Distributions

Returning to the Green's function.
Consider the ODE

$$Ly = -f(x)$$

Define the Green's functions as the solution to

$$LG(x, \xi) = -\delta(x - \xi)$$

Then the solution to the ODE is given by

$$y(x) = \int G(x, \xi) f(\xi) d\xi$$

Since

$$\begin{aligned} Ly &= \int L\left[G(x, \xi)\right] f(\xi) d\xi \\ &= -\int \delta(x - \xi) f(\xi) d\xi \\ &= -f(x) \end{aligned}$$

## Problem: How to we properly define $\delta(x)$

Also not all practical functions have derivatives or Fourier transforms. Point forces are very useful

We wish to extend the definition of a function and its derivative.

Fear: Extending allows for existence but endangers uniqueness

**Notation 1** *support* $(f(x))$ = *suppf(x)*= $\overline{\{x | f(x) \neq 0\}}$

**Definition 2** $C_0^\infty$ = *functions with compact support in* $C^\infty(R)$

**Definition 3** $D(R)$= *space of test functions with the following properties*

- $\varphi \in D \implies \varphi \in C_0^\infty$
- *topology:* $\varphi_n \to \varphi$ *in* $D$ *if*
  - $\{\varphi_n\}$ *have support in the same compact set* $K$
  - $\frac{\partial^r \varphi_n}{\partial x^r} \to \frac{\partial^r \varphi}{\partial x^r}$ *uniformly as* $n \to \infty$

example:

$$\varphi_n \to 0$$
$$\text{if} \quad supp\ \varphi_n \subset K$$
$$\text{and} \lim_{n\to\infty} \left| \frac{\partial^r \varphi_n}{\partial x^r} \right| = 0$$

example

$$\phi(x) = \begin{cases} 0 & |x| \geq a \\ e^{\frac{1}{x^2-a^2}} & |x| < a \end{cases}$$

Then $\phi(x) \in C_0^\infty$ but is not analytic

**Definition 4** *A linear function $q$ on $D(R)$ is a distribution if for every compact set $K \quad \exists C_k > 0$ and nonnegative integer $m$ such that*

$$|q(\phi)| \leq C_k \sup_{\substack{k\leq m \\ x\in K}} |\frac{\partial^r \varphi_n}{\partial x^r}|$$

*i.e. $q$ is a continuous linear functional on $D$*

application: For D'Alembart's formula to be a solution of the wave equation we require that $u \in C^2$.
If the initial condition is smooth except for a finite number of points then the solution has discontinuities along the characteristics.

**Definition 5** *Given linear spaces $U$ and $V$ the space of all continuous linear operations $U \to V$ is a linear space $\mathcal{L}(U,V)$.*
*Dual Space: $U' = \mathcal{L}(I,R)$ We denote $q(\phi) =< q, \phi >$.*

**Definition 6** *$q_n \to q$ in the weak topology $^*$ if*

$$\lim_{n\to\infty} < q_n, \phi >=< q, \phi > \quad all \ \ \phi \in D(R)$$

example of weak convergence

$$S_n(x) = \frac{1}{\pi} \frac{n}{1+n^2x^2}$$
$$b_n = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} S_n(x)\varphi(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} b_n(x)\varphi(x)dx = \varphi(0)$$

**Definition 7** *Dirac delta function*

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0)$$

and we see that the distribution $\delta(x)$ is the weak limit of classical functions

$$\lim_{n\to\infty} (q_n, \varphi) = \int_{-\infty}^{\infty} \delta(x)\varphi(x)dx$$

note: pointwise

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

so $\delta(x)$ is not a classical function

example:

$$c_n = \frac{1}{2} + \tan^{-1} \frac{nx}{\pi}$$

$$\text{then} \quad \lim_{n\to\infty} c_n = H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

$$= \lim_{n\to\infty} \frac{1}{\pi} \int_{-\infty}^{x} \frac{\sin(nt)}{t} dt$$

**Definition 8** *$f$ is locally integrable if $\int_a^b |f(x)|dx < \infty$ for every $[a, b]$*

if $F(\phi)$ is a distribution and $f(x)$ is a classical function then

$$F(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx \leq \sup_K |\phi(x)| \int_K f(x)dx$$

**Definition 9** *$F$ is a regular distribution if it corresponds to a locally integrable function*
*Otherwise $F$ is a singular distribution*

Note: If $f$ and $g$ are distributions $fg$ is not necessarily a distribution.
If $g \in C^{\infty}$ then $fg$ is a distribution.
Assume $f'$ is locally integrable. Then

$$< f', \phi >= \int_{-\infty}^{\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

**Definition 10** *$p$ is the distributional derivative of $q$ if*

$$< p, \phi >= - < q, \phi' > \qquad \textit{for all } \phi \textit{ in } D(R)$$

Note: If $\{q_n\} \to \{q\}$ then $\{q'_n\} \to \{q\prime\}$
**Proof.** $< q'_n, \varphi >= - < q_n, \varphi' >$
So $< q_n, \varphi' > \to < q, \varphi' >= - < q', \varphi >$ ■

Hence, we can derivatives of functions that don't have derivatives in the classical sense.

**Definition 11** *Dipole distribution*

$$< \delta', \varphi >= - < \delta, \varphi' >= -\varphi'(0)$$

**Lemma 12** *If $f$ is an integrable function and we know the functional $\int f\varphi dx$ for all test functions $\varphi(x)$ then we know $f(x)$*

**Proof.** Assume $\int \varphi dx = 1$ (normalization)
Define

$$\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t}) \qquad \text{in } R^n$$

Then

$$\lim_{t \to 0} \int_{R^n} f(s)\varphi_t(x-s)ds = f(x)$$

■
Let

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \qquad t > 0$$

Then

$$\lim_{t \to 0} \int_{-\infty}^{\infty} S(x,t)\varphi(x)dx = \varphi(0)$$

$$\text{or} \qquad S(x,t) \to \delta(x)$$

Similarly for the Dirichlet kernel

$$K_n(\theta) = 1 + 2\sum_{n=1}^{N} \cos(n\theta) = \frac{\sin(N+\frac{1}{2})\theta}{\sin(\theta)} \overset{N\to\infty}{\longrightarrow} 2\pi\delta(x)$$

## Derivatives

**Definition 13** *The derivative $f'$ is defined by*

$$(f', \varphi) = -(f, \varphi')$$

Note: If $f_n \to f$ then $f'_n \to f'$ weakly

example

$$\begin{aligned}
(\delta', \varphi) &= -(\delta, \varphi') = -\varphi'(0) \\
(\delta'', \varphi) &= -(\delta', \varphi') = (\delta, \varphi'') = \varphi''(0) \\
(H^{'}, \varphi) &= -(H, \varphi') = -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx \\
&= -\int_{0}^{\infty} \varphi'(x)dx = \varphi(0) \\
\text{So} \qquad H'(x) &= \delta(x)
\end{aligned}$$

**Theorem 14** *Every distribution has derivatives of all orders*

example

$$|x| = \frac{\pi}{2} - \frac{1}{\pi} \sum_{n \text{ odd}} \frac{4}{n^2} \cos(nx) \qquad -\pi < x < \pi$$

then

$$\int_{-\infty}^{\infty} |x|\varphi(x)dx = \frac{\pi}{2} \int_{-\infty}^{\infty} \varphi(x)dx - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \int_{-\infty}^{\infty} \cos(nx)\varphi(x)dx$$

So this is a distribution. So if we formally differentiate we get in the distributional sense

$$|x|' = \frac{1}{\pi} \sum_{n \text{ odd}} \frac{4}{n} \sin(nx) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

formally differentiate a second time

$$\begin{aligned}
|x|'' &= \frac{4}{\pi} \sum_{n \text{ odd}} \cos(nx) = \delta(x) \\
\text{i.e.} \quad \sum_{n \text{ odd}} \int_{-\pi}^{\pi} \varphi(x) \cos(nx)dx &= \frac{\pi}{2}\varphi(0)
\end{aligned}$$

differential equations

Consider

$$x\frac{du}{dx} = 0$$

The classical solution is $u$ is constant
The distributional solution is

$$u(x) = c_1 H(x) + c_2$$

**Theorem 15** *Ehrenpreis:*
*Let $P$ be a linear differential operator with constant coefficients.*
*Then $Pu = \delta$ has a solution in terms of distributions*

This solution is called the fundamental solution and $u$ is the Green's function.

Laplace Equation

$$\phi(0) = -\iiint \frac{1}{r}\frac{\Delta\phi}{4\pi} dV$$

$$\text{i.e.} \qquad \Delta\left(-\frac{1}{4\pi r}\right) = \delta(x) \qquad \text{in 3D}$$

Poisson Equation

$$\begin{aligned} \Delta u = f & \qquad \text{in } D \\ u = 0 & \qquad \text{on } \partial D \end{aligned}$$

then

$$\begin{aligned} u(x_0) &= \iiint\limits_D \delta(x - x_0) u(x) dx \\ &= \iiint\limits_D \Delta G(x, x_0) u(x) dx \end{aligned}$$

but

$$G(x, x_0) + \frac{1}{4\pi|x - x_0|} \qquad \text{is harmonic in } \ D \text{ including } x_0$$

So

$$\Delta\left(-\frac{1}{4\pi|x - x_0|}\right) = \delta(x - x_0)$$

Consider the Riemann function

$$
\begin{aligned}
\frac{\partial^2 S}{\partial t^2} &= c^2 \Delta S \qquad -\infty < x, y, z < \infty \\
S(x, y, z, 0) &= 0 \\
\frac{\partial S}{\partial t}(x, y, z, 0) &= \delta(x, y, z) = \delta(x)\delta(y)\delta(z)
\end{aligned}
$$

Let $\varphi(x)$ be a test function and (consider for simplicity 1D)

$$
u(x, t) = \int S(x - y, t)\varphi(y)dy
$$

then

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \Delta u \qquad -\infty < x < \infty \\
u(x, 0) &= 0 \\
\frac{\partial u}{\partial t}(x, 0) &= \varphi(x)
\end{aligned}
$$

In 1D we have D'Alambert's formula. So

$$
\begin{aligned}
\int_{-\infty}^{\infty} S(x - y, t)\varphi(y)dy &= \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(y)dy \\
&= \frac{1}{2c}\int_{-\infty}^{\infty} \frac{H(x + ct) - H(x - ct)}{2}\varphi(y)dy
\end{aligned}
$$

Thus, we see that

$$
\begin{aligned}
S(x, y) &= \begin{cases} \frac{1}{2c} & |x| < ct \\ 0 & |x| > ct \end{cases} \\
&= \frac{1}{2c}H(c^2t^2 - x)sgn(t) \qquad c^2t^2 \neq x
\end{aligned}
$$

and so the solution is discontinuous along the characteristics
Define

$$
u(x, t) = \begin{cases} \frac{H(x+ct)-H(x-ct)}{2} & t > 0 \\ 0 & t < 0 \end{cases}
$$

then

$$
\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \delta(x) \qquad -\infty < x < \infty
$$

In three dimensions

$$\begin{aligned} S(x,t) &= \frac{1}{4\pi c^2 t}\delta(ct - |x|) \qquad t > 0 \\ &= \frac{1}{2\pi c}\delta(c^2t^2 - |x|^2)sgn(t) \end{aligned}$$

and

$$u(x,t) = \frac{1}{4\pi c^2 t}\iiint\limits_{|x-y|=ct} \varphi(y)dy$$

Diffusion equation

$$\frac{\partial^2 S}{\partial t^2} = k\Delta S \qquad -\infty < x < \infty$$
$$S(x,0) = \delta(x)$$

Let

$$R(x,t) = \begin{cases} S(x - x_0, t - t_0) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

then

$$\frac{\partial^2 R}{\partial t^2} - k\Delta R = \delta(x - x_0, t - t_0) \qquad -\infty < x < \infty$$