Total Variation Blind Deconvolution
Outline

• Blind Deconvolution formulation
• Numerical methods solving PDE’s
• Choice of regularization parameters
• Results
• Summary
Image Restoration With Known PSF

Blurring model: \[ z = h \ast u + \eta \]

\[
\min_{u} f(u) = \min_{u} \left\{ \frac{1}{2} \left\| h \ast u - z \right\|_{L^2(\Omega)}^2 + \alpha \int_{\Omega} |\nabla u| \, dx \, dy \right\}
\]

Rudin, Osher, Fetami 1992
Blind Deconvolution

- Space invariant image restoration (of \( u \) and \( h \)) without any a priori knowledge of the PSF
- Assuming PSF is piecewise smooth

Atmospheric turbulence  
Motion blur  
Out of focus
Blind Deconvolution Formulation

\[
\min_{u,h} f(u, h) = \min_{u,h} \frac{1}{2} \left\| h * u - z \right\|^2_{L^2(\Omega)} + \\
\alpha_1 \int_{\Omega} |\nabla u| dxdy + \alpha_2 \int_{\Omega} |\nabla h| dxdy
\]

Euler-Lagrange optimization yields:

\[
\frac{\delta L}{\delta h} = (u * h - z) * u(-x,-y) - \alpha_2 \nabla \cdot \left( \frac{\nabla h}{|\nabla h|} \right) = 0
\]

\[
\frac{\delta L}{\delta u} = (u * h - z) * h(-x,-y) - \alpha_1 \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) = 0
\]
Convexity

Does the problem have global or local minima?

\[ f(u, h) = \int_\Omega (h * u - z)^2 + \alpha_1 |\nabla u| + \alpha_2 |\nabla h| \, dx \, dy \]

\( f(u, h) \) is not jointly convex, but for a given \( u \), \( f(u, \cdot) \) is convex with respect to \( h \) and vise versa.
Minimization problem may not have a unique solution [e.g. \((-u, -h)\) is also solution], therefore the following conditions are imposed:

\[
\min_{u, h} f(u, h) = \min_{u, h} \frac{1}{2} \| h * u - z \|^2_{L^2(\Omega)} + \\
\alpha_1 \int_\Omega |\nabla u| dx dy + \alpha_2 \int_\Omega |\nabla h| dx dy
\]

Minimization problem may not have a unique solution [e.g. \((-u, -h)\) is also solution], therefore the following conditions are imposed:

\[
\int_\Omega h(x, y) dx dy = 1 \\
u(x, y), h(x, y) \geq 0 \\
h(x, y) = h(-x, -y)
\]
Numerical Methods

AM – Alternate Minimization

Initialization: \( u^0 = z, h^0 = \delta(x, y) \)

\[
f(u^0, h^1) = \min_h f(u^0, \cdot)
\]

\[
f(u^1, h^1) = \min_u f(\cdot, h^1)
\]

The convexity property yields that the function \( f(u^n, h^n) \) always decreases when \( n \) increases

\[
Solve h^{n+1}: u^n (-x, -y) \ast (u^n \ast h^{n+1} - z) - \alpha_2 \nabla \cdot \left( \frac{\nabla h^{n+1}}{\nabla h^{n+1}} \right) = 0
\]

\[
Solve u^{n+1}: h^{n+1} (-x, -y) \ast (h^{n+1} \ast u^{n+1} - z) - \alpha_1 \nabla \cdot \left( \frac{\nabla u^{n+1}}{\nabla u^{n+1}} \right) = 0
\]
Linearization of PDE’s
Lagged Diffusitivity Fixed Point (FP) , Vogel and Oman 1996

Coefficients are lagged by one iteration and then linear equations are solved:

\[ Solve \ h^{n+1} : u^n (-x, -y) \ast (u^n \ast h_{i+1}^{n+1} - z) - \alpha_2 \nabla \cdot \left( \nabla h_{i+1}^{n+1} \right) = 0 \]

\[ Solve \ u^{n+1} : h^{n+1} (-x, -y) \ast (h^{n+1} \ast u_{i+1}^{n+1} - z) - \alpha_1 \nabla \cdot \left( \nabla u_{i+1}^{n+1} \right) = 0 \]
Impose:

\[ h^{n+1}(x, y) = \begin{cases} 
  h^{n+1}(x, y), & h^{n+1}(x, y) > 0 \\
  0 & \text{otherwise}
\end{cases} \]

\[ h(x, y) = \frac{[h(x, y) + h(-x, -y)]}{2} \]

\[ h_{n+1} = \frac{\int_{\Omega} h^{n+1}(x, y) \, dx, dy}{h^{n+1}} \]
Choice of Regularization Parameters

Consider the noise-constraint minimization problem:

$$\min_{u,h} \iint |\nabla u| + \alpha |\nabla h| \, dx \, dy \quad \text{subject to } \left\| h \ast u - z \right\|^2 = \sigma^2$$

Using Lagrange multiplier notation,

$$f(u, h) = \iint |\nabla u| + \alpha |\nabla h| \, dx \, dy + \frac{\lambda}{2} \left( \left\| h \ast u - z \right\|^2 - \sigma^2 \right)$$

On the other hand

$$f(u, h) = \iint \alpha_1 |\nabla u| + \alpha_2 |\nabla h| \, dx \, dy + \frac{1}{2} \left\| h \ast u - z \right\|^2$$

$$\Rightarrow \alpha_1 = \frac{2}{\lambda} \quad \alpha_2 = \frac{2\alpha}{\lambda}$$
Choice of Parameters

\( \alpha_1 \) depends on the noise level

\[
\text{SNR} \downarrow \Rightarrow \sigma \uparrow \Rightarrow \lambda \downarrow
\]

\[
\alpha_1 = \frac{2}{\lambda}, \quad = C \sigma
\]

\( \alpha_2 \) controls the support of the spread of PSF (desired deblurring)

\[
\alpha_2 \uparrow \Rightarrow \int |\nabla h|dx\,dy \downarrow, \int hdx\,dy = 1 \Rightarrow \text{psf spread out}
\]
Numerical Results

• There were 10 iterations in every FP stage.

• Within each FP iteration there is a linear system which was solved by \textbf{Conjugate Gradient} method (Hestenes and Stiefel 1952), which converges in a finite number of iterations.

\[
f(x) = \frac{1}{2} x^T A x - b^T x - c
\]

\[
\min f(x) \iff A x = b \quad \forall A \text{ Symmetric Positive Definite}
\]
Results

Test Image 127x127

Out of Focus blur  blurred image
\[ \alpha_2 = 10^{-7} \]

\[ \alpha_2 = 10^{-6} \]

\[ \alpha_2 = 10^{-5} \]

\[ \alpha_2 = 10^{-4} \]

\[ \alpha_1 = 2 \cdot 10^{-6} \]
Results After Three AM Iterations

\[ \alpha_1 = 2 \cdot 10^{-6}, \alpha_2 = 1.5 \cdot 10^{-5}, SNR = \infty \]

\[ \alpha_1 = 2 \cdot 10^{-5}, \alpha_2 = 1.5 \cdot 10^{-5}, SNR = 5 \]
TV Norm in $u$ and $L^2$ Norm in $h$

\[ \alpha_2 = 10^{-5} \]

\[ \alpha_2 = 10^{-4} \]

\[ \alpha_2 = 10^{-3} \]
Summary

• We used TV norm for regularizing u and h in the blind deconvolution problem
• AM/FP algorithm was proposed with choice of parameters heuristics
• Algorithm found to be robust and efficient, recovered images are as good as that recovered with the exact PSF