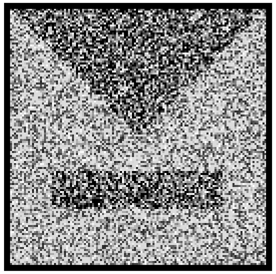
Multiscale image segmentation and anisotropic diffusion

Diffusion approach



Denoising - How can we get a cleaner image?

$$\frac{\partial}{\partial t} U = div(D \cdot \nabla U)$$

$$\nabla U = \begin{pmatrix} U_x \\ U_y \end{pmatrix}$$

- D a scalar
- D_{2X2} a diffusion tensor
  a positive definite symmetric matrix

D determines the behavior of the diffusion process

Diffusion examples

Vector field $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ $\qquad div\ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$

Ex. 1: Isotropic diffusion (Heat Equation)

$\boldsymbol{D \equiv 1}$

$$\frac{\partial}{\partial t} U = div(\nabla U) \equiv \Delta U$$

$$U(x, y, 0) = f(x, y)$$

$$\Rightarrow \boldsymbol{U(x, y, t) = G_{\sqrt{2t}} * f}$$

$$\boldsymbol{U(x, y, t+1) = \begin{bmatrix} & \tau & \\ \tau & 1-4\tau & \tau \\ & \tau & \end{bmatrix} \otimes U(x, y, t)}$$

The solution at every time t is the **input image** f convolved with a Gaussian with variance $2t$

Diffusion Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty$$
$$u(x,0) = f(x)$$

Properties of Diffusion equation

Maximum Principle

Theorem 1 • *Maximum Principle For* $0 \le x, y \le l \quad 0 \le t \le T$ *the maximum of* $u(x,t)$ *occurs either at* $t = 0$ *or along the boundaries* $x = 0$ *or* $x = l$ *or* $y = 0$ *or* $y = l$

(proof at the maximum $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ *and* $\frac{\partial^2 u}{\partial x^2} \le 0$ *and* $\frac{\partial^2 u}{\partial y^2} \le 0$ *)*

- *The solution exists for all initial data* f *and is unique*
- *If* $u(x,0) = f(x)$ *and* $v(x,0) = g(x)$

  Then

  $\int\int(u(x,t) - v(x,t))^2 dx \le \int\int(f(x) - g(x))^2 dx$

Fourier integral in x

$$\widehat{u}(\xi,t) = \int\int_{-\infty}^{\infty} e^{-ix\xi} u(x,t)dx$$

Then

$$\widehat{u}_t = -k\xi^2\widehat{u}$$

So

$$\widehat{u}(\xi,t) = \widehat{u}(\xi,0)e^{-k\xi^2 t}$$

So

(a) $k > 0$ *gives exponential decay*

(b) $k < 0$ *gives exponential growth* *not well posed !!*

explicit solution

For $-\infty < x < \infty$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)dy$$

So

(a) $k > 0$ *diffusion operator is smoothing*

b $k < 0$ *(-diffusion operator) is sharpening*

Definition 2

$$K_\sigma = \frac{1}{2\pi\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Then the solution to the diffusion equation is

$$u(x,t) = \begin{cases} f(x) & t = 0 \\ (K_{\sqrt{2t}} * f)(x) & t > 0 \end{cases}$$

and

$$|u(x,t)| \leq M e^{ax^2}$$

i.e.

$$\sigma = \sqrt{2t}$$
$$T = \frac{1}{2}\sigma^2$$

Note:

$$\mathcal{F}(K_\sigma) = e^{-\frac{\frac{\omega^2}{2}}{\sigma^2}}$$

Scale Space

Image representation at a continuum of scales.
Consider a sequence $T_t f$ with

$$
\begin{aligned}
T_0 f &= f \\
T_{t+s} &= T_t(T_s f) \qquad \text{semi-group}
\end{aligned}
$$

We want the following properties

- no new level curves
- non-enhancement of local extrema
- decreasing number of extrema
- maximum principle

Definition 3 *Gaussian Scale Space: Convolution of the picture with a Gaussian with increasing σ^2.*

Note that this is equivalent to the diffusion equation with $t \to 0$.

Theorem 4 *Let*

$$\Phi(f, x, \sigma) = \int \int_{-\infty}^{\infty} \phi(f(x'), x, x', \sigma) dx'$$

Assume

- $\Phi(Af, x, \sigma) = A\Phi(f, x, \sigma)$
- $\Phi(f(x^{'} - a), x, \sigma) = \Phi(f, x - a, \sigma)$
- $\Phi(f(\frac{x'}{\lambda}, x, \sigma) = \Phi(f, \frac{x}{\lambda}, \sigma)$
- $\Phi\left[\Phi(f(x''), x', \sigma_1), x, \sigma_2\right] = \Phi(f(x''), x, \sigma_3)$
- $\Phi(f, x, \sigma) > 0 \qquad$ *when* $f > 0 \quad$ *and* $\sigma > 0$

Then

$$\Phi(f, x, \sigma) = \int_{-\infty}^{\infty} f(x') e^{-\frac{(x-x')}{4\sigma^2}} dx' = K_{\frac{\sigma}{\sqrt{2}}} * f$$

Image Segmentation:

1. Compute edge regions by threshold on a gradient

2. Thin the edges

3. Close the gap between edge segments

4. Determine regions by connectivity

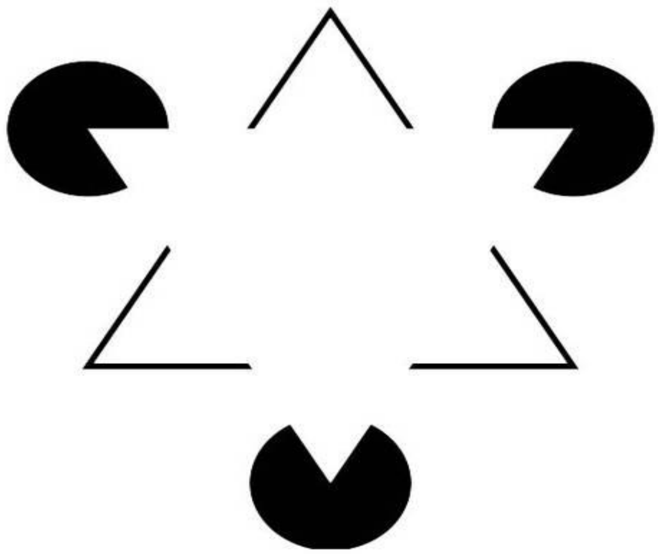
5. Eliminate small regions

6. Shrink again edge regions

7. again add edge points to close gaps

8. eliminate small edge regions

9. calculate properties of uniform untensity regions



Importance of Variational Formulation

1. Axiomatic

2. comparison between different segmentations

3. Most practical methods lead to a minimization problem

Multiscale Analysis

Given an image u_0 we wish to construct a sequence of simpler images u_λ where as λ increases the image becomes coarser. u_λ keeps edges whose scale exceeds λ. $(K_\lambda, u_\lambda): S_\lambda(u_0) = u_\lambda$ where K_λ is the set of edges at scale λ.

basic properties

1. Fidelity: $\lim_{\lambda \to 0} u_\lambda = u_0$

2. Causality: $S_\lambda(u_0)$ depends only on $u_{\lambda'}$ where $\lambda' > \lambda$

3. Euclidean Invariance: If A is an isometry then $S_\lambda(u_0 \circ A) = S_\lambda(u_0) \circ A$

4. Strong Causality: $K_\lambda \subset K_{\lambda'}$ where $\lambda' > \lambda$

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Ex. 2: Anisotropic diffusion

Perona-Malik 1990

$$\frac{\partial}{\partial t}U = div(c\,\nabla U) \equiv \nabla c \cdot \nabla U + c\,\Delta U$$

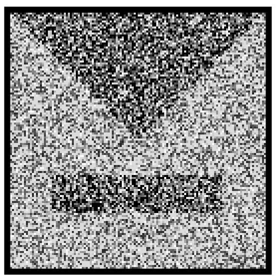
$\boldsymbol{D = c(x,y,t)}$ controls the diffusion rate. Typically a function of the local image gradient to preserve edges

Perona-Malik kernel: $c(\|\nabla U\|) = \frac{1}{1+[\frac{\|\nabla U\|}{K}]^2}$

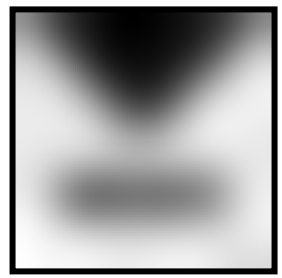
Exponential kernel: $c(\|\nabla U\|) = e^{-(\|\nabla U\|/K)^2}$

The extension of the Perona-Malik kernel for multi-channel is the well-known Beltrami flow (Kimmel, Malladi, Sochen, 98)

Diffusion: restoration properties



$t = 80$



Isotropic diffusion

$t = 80$

Anisotropic diffusion

Trade off - "smoothing" and "edge preservation"

Courtesy of Weickert

Hildreth-Marr

$$E_\lambda(u)=\lambda^2\int_\Omega|\nabla u|^2\,dx+\int_\Omega(u-u_0)^2dx$$

Peronna-Malik

$$\begin{aligned}&\frac{\partial u}{\partial t}=div(f(|\nabla u|^2)\nabla u)\\&=2\left(u_x{}^2u_{xx}+u_y{}^2u_{yy}+2u_xu_yu_{xy}\right)f^{'}(|\nabla u|^2)+f(|\nabla u|^2)\left(u_{xx}+u_{yy}\right)\\&u(0)=u_0\end{aligned}$$

Define $\mathrm{g}(\mathrm{s})\ =\ \mathrm{f}(\mathrm{s})+2\mathrm{sf}^{'}(s),\quad b>0$. Then this is equivalent to

$$\frac{\partial u}{\partial t}=f(|\nabla u|^2)u_{TT}+g(|\nabla u|^2)u_{NN}$$

Giving a normal and tangential diffusion. We wish to smooth more in the tangential direction T and so we demand

$$\lim_{s\to\infty}\frac{g(s)}{f(s)}=0\quad\text{or equivalently}\quad\lim_{s\to\infty}\frac{sg^{'}(s)}{g(s)}=-\frac{1}{2}$$

This implies $g(s)\approx\frac{1}{\sqrt{s}}$ as $s\to\infty$

Set $F(t)=\int_0^t f(s)ds$ and $E(u)=\frac{1}{2}\int_\Omega F(|\nabla u|^2)dx$.

Then steepest descent gives Peronna-Malik. So

$$E_\lambda(u)=\lambda^2\int_\Omega|F(\nabla u|^2)dx+\int_\Omega(u-u_0)^2dx$$

yields

$$\begin{aligned}&\frac{\partial u}{\partial t}=div(f(|\nabla u|^2)\nabla u)+(u_0-u)\\&u(0)=u_0\end{aligned}$$

Choosing $f(x)=1$ recovers the heat equation. We now assume $f(0)=1$ and f is a decreasing function with $\lim_{s\to\infty} f(s)=0$

Then

1. In regions where the gradient is small we get the heat equation and so isotropic diffusion.

2. Near boundaries where the gradient is large there is no regularization and the edges are preserved.

Difficulties:

1. Noise can introduce large oscillations in ∇u and noise edges will be kept.

2. Choosing $f(s)=\frac{1}{1+s^2}$ no theory is available.

Combine Marr-Hildreth with Perona-Malik

$$\frac{\partial u}{\partial t}=div(f(|\nabla G_\sigma * u|)\nabla u)+(u_0-u) \quad G_\sigma(x)=\frac{1}{4\pi\sigma}e^{-\frac{\|x\|^2}{4\sigma}}$$
$$u(0)=u_0$$

Note, that $|\nabla G_\sigma * u|(x,t)$ is the gradient of the solution of the heat equation.

We would like to have diffusion only in the direction of the edge. i.e. let ξ be the directional orthogonal to ∇u. Then $\xi=\xi(\nabla u)$ and the heat equation becomes

$$\frac{\partial u}{\partial t}=u_{\xi\xi}$$

or

$$\frac{\partial u}{\partial t}=|\nabla u|\, div\left(\frac{\nabla u}{|\nabla u|}\right)=\frac{u^2_y u_{xx}-2u_x u_y u_{xy}+u^2_x u_{yy}}{u^2_x+u^2_y}$$

This models the mean curvature motion.

Perona-Malik

Gaussian (linear) diffusion smooths noise but also smooths edges
We want a process that reduces diffusion at the edges (or even sharpens them)
So we have piecewise smoothing between the edges.

Consider

$$\frac{\partial u}{\partial t} = div\left[g(|\nabla u|^2)\ grad(u)\right]$$

with

$$g(s^2) = \frac{1}{1+\frac{s^2}{\lambda^2}} \qquad \lambda > 0 \text{ a parameter}$$

In one dimension

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} g\left(\left(\frac{\partial u}{\partial x}\right)^2\right) \frac{\partial u}{\partial x} \\
&= g\left(\left(\frac{\partial u}{\partial x}\right)^2\right) \frac{\partial^2 u}{\partial x^2} + g' 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} \\
&= \left[g + 2\left(\frac{\partial u}{\partial x}\right)^2 g'\right] \frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Define

$$\Phi(s) = sg(s^2)$$

then

$$\frac{\partial u}{\partial t} = \Phi'(|\nabla u|^2)\Delta u$$

For Perona-Malik

$$\begin{aligned}
\Phi(s) &= sg(s^2) = \frac{s}{1+\frac{s^2}{\lambda^2}} \\
\Phi'(s) &= \frac{1-\frac{s^2}{\lambda^2}}{\left(1+\frac{s^2}{\lambda^2}\right)^2} \\
&= \begin{cases} \text{positive} & s < \lambda \\ \text{negative} & s > \lambda \end{cases}
\end{aligned}$$

To see maximum principle we consider simpler equation

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 &= \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) \\
&= \Phi'' \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \Phi' \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3}
\end{aligned}$$

Now assume that $\left(\frac{\partial u}{\partial x}\right)^2$ has a maximum at (x_0, t).Then

$$0 = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} < 0$$

Also $\displaystyle \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + \left(\frac{\partial^2 u}{\partial x^2}\right)^2$

So

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 = \Phi'\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} = \Phi' \cdot \text{negative}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 |_{x=x_0} = \begin{cases} \text{negative} & s < \lambda \\ \text{positive} & s > \lambda \end{cases}$$

Regularization: We frequently replace Perona-Malik by

$$\frac{\partial u}{\partial t} = \nabla\cdot(g(|\ \nabla G_\sigma * u\ |^2)\nabla u) \qquad G_\sigma \text{ is a Gaussian with mean zero and variance } \sigma^2$$

Two Dimensions

$$\frac{\partial u}{\partial t} = \Phi'(|\nabla u|^2)\frac{\partial^2 u}{\partial \eta^2} + g(|\nabla u|^2)\frac{\partial^2 u}{\partial \xi^2}$$

where ξ direction is perpendicular to ∇u (direction of greatest change in u) and η is parallel to ∇u

So
ξ direction is lines of constant gray level
η direction is lines of maximal change in gray levels

So we have forward diffusion in the ξ direction and
backward diffusion (sharpening) in the η direction

Well-posedness

- Existence of a weak solution in c^1 for finite time.
  Maximum principle
- Unstable with respect to perturbations
- staircasing - constant states divided by edges
  finer discretization gives more edges

Anisotropic diffusion

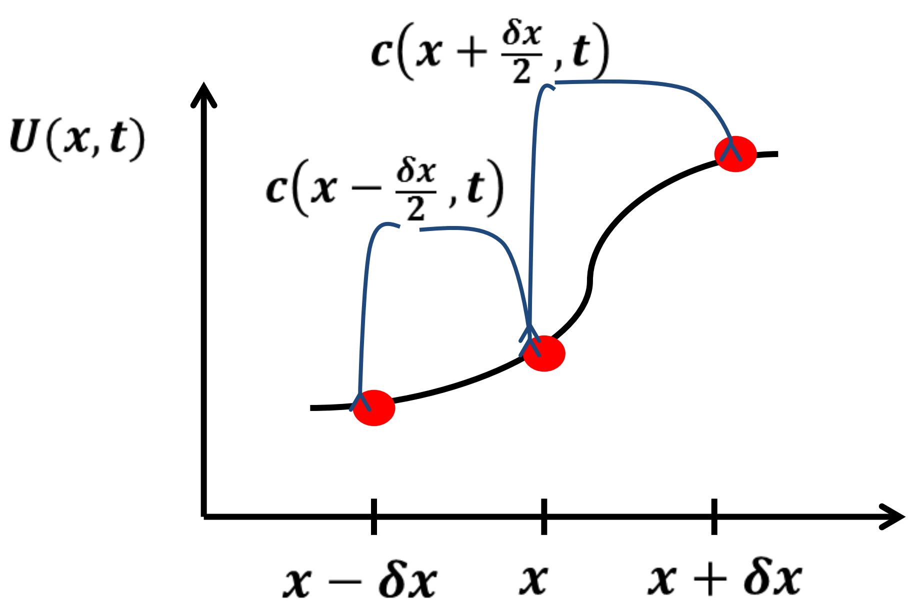
Discretization

1D:

$$\frac{\partial}{\partial t}U(x,t)=\frac{\partial}{\partial x}\left(c(x,t)\frac{\partial}{\partial x}U(x,t)\right)$$

$$\frac{U(x,t+\delta t)-U(x,t)}{\delta t}=\frac{1}{(\delta x)^2}\left(c\left(x-\tfrac{\delta x}{2},t\right)U(x-\delta x\,,t)+c\left(x+\tfrac{\delta x}{2},t\right)U(x+\delta x,t)\right)+$$

$$+\frac{1}{(\delta x)^2}\left[\left(-c\left(x-\tfrac{\delta x}{2},t\right)-c\left(x+\tfrac{\delta x}{2},t\right)\right]U(x\,,t)\right.$$



$$U(x,t)\rightarrow U(x,t+1)$$

Alvarez-Lions-Morel

$$\frac{\partial u}{\partial t} = f(|\nabla G_\sigma * u|)\,|\nabla u|\,div\left(\frac{\nabla u}{|\nabla u|}\right)$$

$$u(x,0) = u_0$$

The term

$$|\nabla u|\,div\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta u - \frac{D^2 u(\nabla u, \nabla u)}{|\nabla u|^2} \quad D^2 \text{ in direction of the gradient}$$

represents a diffusion in the direction normal to the gradient ∇u and does not diffuse in the direction parallel to the gradient. So u is smooth on both sides of an edge.

The term $f\left(|\nabla G_\sigma * u|\right)$ is used for enhancement of the edges. It controls the speed of the diffusion. If ∇u is small then then this point is considered an interior point of a smooth region and the diffusion is large. If ∇u is large then the point is considered an edge point and the diffusion is lowered since $f(s)$ is small for large s.

This depends on

1. The function f which decides whether a detail is shape enough to be kept, i.e. contrast. This is similar to the threshold in Marr-Hildreth

2. A scale parameter given by the variance of G. This gives the minimal size of details kept in the processed picture.

Weickert

A generalization of the above methods considers

$$\frac{\partial u}{\partial t} = div(D(J_{\rho} \, |\nabla u|_{\rho}))\nabla u) \;\; \text{in } (0,T]\times\Omega$$
$$u(0,x) = u_0(x) \qquad \text{in } \Omega$$
$$\left(D(J_{\rho} \, |\nabla u|_{\rho}))\nabla u, N\right) = 0 \quad \text{on } (0,T]\times\partial\Omega \quad \text{N is outward normal}$$

Theorem:

Assume

1. $D = \left(d_{ij}\right) \varepsilon \, C^{\infty}(S^2, S^2) \quad S^2 = \text{set of symmetric matrices}$

2. D is uniformly positive definite

Then for all $u_0 \; \varepsilon \; L^{\infty}(\Omega)$ there is a unique solution and this depends continuously on u_0 in the L^2 norm and satisfies a maximum principle

$$\inf_{\Omega} u_0(x) \le u(x,t) \le \sup_{\Omega} u_0(x)$$

We now show examples of the eigenvalues of D

edge enhancing
$$\lambda_1 = \begin{cases} 1 & \text{if } \mu_1 = 0 \\ 1 - e^{-\frac{3.315}{\mu^4_1}} & \text{otherwise} \end{cases}$$
$$\lambda_2 = 1$$

coherence enhancing
$$\lambda_1 = \alpha$$
$$\lambda_2 = \begin{cases} \alpha & \text{if } \mu_1 = \mu_2 \\ \alpha + \left(1-\alpha\right) e^{-\frac{1}{(\mu_1-\mu_2)^2}} & \text{otherwise} \end{cases}$$
α is a small positive parameter

This enhance flow-like structures and closes interrupted lines.

Osher-Rudin

Choose $f(s) = \frac{1}{s}$ and $\sigma = 0$. Then

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|}\right)$$

which is a particular case of Perona-Malik. This corresponds to minimizing the energy

$$E(u) = \int_{\Omega} |\nabla u|\, dx$$

We again generalize this to

$$E(u) = \lambda^2 \int_{\Omega} |\nabla u|\, dx + \int_{\Omega} (u - u_0)^2 dx$$

Remember: Hildreth-Marr was: $E_\lambda(u) = \lambda^2 \int_{\Omega} |\nabla u|^2\, dx + \int_{\Omega} (u - u_0)^2 dx$

So we have replaced $|\nabla u|^2$ by $|\nabla u|$ in the first integral

Total Variation (TV) schemes
Rudin-Osher-Fatemi

Definition 5

$$TV(u) = \int_\Omega |\nabla u| dx = \int_\Omega \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} dxdy$$

Rudin-Osher-Fatemi

$$\frac{\partial u}{\partial t} = -|\nabla u| F(\mathcal{L}(u))$$
$$u(x, o) = f(x) \qquad \text{given image}$$

$\mathcal{L}(u)$ is second order elliptic PDE whose zerocrossings correspond to edges
For example $\mathcal{L}(u) = \Delta u$
or else $\mathcal{L}(u) = \frac{\partial^2 u}{\partial \eta^2}$ with η normal to the edge i.e. $\eta \parallel \nabla u \parallel$

Minimization

Consider an error function

$$E_{\lambda,p}[z \mid x_0] = \int_0^1 \mid z^{'}(t) \mid dt + \lambda \int_0^1 \mid z(t) - x_0(t) \mid^p dt \qquad p \geq 1$$

The ROF TV denoising model is

$$E_{TV}[u|u_0] = \int_\Omega \mid \nabla u \mid dx \ + \frac{\lambda}{2} \int_\Omega (u_0 - u(x))^2 dx$$

To minimize $E_{TV}[u|u_0]$ we consider the Euler-Lagrange equations. This leads to the PDE and steepest descent

$$-\frac{\partial E_{TV}}{\partial t} = \frac{\partial u}{\partial t} = \nabla \cdot \left[\frac{\nabla u}{\mid \nabla u \mid}\right] - \lambda(u(x,t) - u_0(x))$$

with Neumann homogeneous boundary conditions. λ is a Lagrange multiplier.
To avoid a zero denominator $\mid \nabla u \mid^{-1}$ is frequently replaced by $\sqrt{\mid \nabla u \mid^{-2} + a^2}$ for some smal lparameter a

To improve the well-posedness of the solutions we regularize the equation and consider

$$\frac{\partial u}{\partial t} = div\left(g(|\nabla u_\sigma|)\nabla u\right)$$
$$u_\sigma = K_\sigma * u$$

λ and σ are constants that can be played with.
The filter is insensitive to noise at scales $< \lambda$

Segmentation problem

- Consists in computing a decomposition of the domain of the image ***g(x,y)***

$$R = \bigcup_{i=1}^{n} R_i$$

1. ***g*** varies smootly and/or slowly within R_i
2. ***g*** varies discontinuously and/or rapidly across most of the boundary Γ between regions R_i

Optimal approximation

- Segmentation problem may be restated as
  - finding optimal approximations of a general function ***g***
  - by piece-wise smooth functions ***f***, whose restrictions f_i to the regions R_i are **differentiable**
- **Many other applications:**
  - **Speech recognition**
  - **Sonar, radar or laser range data**
  - **CAT scans**
  - **etc…**

Mumford-Shah

The basic premise of Mumford-Shah is that images can be approximated by piecewise constant/smooth functions with each piece corresponding to an object.

This uses a energy-based variational optimization.

Definition 6 *H^1 norm of u*

$$\parallel u \parallel_{H^1}^2 = \int \left[u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right] dx$$

Definition 7 $E[u, \Gamma] = \alpha \parallel u \parallel_{H^1} + \beta \int_{\Omega \backslash \Gamma} \mid \nabla u \mid^2 dx$ *where Γ is an edge set.*

Definition 8 *Assume the given image is $u_0 = n + K[u]$ K=blur*
Assume additive noise n with variance $\sigma^2 = \frac{1}{|\Omega|} \int_\Omega (u_0 - K[u])^2 dx$

The Mumford-Shah segmentation estimator is to minimize $E[u, \Gamma]$ subject to the given σ^2 .

Introducing a Lagrange multiplier λ this is equivalent to minimizing

$$\alpha \int \left[u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right] dx + \beta \int_{\Omega \backslash \Gamma} \mid \nabla u \mid^2 dx + \lambda \int_\Omega (u_0 - K[u])^2 dx$$

Segmentation

Mumford-Shah

Segmenting an image means finding its homogeneous regions and their edges.

We shall do this by minimizing a segmentation energy. Mumford-Shah defines the segmentation problem as a joint smoothing/edge-detection problem.

Given the image we seek to find a piecewise smoothed image and a set of discontinuities. So we minimize the functional

$$E(u,k) = \int_{\Omega\backslash K} \left[|\nabla u(x)|^2 + (u-g)^2 \right] dx + length(K)$$

The first term requires that the image is smooth not in K (the edges). The second term says that u approximates the image g. The third term says the edges should be as small as possible, though one needs to define length of a curve for general curves. This will yield a cartoon version of the image. It has not been proven yet that a minimal segmentation exists though weaker versions have been proven.

Consider the one-dimensional case

$$E(u,K) = \int_I \left[\left(u^{'}\right)^2 + \left(u(x) - g(x)\right) \right] dx + Card(K)$$

K denotes the jumps in g(x) and u(x) is piecewise smooth.

To discretize consider points i=1,2, ...,n in I

Define:

$$g^0(t) = \begin{cases} t^2 & \text{if } t < 1 \\ 1 & \text{if } t > 1 \end{cases}$$

Then we discretize E by

$$E(u) = \sum_{i=1}^{n} \left(u_i - g_i\right)^2 + g^0(u_i - u_{i-1})$$

K is the set of points where $|u_i - u_{i-1}| > 1$.

Problem: This is non-convex and has many minima.

So we make the functional more convex. Define

$$g^p(t) = \begin{cases} t^2 & \text{if } |t| < \frac{1}{r} \\ 1 - \frac{\left(|t| - r\right)^2}{4p} & \text{if } \frac{1}{r} < |t| < r \\ 1 & \text{if } |t| > r \end{cases} \qquad r = \sqrt{4p+1}$$

and the new discretization is

$$E^p(u) = \sum_{i=1}^{n} \left(u_i - g_i\right)^2 + g^p(u_i - u_{i-1})$$

If $p \geq 1$ then E^p is convex and there is only one minimum.

An alternative is

1. Initialize the segmentation using some edge detector

2. Move the edges so that the energy decreases

Optimal segmentation

- Mumford and Shah studied 3 functionals which measure the degree of match between an image ***g(x,y)*** and a segmentation.
- First, they defined a general functional ***E*** (the famous Mumford-Shah functional):
  - R_i will be disjoint connected open subsets of the planar domain ***R***, each one with a piece-wise smooth boundary
  - Γ will be the union of the boundaries.

$$R = \coprod_{i=1}^{n} R_i \coprod \Gamma$$

Mumford-Shah functional

- Let ***f*** differentiable on $\cup$***R***$_i$ and allowed to be discontinuous across Γ.

$$E(f,\Gamma) = \mu^2 \iint_R (f-g)^2\,dxdy + \iint_{R-\Gamma} \|\nabla f\|^2\,dxdy + \nu|\Gamma|$$

- The smaller ***E***, the better ***(f, Γ)*** segments ***g***

1. ***f*** approximates ***g***
2. ***f*** (hence ***g***) does not vary much on ***R***$_i$s
3. The boundary Γ be as short as possible.

- Dropping any term would cause ***inf E=0***.

Piecewise Constant Mumford-Shah

We consider the simplest version of Mumford-Shah which is piecewise constant. We now seek to minimize the energy

$\frac{8 osc(g)^2}{\lambda}$

$$E(u,k) = \int_{\Omega \backslash K} (u-g)^2 dx + \lambda \, length(K)$$

and u is piecewise constant in $\Omega \backslash K$. λ is a scale constant and measures the amount of boundary. If λ is small then a lot of boundary is allowed and we get a fine segmentation. If λ is large then the segmentation gets coarser.

Define: $osc(g) = \sup(g) - \inf(g)$

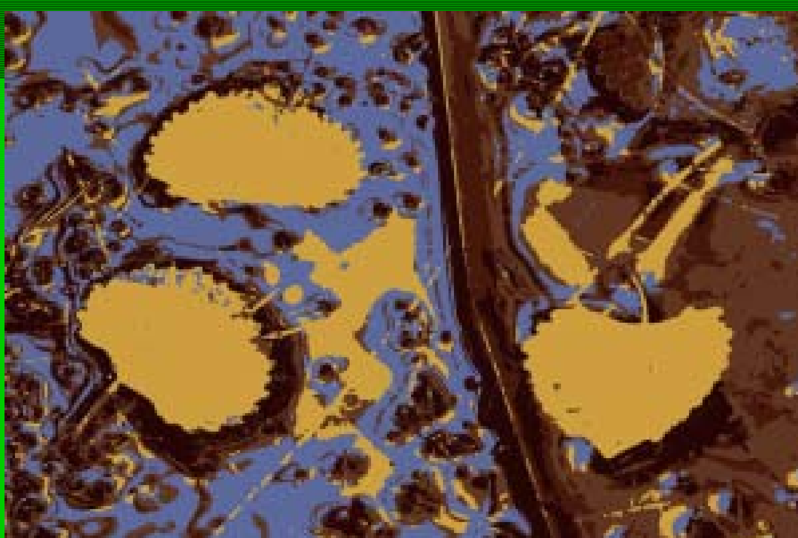
Then we have the theorem:

Let g be a measurable bounded function in Ω. Then the minimum of E(u,K) is attained for some K. Moreover, the minimal boundary sets have the following geometric property: either the points of K are regular, C^1, and with curvature bounded by $\frac{8 osc(g)^2}{\lambda}$, or else the singular points are of two types, either triple points where three branches meet at 120° angles and boundary points where K meets the boundary of Ω at a 90° angle.

Cartoon image

- ***(f, Γ)*** is simply a cartoon of the original image ***g***.
  - Basically ***f*** is a new image with edges drawn sharply.
  - The objects are drawn smootly without texture
  - ***(f, Γ)*** is essentially an idealization of ***g*** by the sort of image created by an artist.
  - Such cartoons are perceived correctly as representing the same scane as ***g*** → ***f*** is a simplification of the scene containing most of its essential features.

Cartoon image example



Piecewise constant approximation

- A special case of ***E*** where ***f=a$_i$*** is constant on each open set ***R$_i$***.

$$\mu^{-2}E(f,\Gamma)=\sum_i \iint_{R_i}(g-a_i)^2\,dxdy+\frac{\nu}{\mu^2}|\Gamma|$$

- Obviously, it is minimized in ***a$_i$*** by setting ***a$_i$*** to the mean of ***g*** in ***R$_i$***:

$$a_i = mean_{R_i}(g) = \iint_{R_i} g\,dxdy \Big/ area(R_i)$$

Piecewise constant approximation

$$E_0(\Gamma) = \sum_i \iint_{R_i} (g - mean_{R_i}(g))^2 \, dxdy + \frac{\nu}{\mu^2} |\Gamma|$$

- It can be proven that minimizing ***E_0*** is well posed:
  - For any continuous ***g***, there exists a Γ made up of finit number of singular points joined by a finit number of arcs on which ***E_0*** atteins a minimum.
- It can also be shown that ***E_0*** is the natural limit functional of ***E*** as ***$\mu \to 0$***

Mumford Shah attempts to find partitions of the image. On the other hand level sets, snakes etc. try to automatically detect contours of objects.

Kass-Witkin-Terzopolus

$\Gamma = \bigcup_{j \in J} C_j$ is the set of image edges where C_j is a piecewise smooth closed curve.

Let g(x) be monotonic decreasing and $g(0) = 1 \quad \lim_{s \to \infty} g(s) = 0$.

Then $g(|\nabla I|)$ is an edge detector, eg $g(s) = \dfrac{1}{1+s^2}$

Define

$$J(c) = \int_a^b |c'(q)|^2 \, dq + \beta \int_a^b |c''(q)|^2 \, dq + \lambda \int_a^b g^2(|\nabla I(x(q))|) dq$$

The first two terms are an internal energy and impose a smoothness constraint. The third term is an external energy attracts curves towards edges..

Unfortunately this energy is non-convex and so no uniqueness is possible.

1. $J(c)$ depends on the parametrization.

2. The model does not handle more than one convex body

3. Numerical problems

Setting $\beta = 0$ we get

$$J_1(c) = \int_a^b |c'(q)|^2 \, dq + \lambda \int_a^b g^2(|\nabla I(x(q))|) dq$$

To make the energy not depend on the parameterization we instead consider

$$J_2(c) = 2\sqrt{\lambda} \int_a^b g(|\nabla I(c(q))|) |c'(q)| \, dq$$

One can then prove $\inf_c J_1(c) = \inf_c J_2(c)$

Level Sets

$$\frac{\partial c}{\partial t} = F\ N$$

i.e. the curve c(q,t) moves along its normal with a speed $F(t,c,c',c'')$

$$c(0,q) = c_0(q)$$

The basic observation is that a curve can be seen as a zero-level of a function in higher dimensions.