Representation Formula

Theorem 1 *If* $\Delta u = 0$ *then*

$$u(x_0) = \frac{1}{4\pi} \iint\limits_{\partial D} \left(\frac{1}{|x - x_0|} \frac{\partial u}{\partial n} - u(x) \frac{\partial}{\partial n} (\frac{1}{|x - x_0|}) \right) dS$$

Proof. By Green's second identity

$$\iiint (u\Delta v - v\Delta u)\, dV = \iint\limits_{\partial D} \left(u(x) \frac{\partial v}{\partial n} - v(x) \frac{\partial u}{\partial n} \right) dS$$

Choose

$$v(x) = -\frac{1}{4\pi|x - x_0|} \sim \frac{1}{r}$$

Then

$$\Delta u = 0 \text{ and } \Delta v = 0 \text{ except at } x = x_0$$

Consider a region $D_\epsilon = D-$circle of radius r about x_0.
So

$$\iint\limits_{\partial D_\epsilon} \left(u(x) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS = 0$$

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Green's function

Consider

$$\begin{aligned} -\Delta u = 0 &\qquad \text{in } D \\ u = f &\qquad \text{on } \partial D \end{aligned}$$

Definition 2 *$G(x, x_0)$is a Green's function for $-\Delta$ in D and $x_0 \in D$ if*

1. $G(x) \in C^2$
2. $-\Delta G(x, x_0) = 0 \quad x \in D \quad x \neq x_0$
3. $G(x, x_0) = 0 \quad x \in \partial D$
4. $\widetilde{G} = G(x, x_0) + \frac{1}{4\pi|x - x_0|}$ *is finite at x_0 and so $\Delta\widetilde{G} = 0$ everywhere in D*

Theorem 3 *$G(x, x_0)$ exists and is unique*

Theorem 4

$$u(x_0) = \iint\limits_{D} u(x) \frac{\partial G(x, x_0)}{\partial n} dS$$

Proof. Using the second Green's identity

$$\iiint (u\Delta H - H\Delta u)\, dV = \iint\limits_{\partial D} \left(u(x)\frac{\partial H}{\partial n} - H(x)\frac{\partial u}{\partial n} \right) dS$$

Let $\Delta u = 0$ and $\Delta H = 0$ then

$$(*) \qquad 0 = \iint\limits_{\partial D} \left(u(x)\frac{\partial H}{\partial n} - H(x)\frac{\partial u}{\partial n} \right) dS$$

Define

$$\begin{aligned} v &= -\frac{1}{4\pi|x - x_0|} \\ \text{so} \quad \Delta v &= \delta(x - x_0) \end{aligned}$$

Then

$$(**) \qquad u(x_0) = \iint\limits_{\partial D} \left(u(x)\frac{\partial v}{\partial n} - v(x)\frac{\partial u}{\partial n} \right) dS$$

Choose

$$G(x, x_0) = H(x) + v(x)$$

adding $(*)$ and $(**)$ we get

$$u(x_0) = \iint\limits_{\partial D} \left(u(x)\frac{\partial G}{\partial n} - G(x)\frac{\partial u}{\partial n} \right) dS$$

but $G = 0$ on the boundary. So

$$u(x_0) = \iint\limits_{\partial D} u(x)\frac{\partial G}{\partial n} dS$$

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Theorem 5

$$G(x, x_0) = G(x_0, x)$$

For a general region G is difficult to find