Image Processing and Deconvolution

Speckle Adaptive Optics Deconvolved

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Outline

- Introductory Mathematics
- Image Formation & Fourier Optics
- Deconvolution Schemes
    - Linear – optimal filtering
    - Non-Linear
        - Conjugate Gradient Minimization
            - - steepest descent search *a la* least squares
        - Lucy Richardson (LR) – Maximum Likelihood
        - Maximum a posteriori (MAP)
    - Regularization schemes
    - Other PSF calibration techniques
- Quantitative Measurements

Why Deconvolution?

- Better looking image
- Improved identification

  Reduces overlap of image structure to more easily identify features in the image (needs high SNR)
- PSF calibration

  Removes artifacts in the image due to the point spread function (PSF) of the system, i.e. extended halos, lumpy Airy rings etc.
- Higher resolution

  In specific cases depending upon algorithms and SNR
- Better Quantitative Analysis

Image Formation

- Image Formation is a convolution procedure for PSF invariance and incoherent imaging.

  Convolution is a superposition integral, i.e.

$$i(r) = \int ds\ o(s)\, p(r-s) = o(r) * p(r)$$

where

i(*r*)	– measured image
p(*r*)	– point spread function (impulse response function)
o(*r*)	– object distribution
*	- Convolution operator

Nomenclature

In this presentation, the following symbols are used:

$g(r)$	– measured image
$h(r)$	– point spread function (impulse response function)
$f(r)$	– object distribution
*	- Convolution operator

Relatively standardized nomenclature in the field.

Inverse Problems

The problem of reconstructing the original target falls into a class of Mathematics known as Inverse Problems which has its own Journal. References in diverse publications such as SPIE Proceedings & IEEE Journals.

Multidisciplinary Field with many applications:

- Applied Mathematics
  - Matrix Inversion (*SIAM*)

- Image and Signal Processing
  - Medical Imaging (*JOSA*, *Opt.Comm*., *Opt Let*.)
  - Astronomical Imaging (*A.J., Ap.J., P.A.S.P., A.&A*.)

- OSA Topical Meetings on *Signal Recovery & Synthesis*

Fourier Transform Theorems

- Autocorrelation Theorem

If $\mathcal{F}\{g(r)\} = G(f)$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} du dv \quad g(u) g^*(u-v)\right\} = |G(f)|^2$$

Similarly,

$$\mathcal{F}\left\{|g(r)|^2\right\} = \int_{-\infty}^{\infty} df dq \quad G(f) G^*(f-q)$$

- Convolution Theorem

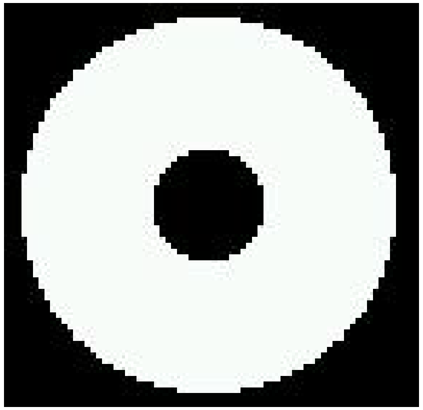
If $\mathcal{F}\{g(r)\} = G(f)$ and $\mathcal{F}\{h(r)\} = H(f)$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} du dv \quad g(u) f(u-v)\right\} = G(f) H(f)$$

Image Formation & Fourier Optics

Fraunhofer diffraction theory (far field):

The observed field distribution (complex wave in the focal plane) *u(r)* is approximated as the Fourier transform of the aperture distribution (complex wave at the pupil) *P(r').*

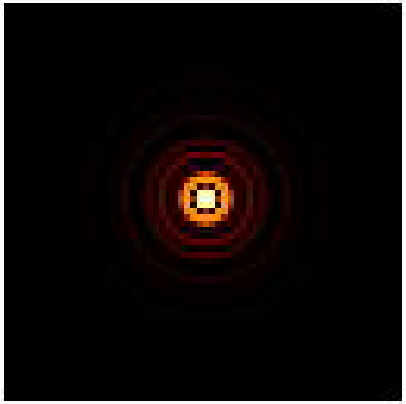


P(r')

The point spread function (impulse response) is the square amplitude of the Fourier Transform of a plane wave sampled by the finite aperture, i.e.

$$h(r) = |u(r)|^2 \quad = |FT\{P(r')\}|^2$$

The power spectral density of the complex field at the pupil.



h(r)

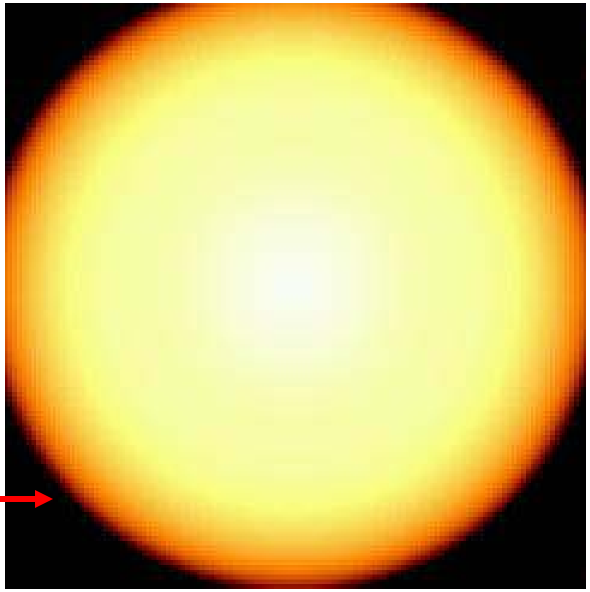
J. Goodman "*Introduction to Fourier Optics*"

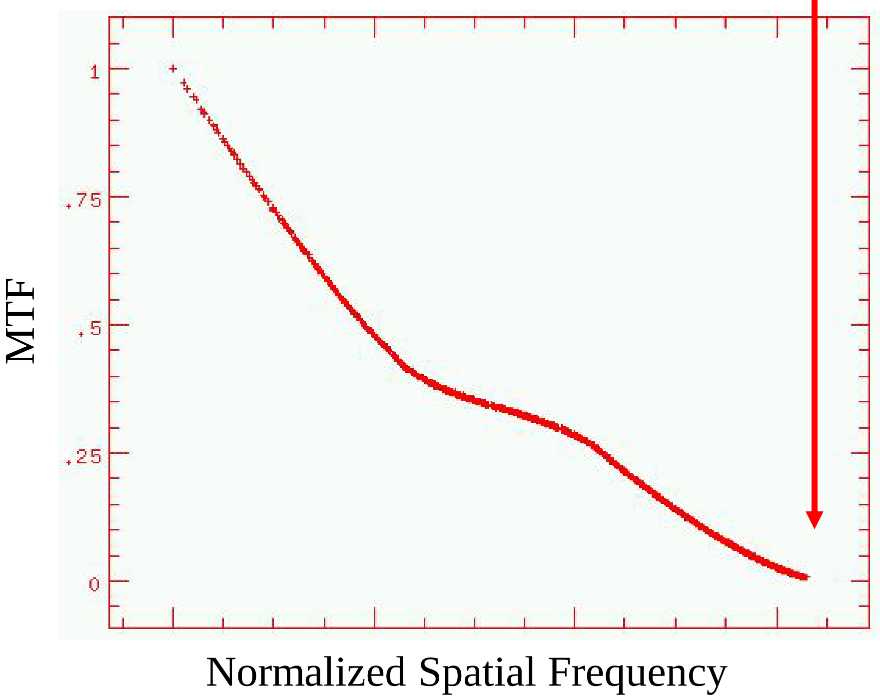
The Transfer Function

The Optical Transfer Function (OTF) is the spatial frequency response of the optical system.

The Modulation Transfer Function (MTF) is the modulus of the OTF and is the Fourier transform of the PSF.

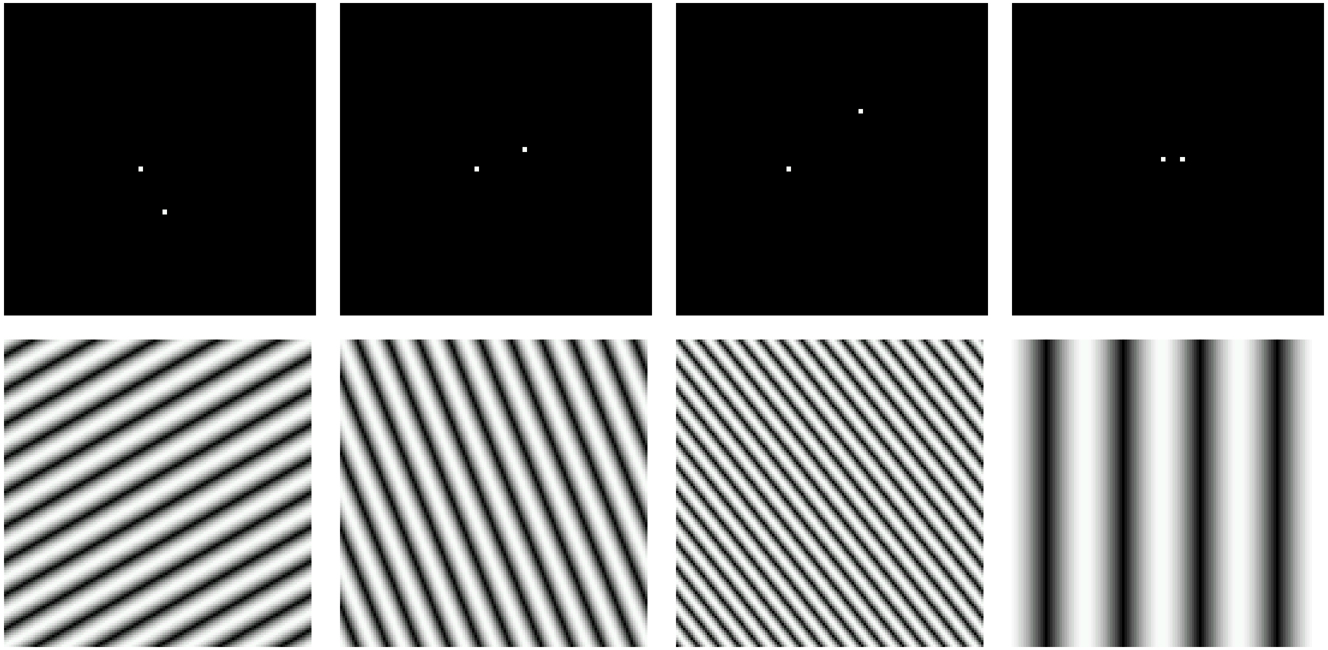
$f_c = \lambda/D$





From the autocorrelation theorem the MTF is the autocorrelation of the complex wavefront at the pupil.

The Fourier Domain

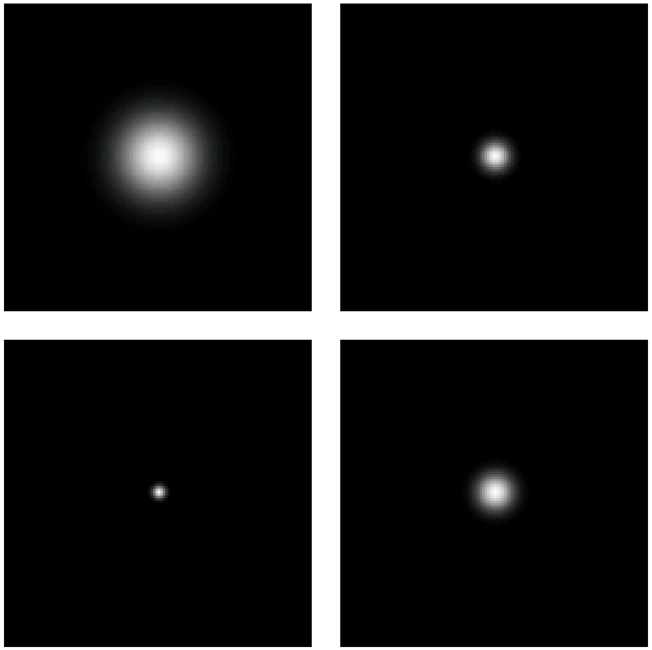


Binary Stars

Fourier Modulus

Two delta functions produce a set of “fringes”, the frequency of which is inversely proportional to the separation and which are oriented along the separation vector. The “visibility” of the fringes corresponds to the intensity differences. How?

The Fourier Domain



Gaussian

Fourier Modulus
(also Gaussian)

These Fourier modulus of a Gaussian produces another Gaussian. A large object comprised of low spatial frequencies produces a compact Fourier modulus and a smaller object with higher spatial frequencies produces a larger Fourier modulus.

Fourier Relationships

- Resolution of an aperture of size D is $\alpha = \dfrac{\lambda}{D}$ radians
- Diffraction limit of an aperture of size D is $f_c = \dfrac{1}{\alpha} = \dfrac{D}{\lambda}$ cycles/radian

  - resolution depends on wavelength and aperture

- Large spatial structures correspond to low-spatial frequencies
- Small spatial structures correspond to high-spatial frequencies

Image Formation - Convolution

Shift invariant imaging equation (Image and Fourier Domains)

Image Domain: $$g(r) = f(r) * h(r)$$

Fourier Domain: $$G(f) = F(f)H(f)$$

- $g(r)$ - Measurement
- $f(r)$ - Object
- $h(r)$ - blur (point spread function)
- $g(r)$ - Noise contamination
- Fourier Transform FT{$g(r)$} = $G(f)$ etc.
- * - convolution

Deconvolution

The convolution equation is inverted.

Given the measurement *g*(*r*) and the PSF *h*(*r*) the object *f*(*r*) is computed.

e.g. $$F(\vec{f}) = \left|F(\vec{f})\right| \exp\left[i\phi(\vec{f})\right] = \frac{G(\vec{f})}{H(\vec{f})}$$

and inverse Fourier transform to obtain *f*(*r*).

Problem:

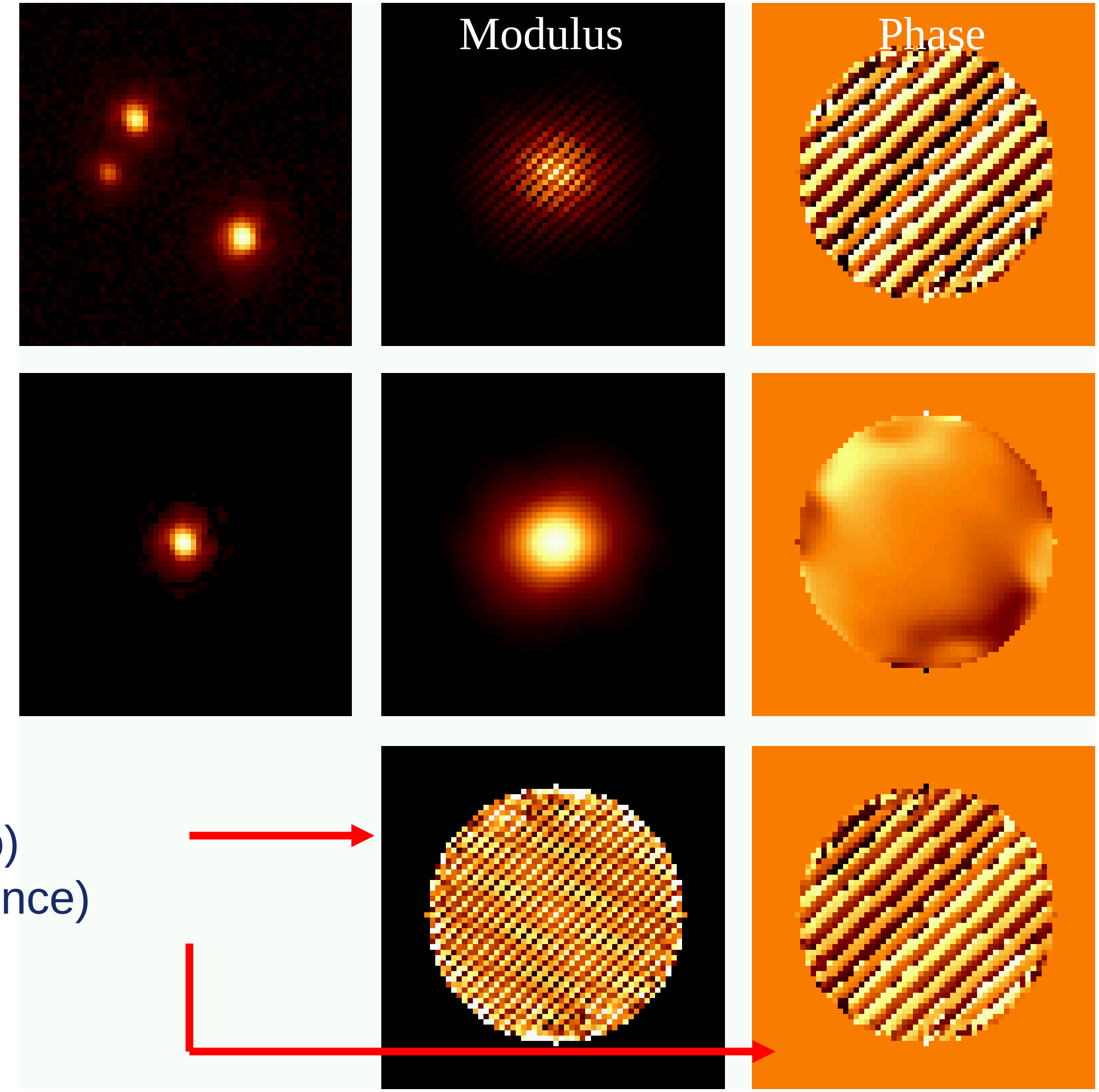
The PSF and the measurement are both band-limited due to the finite size of the aperture.

The object/target is not.

Images & Fourier Components

measurement

PSF



Left: Fourier amplitudes (ratio)
Right: Fourier phases (difference)
for the object.

note circle = band-limit

Deconvolution via Linear Inversion

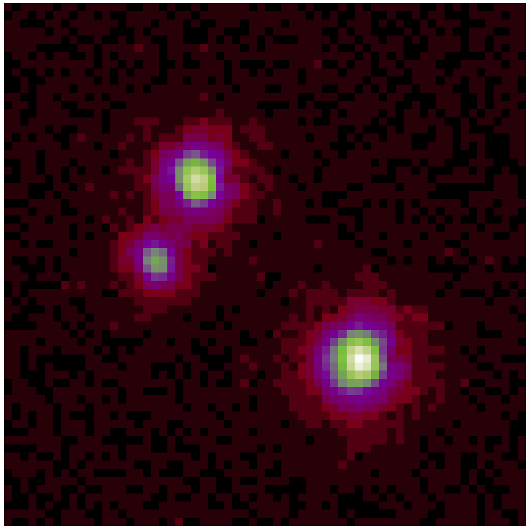
$$f(r) = \mathrm{FT}^{-1}\left\{ \frac{G(f)}{H(f)} \Phi(f) \right\}$$

Inverse Filtering: *F*(*f*) is a bandpass-limited attenuating filter, e.g. a *chat* function where *H*(*f*) = 0 for $f > f_c$.

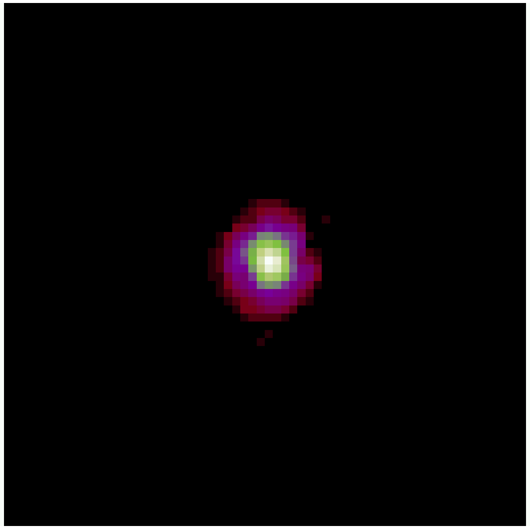
Wiener Filtering: A noise-dependent filter -

$$\Phi(f) \approx \frac{|G(f)|^2 - |N(f)|^2}{|G(f)|^2}$$

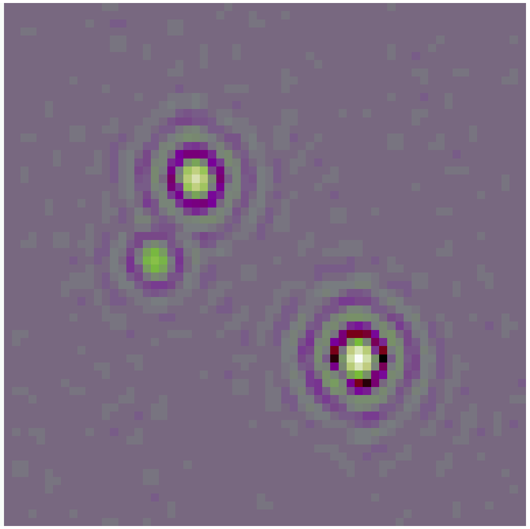
Deconvolution via Linear Inversion with a Wiener Filter - Example



measurement



PSF



reconstruction

Note the negativity in the reconstruction – not physical

Deconvolution
Iterative non-linear techniques

Radio Astronomers, because of working with amplitude and phase signals, have far more experience with image/signal processing.

- Maximum Entropy Method
- CLEAN

• Deconvolution (for visible astronomy)

HST - *The Restoration of HST Images & Spectra, ed. R.J.Hanisch & R.L.White, STScI, 1993*

- Richardson-Lucy
- Pixon - Bayesian image reconstruction
- "Blind/Myopic" Deconvolution – poorly determined or unknown PSFs
- Maximum a posteriori
- Iterative Least Squares

A Simple Iterative Deconvolution Algorithm

- Error Metric Minimization – object estimate & PSF convolve to measurement

$$E = \sum_{g \in R} \left[g_i - \left(\hat{f}_i * h_i \right) \right]^2$$

- Strict positivity constraint reparameterize the variable

$$\hat{f}_i = \left(\phi_i \right)^2$$

- Conjugate Gradient Search (least squares fitting) requires the first-order derivatives w.r.t. the variable, e.g. $\delta E / \delta \phi_i$

- Equivalent to maximum-likelihood (the most probable solution) for Gaussian statistics

- Permits "super-resolution"

Bayes Theorem on Conditional Probability

$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

P – Probabilities
A & *B* – Outcomes of random experiments
$P(A|B)$ - Probability of *A* given that *B* has occurred

For Imaging:

$P(B|A)$ - Probability of measuring image *B* given that the object is *A*

Fitting a probability model to a set of data and summarizing the result by a probability distribution on the model parameters and observed quantities.

Bayes Theorem on Conditional Probability

- Setting up a *full probability model* – a joint probability distribution for all observable and unobservable quantities in a problem,
- Conditioning on observed data: calculating and interpreting the appropriate posterior distribution – the conditional probability distribution of the unobserved quantities.
- Evaluating the fit of the model. How good is the model?

Lucy-Richardson Algorithm

Discrete Convolution

$$g_i = \sum_j h_{ij} f_j \text{ where } \sum_j h_{ij} = 1 \text{ for all } j$$

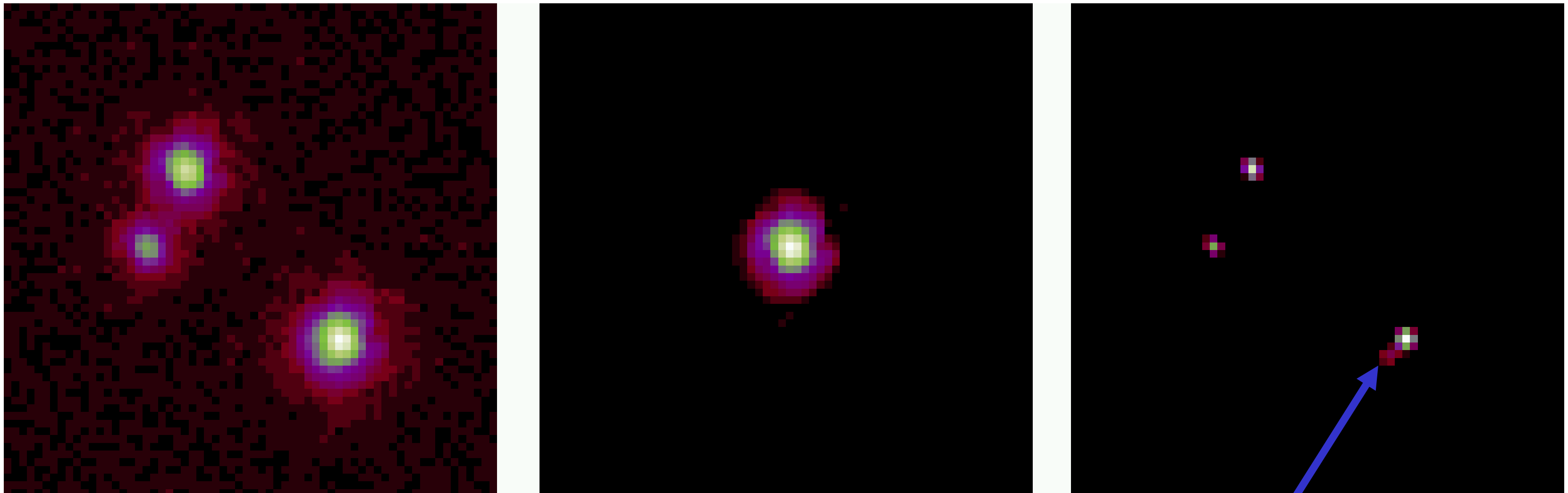
From Bayes theorem $P(g_i|f_j) = h_{ij}$ and the object distribution can be expressed iteratively as

$$f_j = f_j \sum_i \left(\frac{h_{ij} g_i}{\sum_k h_{jk} f_k} \right)$$

so that the LR kernel approaches unity as the iterations progress

Richardson, W.H., "Bayesian-Based Iterative Method of Image Restoration", *J. Opt. Soc. Am.*, **62**, 55, (1972).
Lucy, L.B., "An iterative technique for the rectification of observed distributions", *Astron. J.*, **79**, 745, (1974).

Richardson-Lucy Application
Simulated Multiple Star

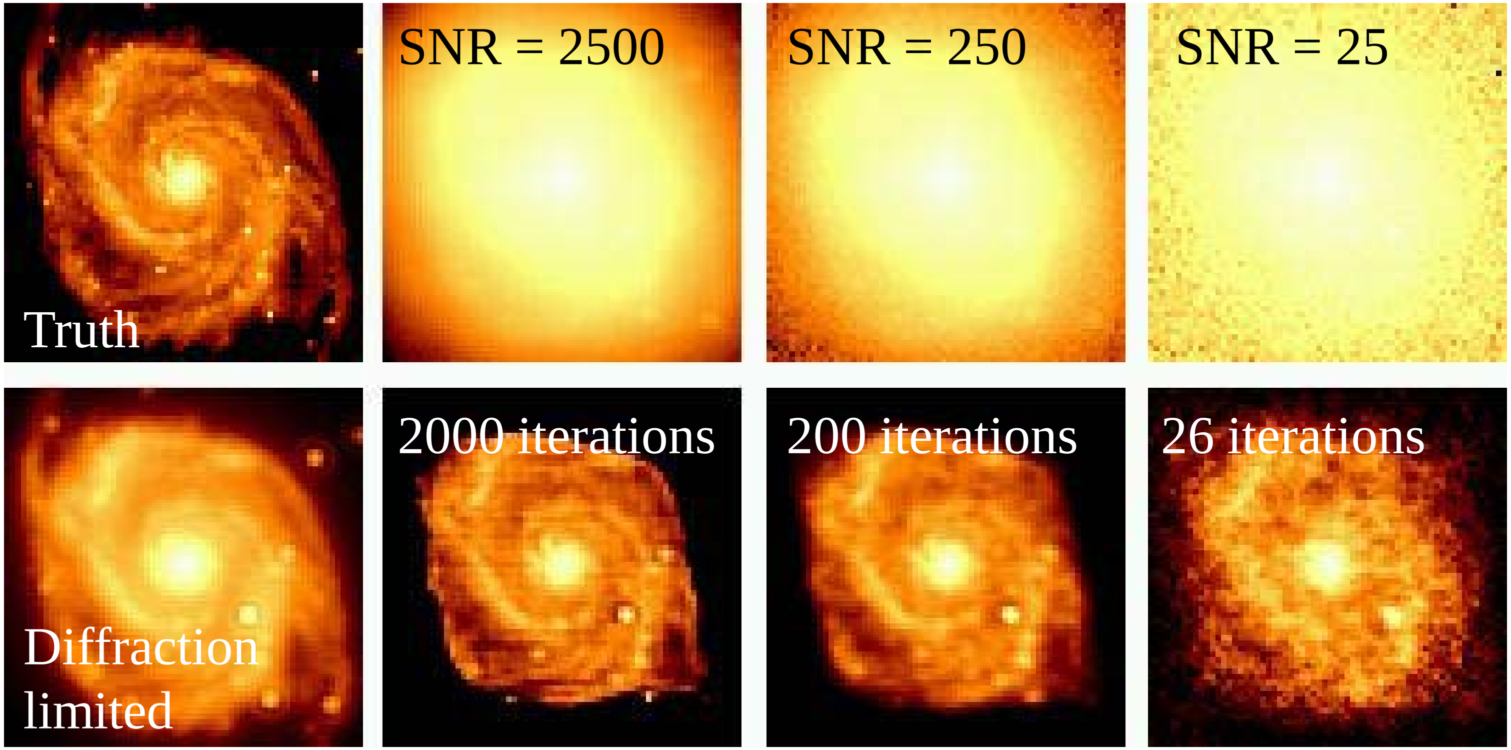


measurement PSF reconstruction

Note – super-resolved result and identification of a 4th component

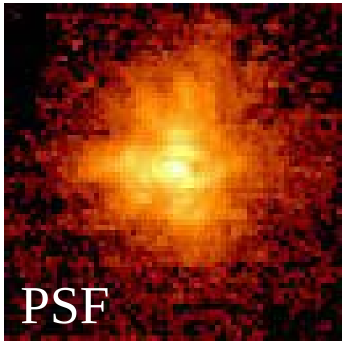
Super-resolution means recovery of spatial frequency information beyond the cut-off frequency of the measurement system.

Richardson-Lucy Application
Simulated Galaxy



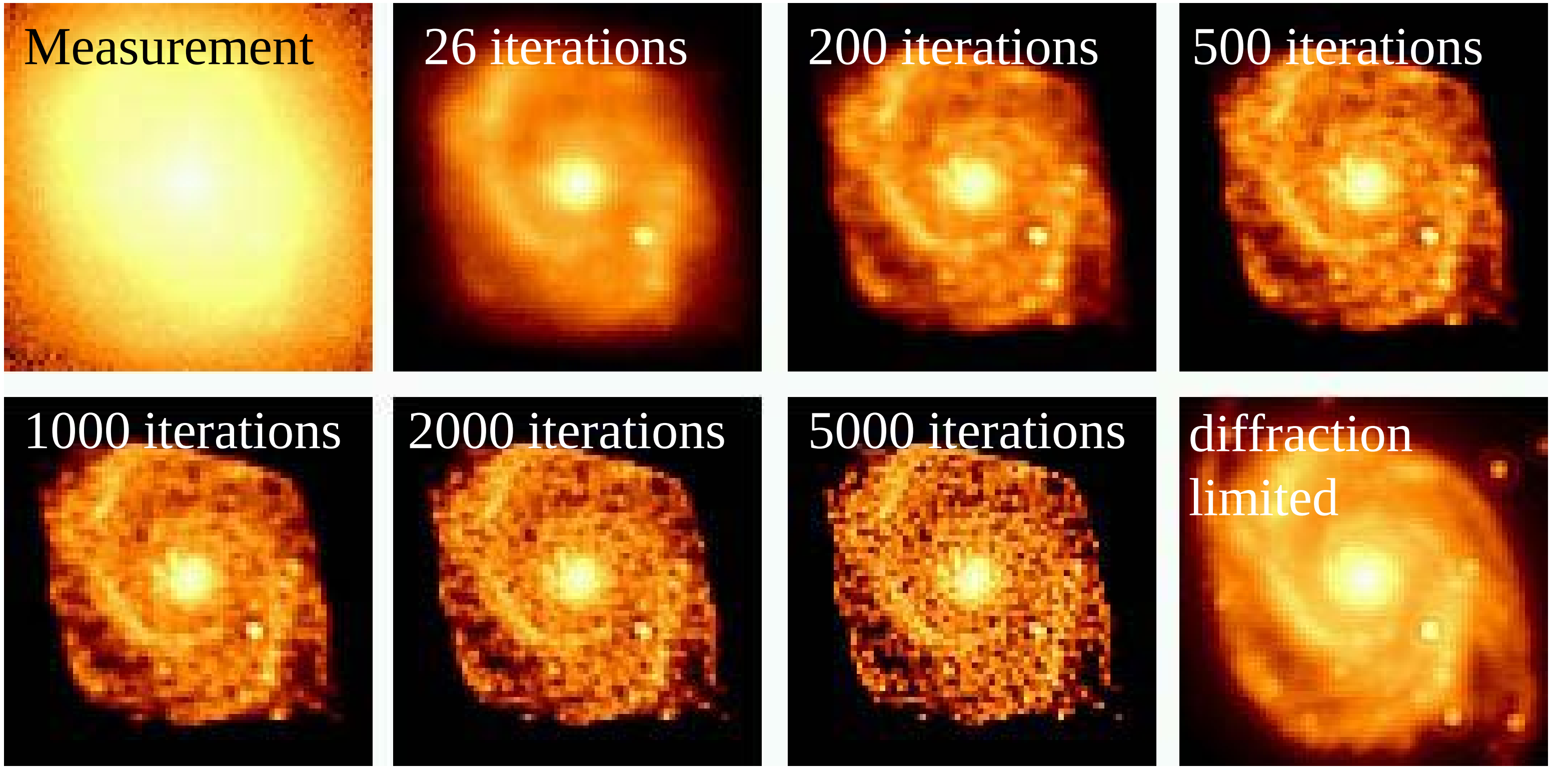
All images on a logarithmic scale

LR works best for high SNR



Richardson-Lucy Application
Noise Amplification

- Maximum-likelihood techniques suffer from noise amplification
- Problem is knowing when to stop
- SNR = 250



All images on a logarithmic scale

Richardson-Lucy Application
Noise Amplification

- For small iterations RL produces spatial frequency components not strongly filtered by the OTF, i.e. the low spatial frequencies.

- Spatial frequencies which are strongly filtered by the OTF will take many iterations to reconstruct (the algorithm is relatively unresponsive), i.e. the high spatial frequencies.

- In the presence of noise, the implication is that after many iterations the differences are small and are likely to be due to noise amplification.

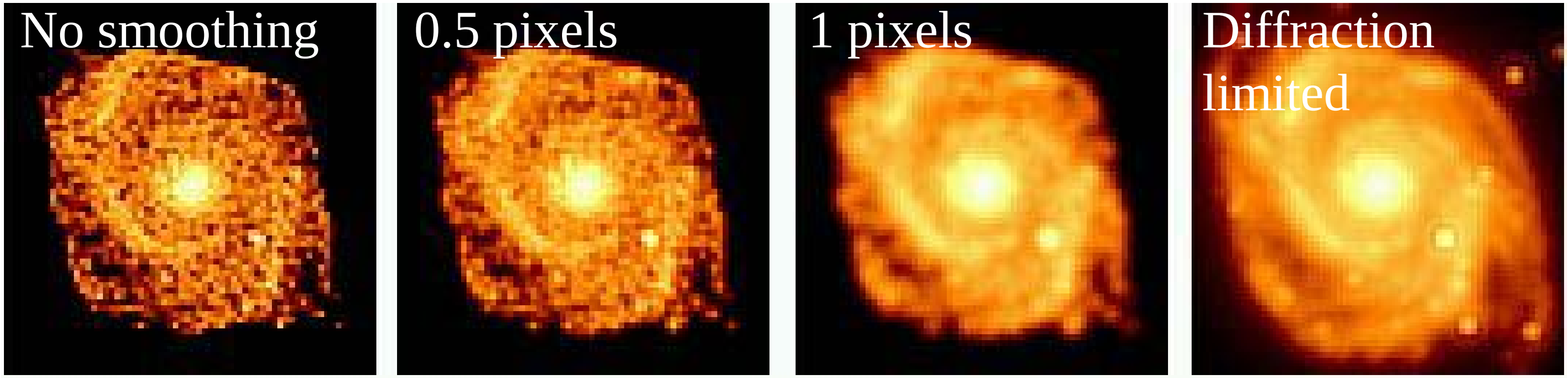
- This is a problem with any of these types of algorithms which use maximum-likelihood approaches including error metric minimization schemes.

Richardson-Lucy Application
Regularization Schemes

Sophisticated and silly!

- Why not smooth the result? – a low-pass filtering!

SNR = 250 – 5000 iterations



- What is the reliability of the high SNR region?
- Is it oversmoothed or undersmoothed?

Maximum *a posteriori* (MAP)

Regularized Maximum-likelihood

The *posterior* probability comes from Bayesian approaches, i.e. the probability of *f* being the object given the measurement *g* is:

$$P(f \mid g) = \frac{P(g \mid f)\ P(f)}{P(g)}$$

where $P(g|f) = \prod \dfrac{\exp\{-\sum_{\mathrm{j}} h_{kj} f_j\}(\sum_j h_{kj} f_j)^{g_k}}{g_k!}$

and $P(f)$ is now the *prior probability distribution* (prior)

Maximum *a posteriori* (MAP)

- Poisson maximum *a posteriori* - Hunt & Semintilli

$$\hat{f}^{n+1} = \hat{f}^{n} \exp\left\{\left(\frac{g(r)}{\hat{f}^{n}(r) * h(r)}\right) \otimes h(r)\right\}$$

  - $*$ denotes convolution
  - $\otimes$ denotes correlation

- Positivity assured by exponential
- Non-linearity permits super-resolution, i.e. recovery of spatial frequencies for $f > f_c$

Other Regularization Schemes

- Physical Constraints
  - Object positivity
  - Object support
    (finite size of the object, e.g. a star is a point)
  - Object model
    - Parametric
    - texture
  - Noise modeling
  - The imaging process

Regularization Schemes

- Reparameterization of the object with a smoothing kernel – (sieve function or low-pass filter).

$$f(r) = \phi(r)^2 * a(r)$$

- Truncated iterations

$$E = \sum_{g \in R} \left[g_i - \left(\hat{f}_i * h_i \right) \right]^2$$

stop convergence when the error-metric reaches the noise-limit, $g_i = g'_i + n_i$, such that $E \to N\sigma_n^2$

Object Prior Information

Prior information about the target can be used to modify the general algorithm.

- Multiple point source field – *N* sources

$$f(r) = \sum_{j}^{N} \left[A_j \delta(r - r_j) \right]$$

Solve for three parameters per component:

amplitude A_j

location r_j

Object Prior Information

Planetary/hard-edged objects (avoids ringing)

(Conan et al, 2000)

Use of the finite-difference gradients $\Delta f(r)$ to generate an extra error term which preserves hard edges in $f(r)$.

α & β are adjustable parameters.

$$E_{\mathrm{FD}} = \alpha \sum_r \left[\frac{\Delta f(r)}{\beta} - \ln\left(1 + \frac{\Delta f(r)}{\beta}\right) \right]$$

Object Prior Regularization - Texture

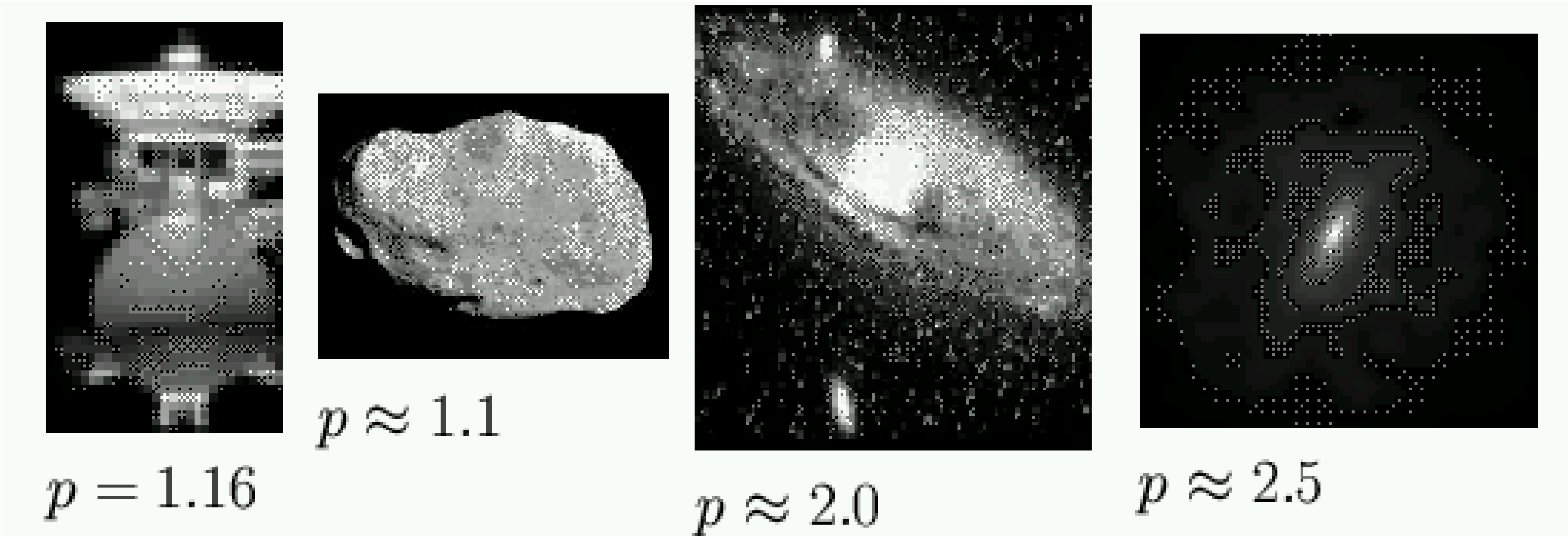
Generalized Gauss-Markov Random Field Model (Jeffs)

a.k.a. Object "texture" – local gradient

$$E_{\mathrm{GM}} = \sum_{i,\, j(i \neq j)} b_{i.\, j} \left| f_i - f_j \right|^p$$

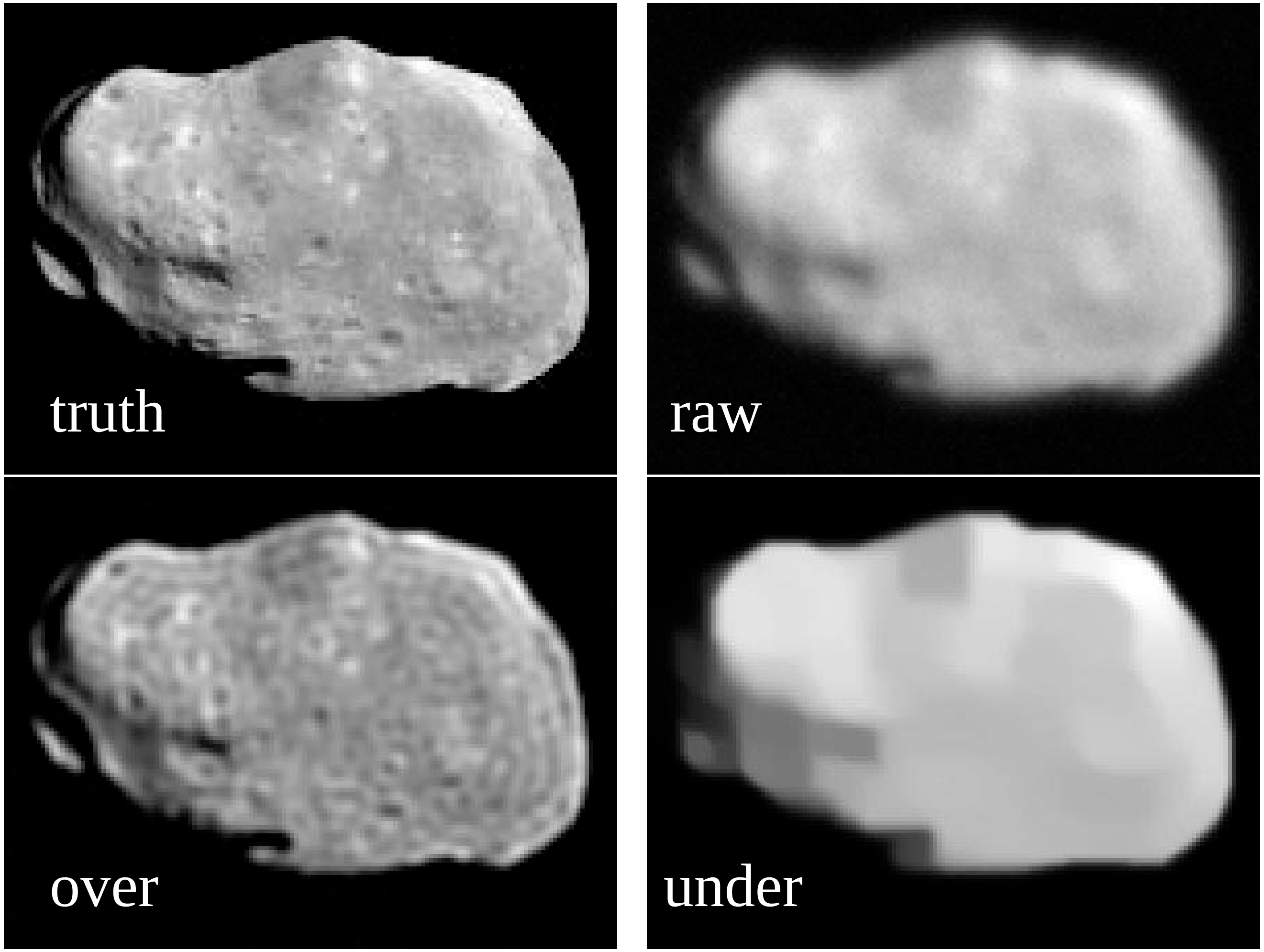
$b_{i,j}$ - neighbourhood influence parameter

p - shape parameter



Object Prior Regularization

Generalized Gauss-Markov Random Field Model



The Imaging Process

Model the preprocessing in the imaging process

- Light from target to the detector
  - Through optical path – PSF
  - Detector
    - Gain – (flat field) $a(r)$
    - Dark current – (darks) $d(r)$
    - Background – (sky) $b(r)$
    - Hot and dead pixels – included in $a(r)$
    - Noise
- Most algorithms work with “corrected” data
- Forward model the estimate to compare with the measurement

$$g''(r) = \left\{ \left[\hat{f}(r) * h(r) \right] + b(r) \right\} a(r) + d(r)$$

PSF Calibration: Variations on a Theme

- Poor or no PSF estimate – Myopic/Blind Deconvolution

- Astronomical imaging typically measures a point source reference sequence with the target.
  - Long exposure – standard deconvolution techniques
  - Short exposure – speckle techniques – e.g. power spectrum & bispectrum

- Deconvolution from wavefront sensing (DWFS)

  Use a simultaneously obtained wavefront to deconvolve the focal-plane data frame-by-frame. PSF generated from wavefront.

- Phase Diversity

  Two channel imaging typically in & out of focus. Permits restoration of target and PSF simultaneously. No PSF measurement needed.

Blind Deconvolution

$$g(\vec{r}) = f(\vec{r}) * h(\vec{r}) + n(\vec{r})$$

Measurement

unknown object

unknown or poorly known PSF

contamination

Need to solve for both object & PSF

“It’s not only impossible, it’s hopelessly impossible”

Blind Deconvolution – Key Papers

Ayers & Dainty, "Iterative blind deconvolution and its applications" , *Opt. Lett*. **13** , 547-549, 1988.

Holmes , "Blind deconvolution of speckle images quantum-limited incoherent imagery: maximum-likelihood approach" , *J. Opt. Soc. Am. A*, **9** , 1052-106, 1992.

Lane , "Blind deconvolution of speckle images" , *J. Opt. Soc. Am. A*, **9** , 1508-1514, 1992 .

Jefferies & Christou, "Restoration of astronomical images by iterative blind deconvolution" , *Astrophys. J.* **415**, 862-874, 1993.

Schultz , "Multiframe blind deconvolution of astronomical images" , *J. Opt. Soc. Am. A*, **10** , 1064-1073, 1993.

Thiebaut & Conan, "Strict *a priori* constraints for maximum-likelihood blind deconvolution" , *J. Opt. Soc. Am. A,* **12** , 485-492, 1995.

Conan et al., "Myopic deconvolution of adaptive optics images by use of object and point-spread function power spectra", *Appl. Opt*., **37**, 4614-4622, 1998 .

Multiple Frame Blind Deconvolution

m independent observations of the same object.

$$g_1(\vec{r}) = f(\vec{r}) * h_1(\vec{r}) + n_1(\vec{r})$$
$$g_2(\vec{r}) = f(\vec{r}) * h_2(\vec{r}) + n_2(\vec{r})$$
$$\vdots$$
$$g_m(\vec{r}) = f(\vec{r}) * h_m(\vec{r}) + n_m(\vec{r})$$

The problem reduces from

1 measurement & 2 unknowns

to

m measurements & *m*+1 unknowns

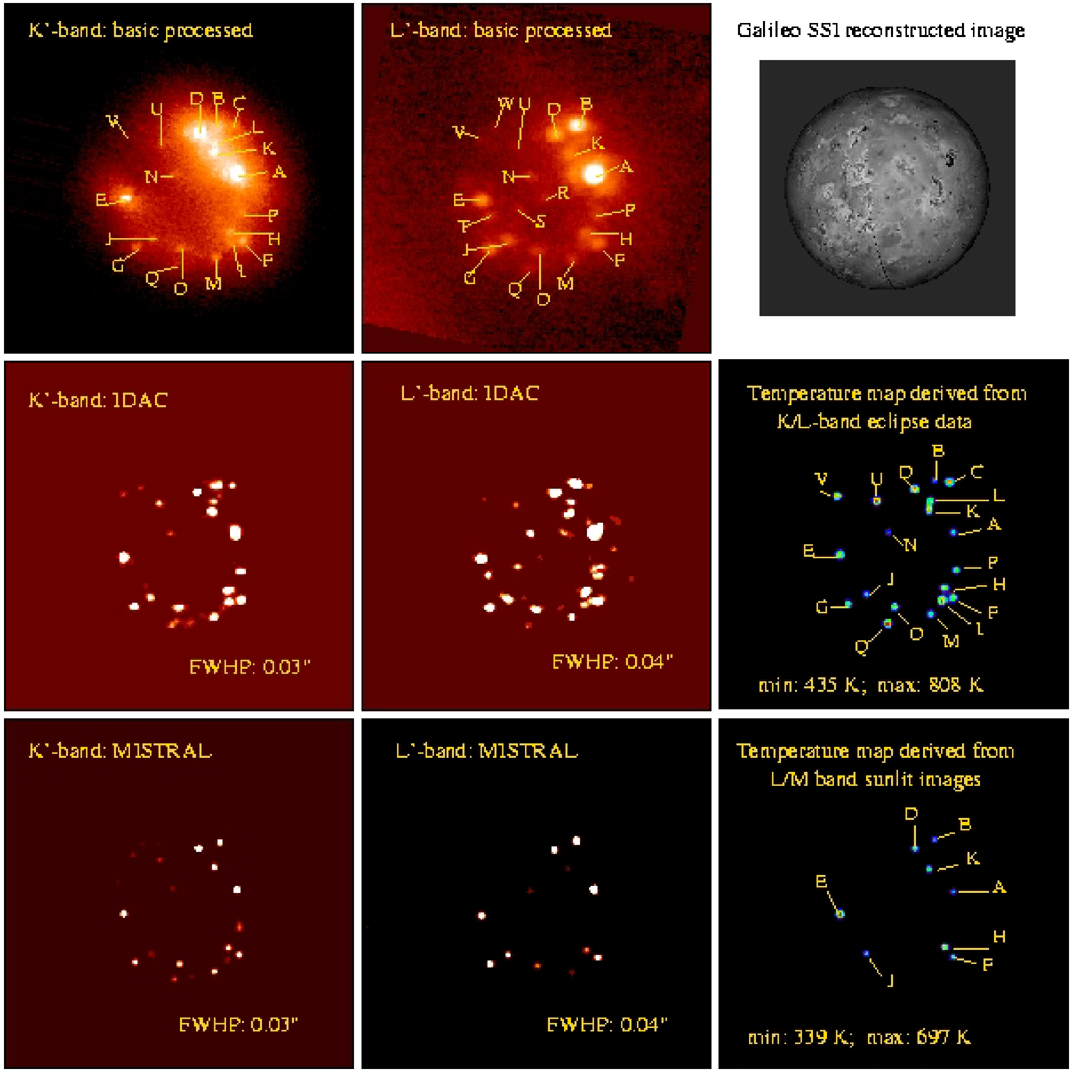
Physical Constraints

- “Blind” deconvolution solves for both *object* & *PSF* simultaneously.
  - Ill-posed inverse problem.
  - Under – determined: 1 measurement, 2 unknowns

- Uses Physical Constraints.
  - $f(r)$ & $h(r)$ are positive, real & have finite support.
    - Finite support reduces # of variables (symmetry breaking)
  - $h(r)$ is band-limited – symmetry breaking

- a priori information - further symmetry breaking ($a * b = b * a$)
  - Prior knowledge
  - PSF knowledge: band-limit, known pupil, statistical derived PSF
  - Object & PSF parameterization: multiple star systems
  - Noise statistics

Io in Eclipse (Marchis et al.)



Two Different BD Algorithms

Keck observations to identify hot-spots.

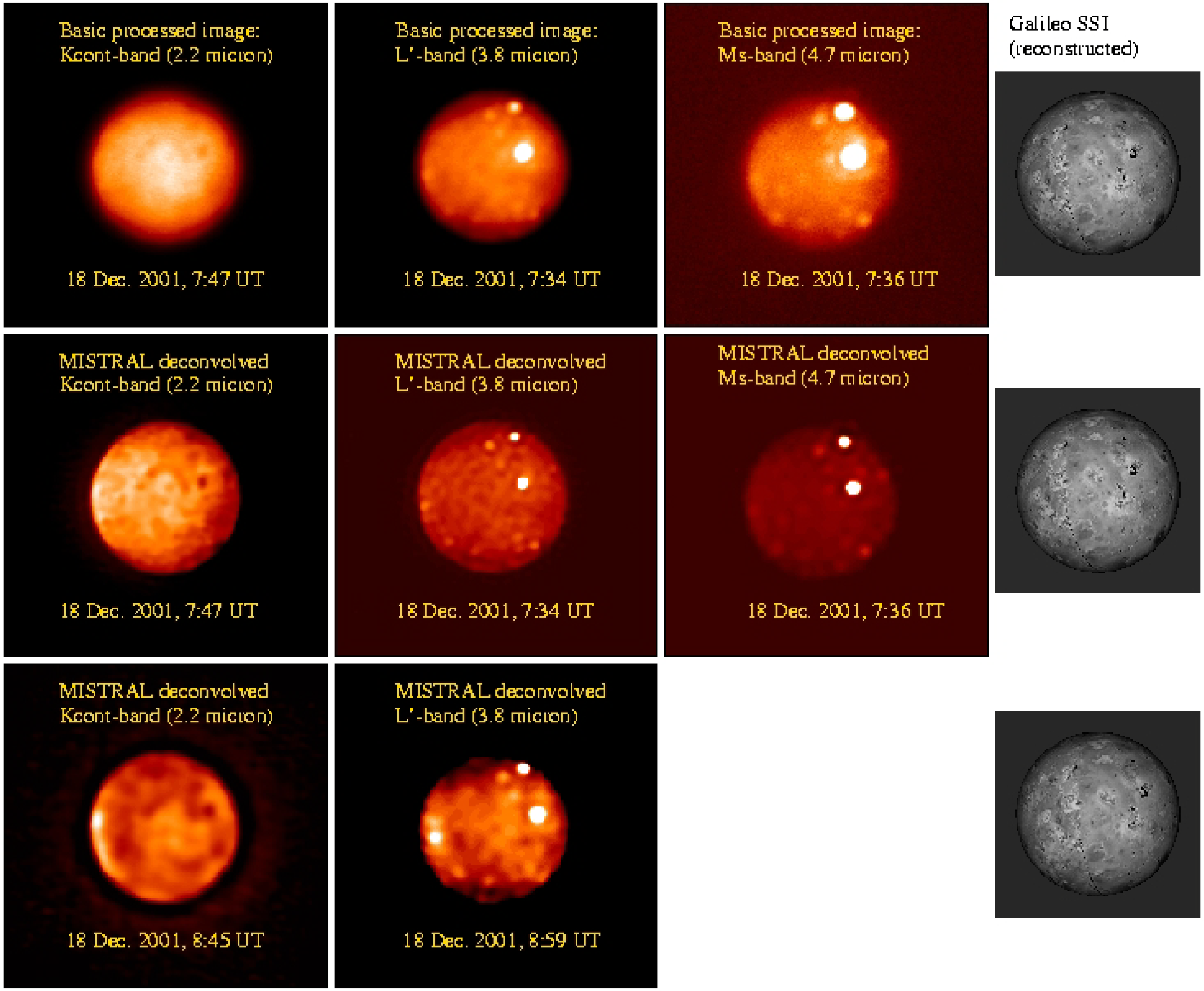
K-Band

19 with IDAC
17 with MISTRAL

L-Band

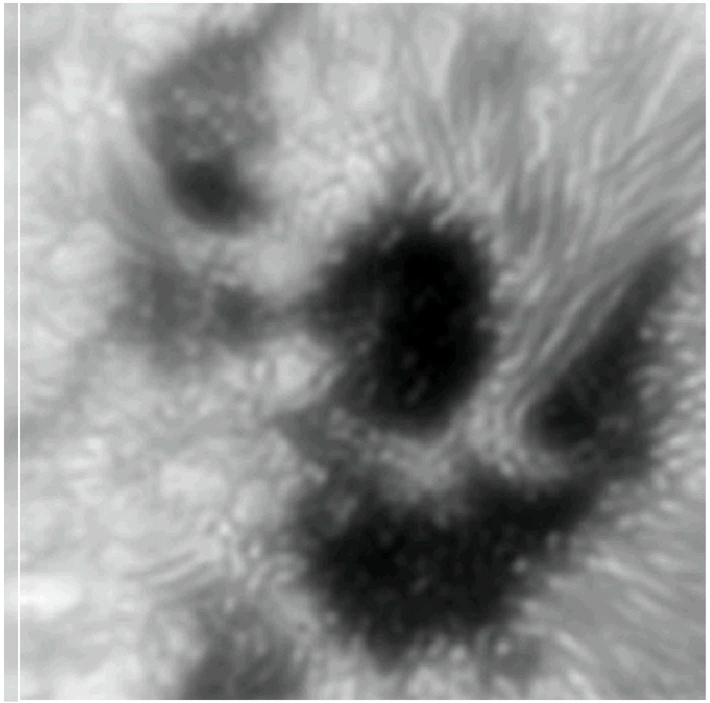
23 with IDAC
12 with MISTRAL

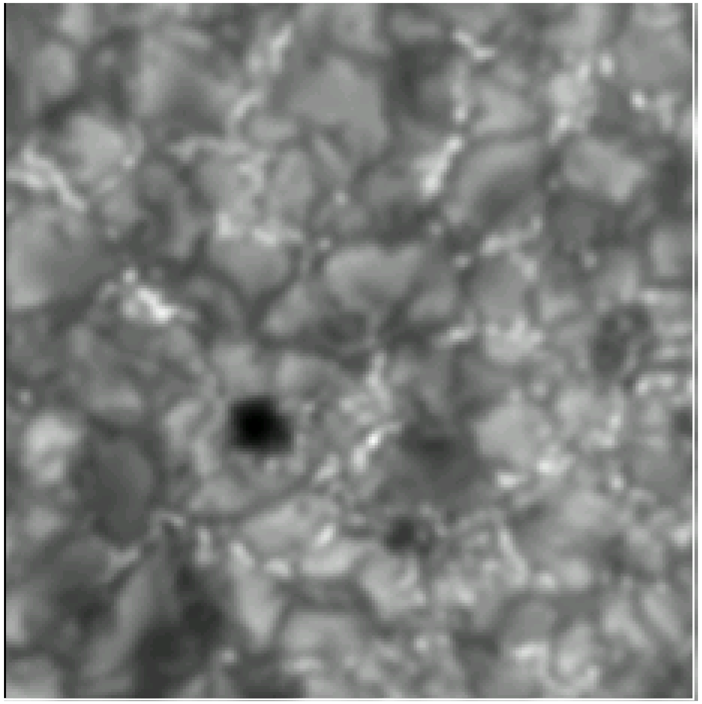
Io in Sunlight (Marchis et al.)



Solar Imaging

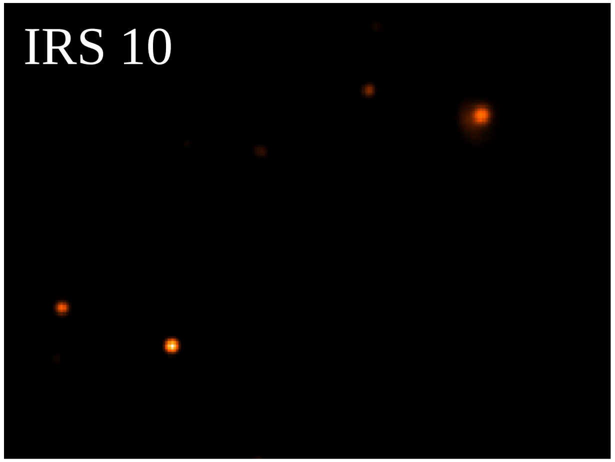
- Rimmele, Marino & Christou
  - AO Solar Images from National Solar Observatory low-order system.

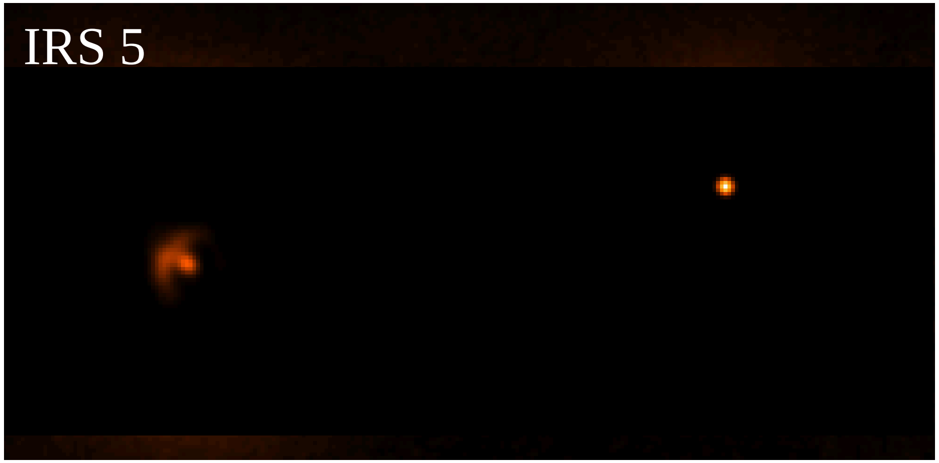


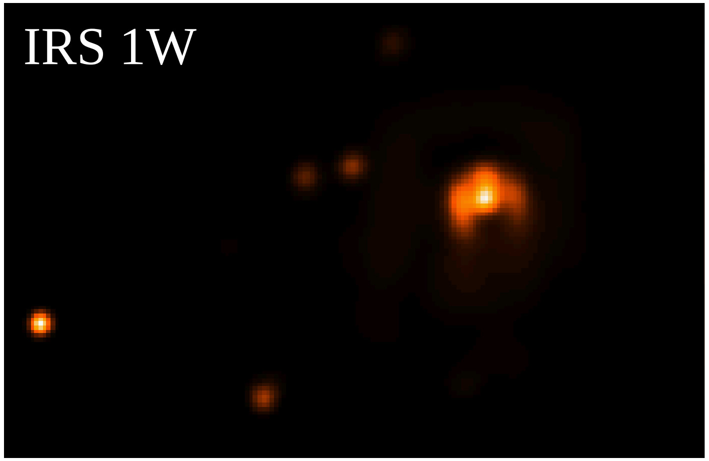


Extended Sources near the Galactic Center

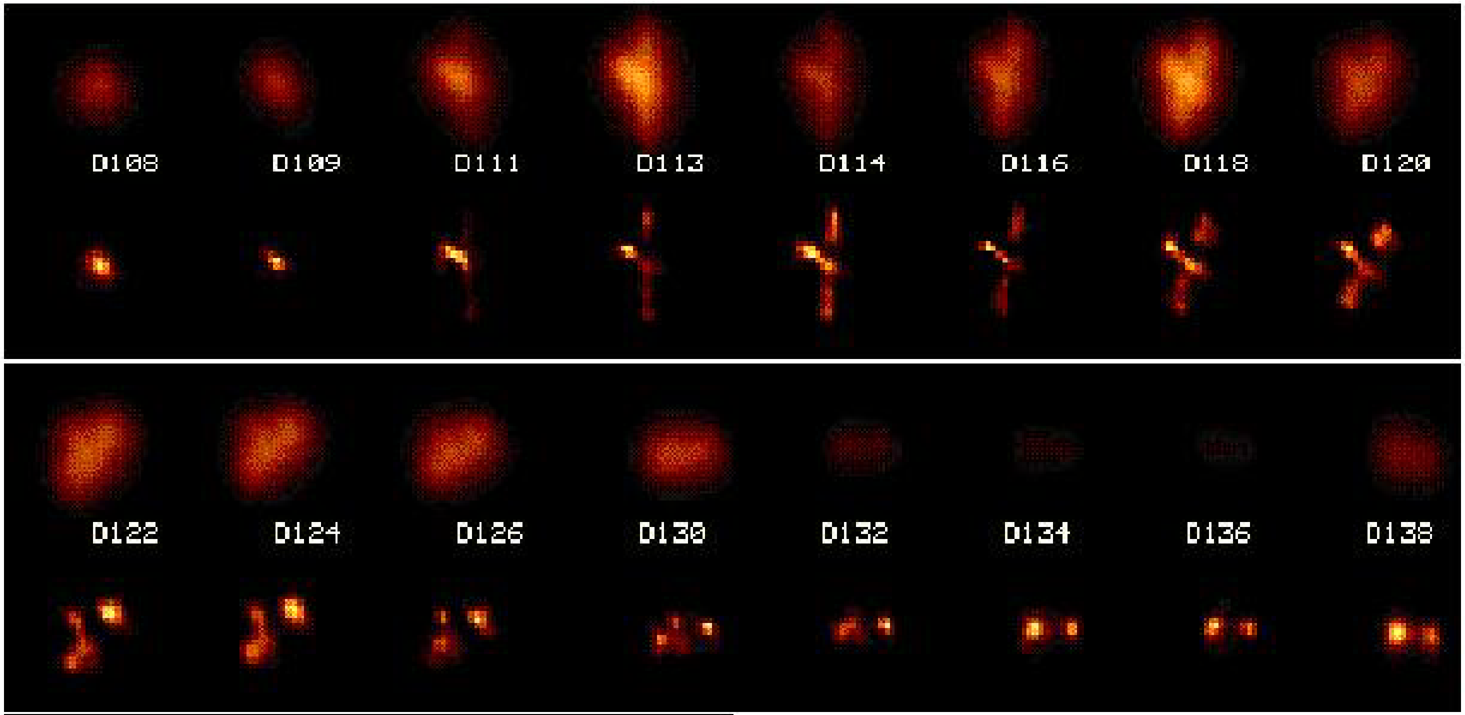
Tanner et al.

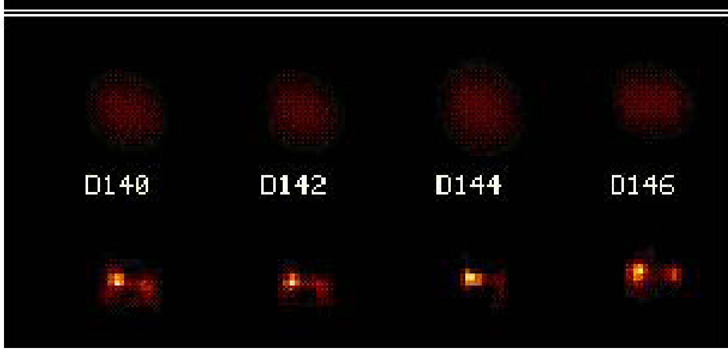


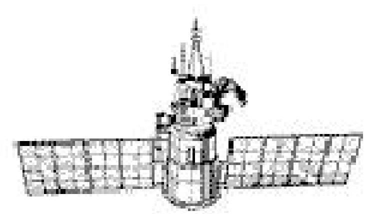




Artificial Satellite Imaging







Deconvolution from Wavefront Sensing

Multiframe deconvolution with a “known” PSF.

The estimate of the Fourier components of the target for a series of short-exposure observations is (Primot et al.) (also see speckle holography)

$$F'(f) = \frac{\left\langle G(f) H^{*}(f) \right\rangle}{\left\langle | H'(f) |^{2} \right\rangle} = F(f) \frac{\left\langle H(f) H'^{*}(f) \right\rangle}{\left\langle | H'(f) |^{2} \right\rangle}$$

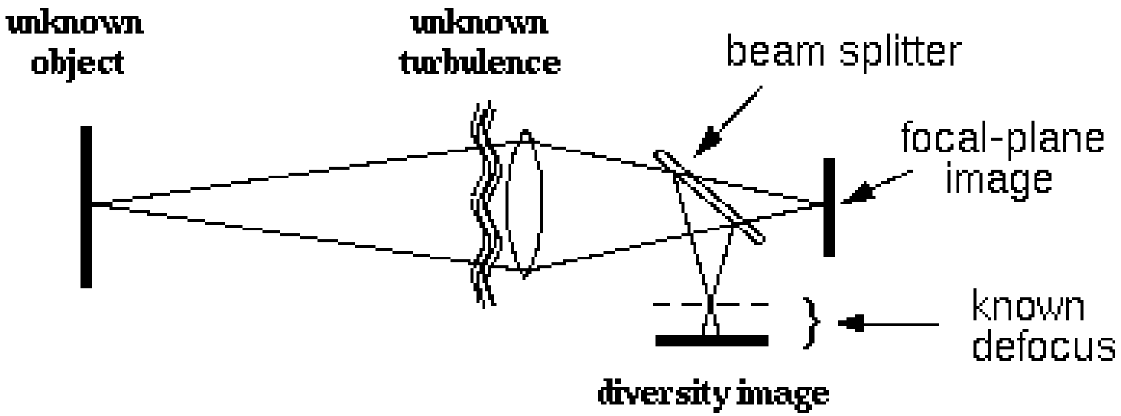
where $| H`(f) |^2 = H`(f)\, H`(f)^*$ and $H`(f)$ is the PSF estimate obtained from the measured wavefront, i.e. the autocorrelation of the complex wavefront at the pupil.

$F`(f) = F(f)$ when $H`(f) = H(f)$

Noise sensitive transfer function. Requires good SNR modeling.

Primot et al. “Deconvolution from wavefront sensing: a new technique for compensating turbulence-degraded images” , *J. Opt. Soc. Am. A*, **7**, 1598-1608, 1990.

Phase Diversity



Measurement of the object in two different channels. No separate PSF measurement.

$$g_1(r) = f(r) * h_1(r) \quad \text{and} \quad g_2(r) = f(r) * h_2(r)$$

Two measurements – 3 unknowns – $f(r), h_1(r), h_2(r)$ but $h_1(r)$ & $h_2(r)$ are related by a known diversity, e.g. defocus. Hence 2 unknowns $f(r), h_1(r)$

$$H_1(f) = |H_1(f)| \exp[i\vartheta(f)] \qquad H_2(f) = |H_1(f)| \exp[i\vartheta(f) + \alpha(f)]$$

Phase Diversity

Phase Diversity restores both the target and the complex wavefront phases at the pupil.

- Solve for the wavefront phases which represent the unknowns for the PSFs

- The phases can be represented as either
  - zonal (pixel-by-pixel)
  - modal (e.g. Zernike modes) – fewer unknowns

- The object spectrum can be written in terms of the wavefront phases, i.e.

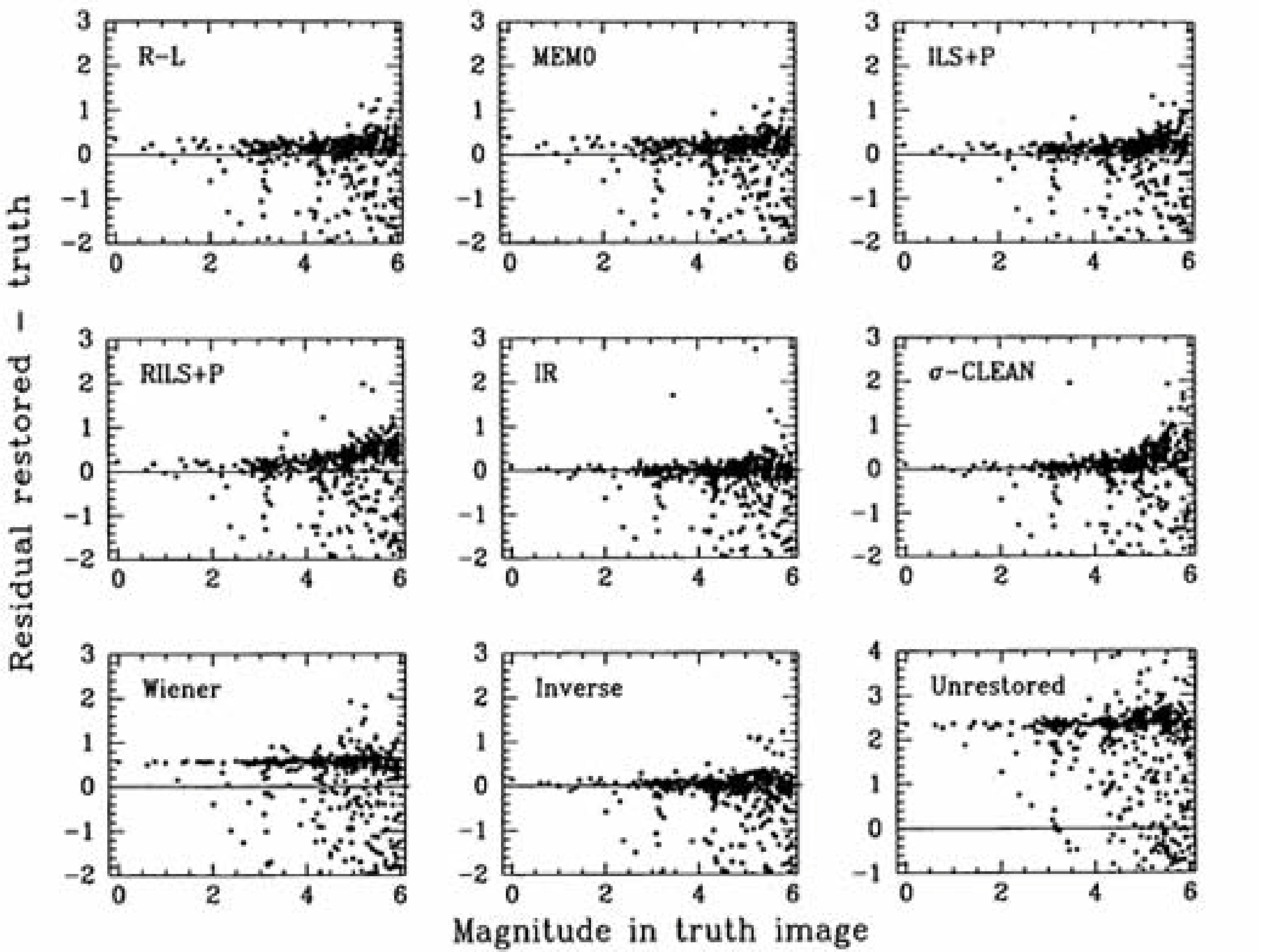
$$F(f) = \frac{\left[G_1(f)H_1^*(f) + G_2(f)H_2^*(f)\right]}{\left[|G_1(f)|^2 + |G_2(f)|^2\right]}$$

- Recent work suggests that solving for the complex wavefront, i.e. modeling scintillation improved PD performance for both object and phase recovery.

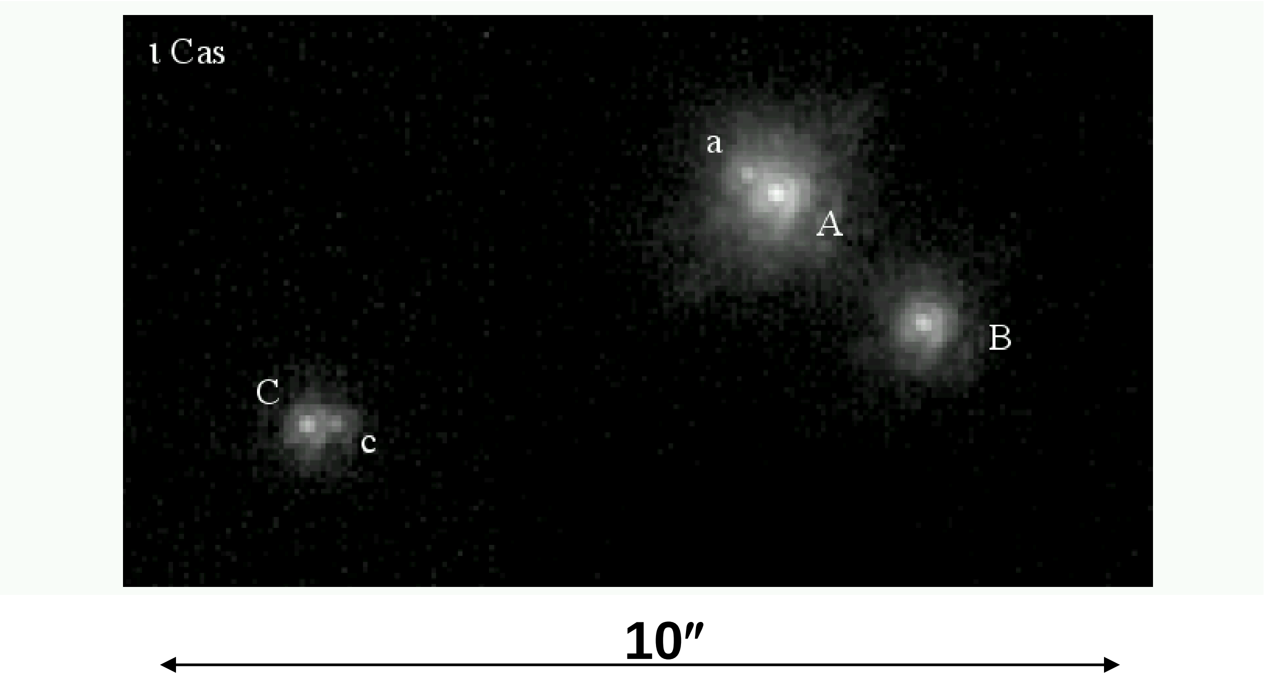
Photometric Quality in Crowded fields

Busko, 1993
HST Tests

“Should stellar photometry be done on restored or unrestored images?”



Photometric Quality in Crowded fields



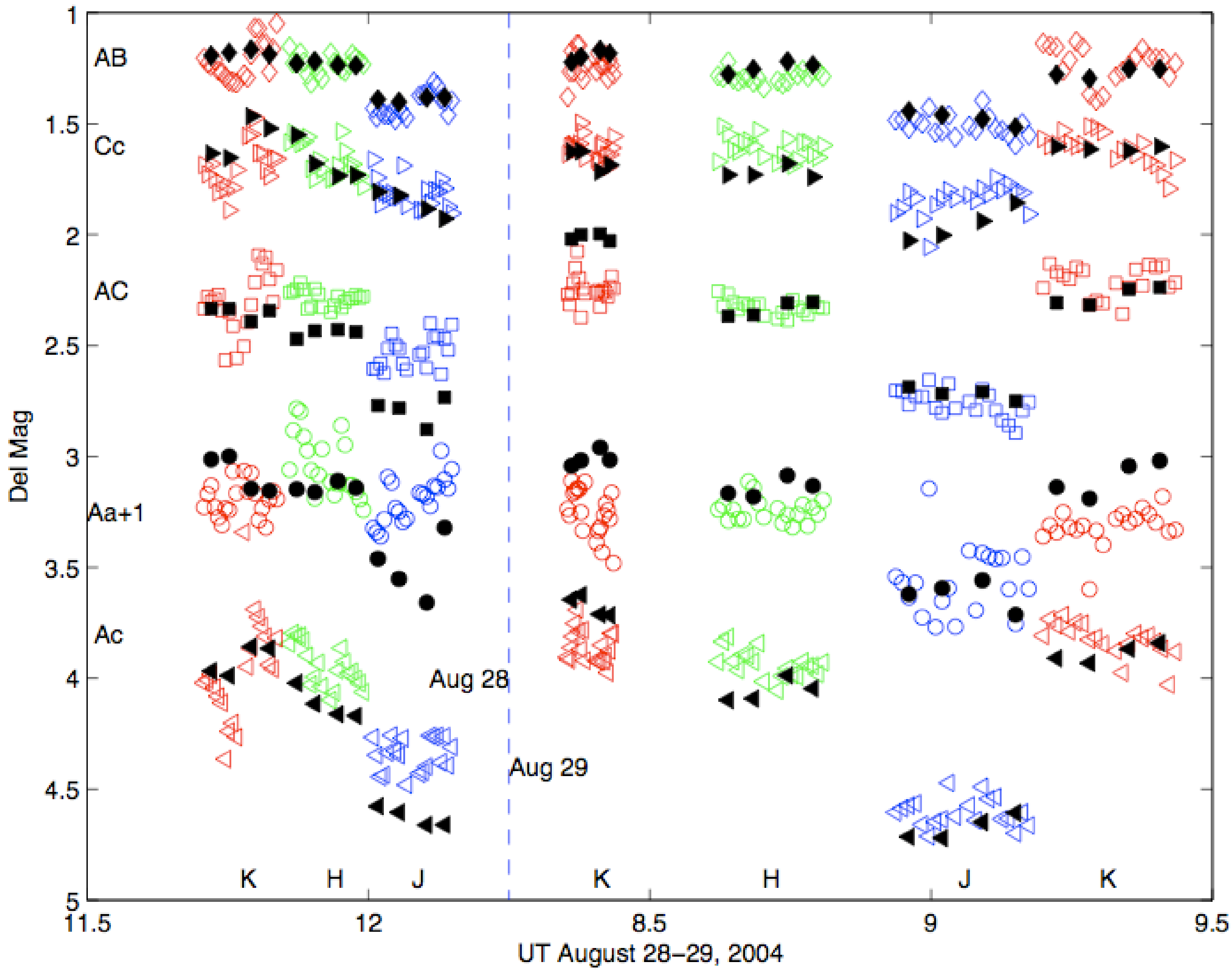
Photometric Quality in Crowded fields

Two Analysis Techniques:

1. Parametric Blind Deconvolution (PBD):
   - o Each star modeled as a 2D elliptical Lorentzian profile in a simultaneous fit
   - o Frame-by-frame
   - o A weighted mean for the separation (sep), position angle (PA) and magnitude difference (Δm) for the components.

2. Multi-Frame Blind Deconvolution (MFBD):
   - o MFBD finds a common solution to a set of independent images of the same field assuming that the PSF varies from one frame to the next.
   - o Multi-frame data subsets
   - o Each component constrained to be Gaussian
   - o 2D Elliptical Gaussian fits give separation (sep), position angle (PA) and magnitude difference (Δm)

Photometric Quality in Crowded fields



Summary

- Deconvolution is necessary for many applications to remove the effects of PSF
  - PSF calibration
  - identification of sources in a crowded field
  - removal of asymmetric PSF artifacts etc.

- A choice of algorithms available
  - Is any one algorithm the best?
    - different algorithms for different applications
    - algorithm comparison by different groups (Busko for HST & Stribling et al. for AFRL applications.
    - Preservation of photometry (radiometry) and astrometry (location) of sources in the image.

- What happens when the PSF is poorly determined?
  - This is a problem for many AO cases.

- What happens when the PSF is spatially variable?(anisoplanatism)

Homework (Fourier Transforms)

- Describe how the PSF & MTF of a telescope changes as the central obscuration gets larger for a given size pupil?

- How does an increase and decrease in the size of the telescope pupil affect the resolution and cut-off frequency?

- The Fourier transform is $G(u) = \mathrm{FT}\{g(x)\} = \int_{-\infty}^{\infty} dx\, g(x) \exp[-i2\pi(ux)]$
  and the Fourier modulus is $|G(u)|$

  a. Compute the Fourier modulus of $\delta(x-a)$?

  b. Compute the Fourier modulus of $A\delta(x-a) + B\delta(x-b)$?

- If the PSF of an optical system is described as a Gaussian, i.e.

$h(r) = \exp\left[-\left(\frac{r}{2\sigma}\right)^2\right]$ and the object as $h(r) = A\delta(r-a) + B\delta(r-b)$, what is the expression for the measurement $g(r)$?

Reference Material

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