Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l \quad 0 < t < \infty$$
$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$
$$u(0,t) = 0 \quad u(l,t) = 0$$

Heat Equation

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l \quad 0 < t < \infty$$
$$u(x,0) = \varphi(x)$$
$$u(0,t) = 0 \quad u(l,t) = 0$$

We assume that u can be expressed as a Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

This automatically satisfies the boundary conditions. Also since sin is a solution to both the wave and heat equations it will satisfy the PDE. Thus the main question is the intial conditions. Formally put in the equation.

$$\frac{\partial^2 u}{\partial x^2}(x,t) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 A_n(t) \sin\left(\frac{n\pi x}{l}\right)$$
$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} A_n'(t) \sin\left(\frac{n\pi x}{l}\right)$$
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \sum_{n=1}^{\infty} A_n''(t) \sin\left(\frac{n\pi x}{l}\right)$$

So

$$\sum_{n=1}^{\infty} \left[A_n'(t) + \left(\frac{n\pi}{l}\right)^2 A_n(t)\right] \sin\left(\frac{n\pi x}{l}\right) = 0 \qquad \text{heat equation}$$
$$\sum_{n=1}^{\infty} \left[B_n''(t) + \left(\frac{n\pi}{l}\right)^2 B_n(t)\right] \sin\left(\frac{n\pi x}{l}\right) = 0 \qquad \text{wave equation}$$

or setting the coefficient of each $\sin\left(\frac{n\pi x}{l}\right)$ to zero we get

$$A_n'(t) + \left(\frac{n\pi}{l}\right)^2 A_n(t) = 0 \qquad \text{all n} \quad \text{heat equation}$$
$$B_n''(t) + \left(\frac{n\pi}{l}\right)^2 B_n(t) = 0 \qquad \text{all n} \quad \text{wave equation}$$

so

$$A_n(t) = A_n(0)\ e^{-\left(\frac{n\pi}{l}\right)^2 t} \qquad \text{all n} \quad \text{heat equation}$$
$$B_n(t) = B_n(0)\cos(\frac{n\pi}{l}t) + \frac{l}{n\pi}B'_n(0)\ \sin(\frac{n\pi}{l}t) \qquad \text{all n} \quad \text{wave equation}$$

So we have solved the PDE and boundary conditions. What remains is the initial conditions, i.e. find $A_n(0)$, $B_n(0)$, $B_n^{(}0)$ so that

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n(0)\sin\left(\frac{n\pi x}{l}\right) \quad \text{heat equation}$$
$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} B_n(0)\sin\left(\frac{n\pi x}{l}\right) \quad \text{wave equation}$$
$$\psi(x) = u(x,0) = \sum_{n=1}^{\infty} B_n'(0)\sin\left(\frac{n\pi x}{l}\right) \quad \text{wave equation}$$

Sine Series

Theorem:

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n \end{cases}$$

If

$$\varphi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \qquad 0 < x < l$$

then

$$B_n = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Cosine Series

Theorem:

$$\int_0^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n \neq 0 \\ l & m = n = 0 \end{cases}$$

If

$$\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \qquad 0 < x < l$$

then

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Full interval Series

Theorem:

$$\int_{-l}^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & m \neq n \\ l & m = n \end{cases}$$

$$\int_{-l}^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \\ 2l & m = n = 0 \end{cases}$$

$$\int_{-}^l \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = 0 \qquad \text{all m,n}$$

If

$$\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right) \qquad -l < x < l$$

then

$$A_n = \frac{1}{l}\int_{-l}^l \varphi(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$B_n = \frac{1}{l}\int_{-l}^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note: sine series corresponds to odd extension i.e. Dirichlet bc
cosine series corresponds to even extension i.e. Neumann bc

Robin Condition

$$
\begin{aligned}
X^{''} + \beta^2 X &= 0 \\
u_x - a_0 u &= 0 \qquad x = 0 \\
u_x + a_l u &= 0 \qquad x = l
\end{aligned}
$$

Assume

$$
\begin{aligned}
X(x) &= C\cos(\beta x) + D\sin(\beta x) \\
X^{'}(x) &= \beta\left[-C\sin(\beta x) + D\cos(\beta x)\right] \\
X^{'}(x) + aX &= (\beta D + aC)\cos(\beta x) + (-\beta C + aD)\sin(\beta x)
\end{aligned}
$$

So

$$
\begin{aligned}
\beta D + a_0 C &= 0 \qquad \text{at } x = 0 \\
(\beta D + a_l C)\cos(\beta l) + (-\beta C + a_l D)\sin(\beta l) &= 0 \qquad \text{at } x = l
\end{aligned}
$$

To have a nontrivial solution the determinant of the system needs to be nonzero

$$
-a_0\left[\beta\cos(\beta l) + a_l\sin(\beta l)\right] - \beta\left[a_l\cos(\beta l) - \beta\sin(\beta l)\right] = 0
$$

or

$$
\tan(\beta l) = \frac{(a_0 + a_l)\,\beta}{\beta^2 - a_0 a_l}
$$

Example sin(x) expansion

$$\phi(x) = 1 \qquad 0 < x < l$$

Then

$$\begin{aligned} B_m &= \frac{2}{l}\int\limits_0^l \sin\left(\frac{m\pi x}{l}\right)dx \\ &= \left[-\frac{2}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\right]_0^l \\ &= \frac{2}{m\pi}\left[1-\cos\left(m\pi\right)\right] = \frac{2}{m\pi}\left[1-(-1)^m\right] \\ &= \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{l}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{l}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{l}\right)+...\right] \end{aligned}$$

$\phi(x)$is discontinuous at x=0
Hence, we have proven that

$$\frac{\pi}{4} = \sin\left(\frac{\pi x}{l}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{l}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{l}\right)+...$$

Example: sine series

$$\phi(x) = x \qquad 0 < x < l$$

Then

$$\begin{aligned}
B_m &= \frac{2}{l}\int_0^l x \sin\left(\frac{m\pi x}{l}\right) dx \\
&= \left[-\frac{2x}{m\pi}\cos\left(\frac{m\pi x}{l}\right) + \frac{2l}{m^2\pi^3}\sin\left(\frac{m\pi x}{l}\right)\right]_0^l \\
&= -\frac{2l}{m\pi}\cos(m\pi) = (-1)^{m+1}\,\frac{2l}{m\pi} \\
&= \frac{2l}{\pi}\left[\sin\left(\frac{\pi x}{l}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) - ...\right]
\end{aligned}$$

This is discontinuous at $x = l$

$$\phi(x) = x \qquad -l < x < l$$

Then $\phi(x)$ is ann odd function and $A_n = 0$

$$\begin{aligned}
B_m &= \frac{1}{l}\int_{-l}^l x \sin\left(\frac{m\pi x}{l}\right) dx = \frac{2}{l}\int_0^l x \sin\left(\frac{m\pi x}{l}\right) dx_0^l \\
&= (-1)^{m+1}\,\frac{2l}{m\pi}
\end{aligned}$$

Example - PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad 0 < x < l \\ u(x,0) &= x \\ u_t(x,0) &= 0 \\ u(0,t) &= u(l,t) = 0 \end{aligned}$$

Then

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \right] \sin\left(\frac{n\pi x}{l}\right) \\ u_t(x,t) &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} \left[-A_n \sin\left(\frac{n\pi ct}{l}\right) + B_n \cos\left(\frac{n\pi ct}{l}\right) \right] \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

set $t = 0$. Then

$$\begin{aligned} x &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \\ 0 &= \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

So

$$u(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$