Conservation Laws

Let u be a density and f be a flux. We define a conservation law (1D) as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Integrating with respect to x we get

$$\frac{d}{dt}\int_a^b u(x,t)dx = -\int_a^b \frac{\partial f(u)}{\partial x}dx = f(a) - f(b)$$

Hence, the change in the total "mass" changes only through fluxes that enter and leave the boundary. In several dimesnions this generalizes to

$$\frac{\partial u}{\partial t} + div(f) = 0$$

and

$$\frac{d}{dt}\int\int_D u(\vec{x},t)d\vec{x} = -\int\int div(f)d\vec{x} = \oint fds$$

Generalized solution

For a linear problem discontinuities occur only across characteristics. For nonlinear problems we can have discontinuities anywhere. We need to define a solution when it is discontinuous

$$\int_{x_L}^{x_R} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \, dx = \frac{d}{dt} \int_{x_L}^{x_R} \frac{\partial u}{\partial t} \, dx + f_R - f_L$$

Hence, space derivatives no longer appear. More generally let $\phi \in C^\infty$ (compact support and infinite number of derivatives). Then

$$(*) \qquad \int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \phi \, dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \phi \, dx - \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} f \, dx$$

Def: A solution is a generalized solution if (*) is true for every ϕ in some class of functions

Note: A classical solution is a generalized solution but not necessarily conversely. Uniqueness may be lost when including generalized solutions !!

Rankine-Hugoniot

Define:

$$I(t) = \int_a^b u(x,t)dx = \int_a^{y(t)} u(x,t)dx + \int_{y(t)}^b u(x,t)dx$$

Denote: $\quad u_l = u(y^-) \quad u_r = u(y^+) \quad s = \frac{dy}{dt}$

$$\frac{dI}{dt} = \int_a^y \frac{\partial u}{\partial t}(x,t)dx + u_l s + \int_y^b \frac{\partial u}{\partial t}(x,t)dx - u_r s$$

but $\frac{du}{dt} = \frac{\partial f}{\partial x}$. So

$$\begin{aligned} \frac{dI}{dt} &= - \int_a^y \frac{\partial f}{\partial x}(x,t)dx - \int_y^b \frac{\partial f}{\partial x}(x,t)dx + (u_l - u_r)\, s \\ &= f_a - f_l - f_b + f_r + (u_l - u_r)\, s \end{aligned}$$

but $\frac{dI}{dt} = \int_a^b \frac{\partial f}{\partial x}(x,t)dx = f_a - f_b$
So

$$\begin{aligned} f_r - f_l &= (u_r - u_l)s \\ s\,[u] &= [f] \end{aligned}$$

example

$$f(u) = \frac{u^2}{2}$$

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + u\ u_x = 0$$

So the slope of the characteristic curves are: $\frac{dt}{dx} = \frac{1}{u}$.

Choose as initial data

$$u_0(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & 1 \leq x \end{cases}$$

Until t=1 the solution is single valued.
To choose the solution after t=1 we use the Rankine-Hugoniot condition

$$s = \frac{[f]}{[u]} = \frac{\frac{u_r^2}{2} - \frac{u_l^2}{2}}{u_r - u_l} = \frac{\frac{0-1}{2}}{0-1} = \frac{1}{2}$$

So for $t > 1$ we choose the solution

$$u(x,t) = \begin{cases} 1 & x \leq \frac{1+t}{2} \\ 0 & \frac{1+t}{2} \leq x \end{cases}$$

Entropy Condition

Characteristics on either side of the discontinuity in the direction of increasing t intersect the line of discontinuity. Hence, we can connect every point on the discontiuity back to the initial line. Equivalently

$$a(u_l) > s > a(u_r) \qquad a = \frac{\partial f}{\partial u}$$

An equivalent formulation is

$$u_t^\varepsilon + f(u^\varepsilon)_x = (\varepsilon u_x)_x \quad \text{and} \qquad \varepsilon \to 0$$

example

$$f(u) = \frac{u^2}{2}$$

Choose as initial data

$$u_0(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$$

Instead of characteristics colliding we have a gap. Choose

$$u(x,t) = \begin{cases} 0 & x \leq \frac{t}{2} \\ 1 & x > \frac{t}{2} \end{cases}$$

This satisfies the Rankine-Hugoniot condition.
Consider

$$u(x,t) = \frac{x}{t}$$

This also satisfies the R-H condition. However, only the second one satisfies the entropy condition.

Definition: A solution is a shock if it satisfies bothe Rankine-Hugoniot condition and the entropy condition