Geometric Approach

Given

$$au_x + bu_y = 0$$

a,b constants. $au_x + bu_y$ is a directional derivative in the direction

$$V = (a, b) = ai + bj$$

Hence, u is constant in the direction V. Lines parallel to V have the formula $bx - ay$ =constant (i.e. slope $\frac{b}{a}$) = characteristic lines. Hence, u is a function of $bx - ay$.

$$u(x, y) = f(bx - ay)$$

Hence, on the characteristic lines $c = bx - ay$ is constant and $u = f(c)$. The vector $(b, -a)$ is orthogonal to V. Hence, u varies along these lines.

Method #2.

Let $x^{'} = ax + by$ $y^{'} = bx - ay$

By the chain rule $u_x = au_{x'} + bu_{y'}$ $u_y = bu_{x'} - au_{y'}$.

So

$au_x + bu_y = (a^2 + b^2)u_{x'}$. Hence, thr original equation is equivalent to $u_{x'} = 0$.

Conclusion:

$$u(x, y) = f(y^{'}) = f(bx - ay)$$

general geometric approach

$$(*) \qquad a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

initial condition: curve Γ in (x,y,u).
Parameterize: $x = x(s) \qquad y = y(s) \qquad u = u(s)$
Solution: a surface $u = u(x,y)$ that passes through the initial curve.

Given a solution surface $F(x,y,u) = 0$.
$\nabla F = (F_x, F_y, F_z)$ is normal to this surface.
Consider $z = u(x,y)$ or equivalently $F(x,y,u) = u(x,y) - z$
Then $\nabla F = (u_x, u_y, -1)$.
We rewrite $(*)$ as $(a,b,c) \cdot (u_x, u_y, -1) = 0$.
So (a,b,c) is normal to the gradient which itself is normal to the surface.
Hence, (a,b,c) lies in the tangent plane to the solution surface.