versa

We get equality if $f'(x) = const\ xf(x)$ or $f(x) = e^{-\alpha x^2}$. Hence, the Gaussian is the most concentrated function.



$k = 1 : \mathbf{w}_{1,j} = \sin\left(\frac{j\pi}{12}\right)$, $l = 2 = 24h$

$k = 2 : \mathbf{w}_{2,j} = \sin\left(\frac{j\pi}{6}\right)$, $l = 1 = 18h$

$k = 3 : \mathbf{w}_{3,j} = \sin\left(\frac{j\pi}{4}\right)$, $l = \frac{2}{3} = 8h$

$k = 4 : \mathbf{w}_{4,j} = \sin\left(\frac{j\pi}{3}\right)$, $l = \frac{1}{2} = 6h$

$k = 6 : \mathbf{w}_{6,j} = \sin\left(\frac{j\pi}{2}\right)$, $l = \frac{1}{3} = 4h$

$k = 8 : \mathbf{w}_{8,j} = \sin\left(\frac{2j\pi}{3}\right)$, $l = \frac{1}{4} = 3h$

$k = 9 : \mathbf{w}_{9,j} = \sin\left(\frac{3j\pi}{4}\right)$, $l = \frac{2}{9} = \frac{8}{3}h$

FIG. 8. Graphs of the Fourier modes of A on a grid with $N = 12$. Modes with wavenumbers $k = 1, 2, 3, 4, 6, 8, 9$ are shown. The wavelength of the kth mode is $l = 24h/k$.

$w_{N-k,j} = (-1)^{j+1} w_{k,j}$