9.4 The hit-or-miss transformation

Illustration...
• Objective is to find a disjoint region (set) in an image

• If \( B \) denotes the set composed of \( X \) and its background, the match/hit (or set of matches/hits) of \( B \) in \( A \), is

\[
A \otimes B = (A \ominus X) \cap [A^c \ominus (W - X)]
\]

• Generalized notation: \( B = (B_1, B_2) \)
  
  - \( B_1 \): Set formed from elements of \( B \) associated with an object
  
  - \( B_2 \): Set formed from elements of \( B \) associated with the corresponding background

  [Preceeding discussion: \( B_1 = X \) and \( B_2 = (W - X) \)]

• More general definition:

\[
A \otimes B = (A \ominus B_1) \cap [A^c \ominus B_2]
\]

• \( A \otimes B \) contains all the origin points at which, simultaneously, \( B_1 \) found a hit in \( A \) and \( B_2 \) found a hit in \( A^c \)
• Alternative definition:

\[ A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2) \]

• A background is necessary to detect disjoint sets

• When we only aim to detect certain patterns within a set, a background is not required, and simple erosion is sufficient

**9.5 Some basic morphological algorithms**

When dealing with binary images, the principle application of morphology is extracting image components that are useful in the representation and description of shape

**9.5.1 Boundary extraction**

The boundary \( \beta(A) \) of a set \( A \) is

\[ \beta(A) = A - (A \ominus B), \]

where \( B \) is a suitable structuring element
Illustration...

**Example 9.5:** Morphological boundary extraction

**Figure 9.14**
(a) A simple binary image, with 1’s represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).
9.5.2 Region filling

• Begin with a point $p$ inside the boundary, and then fill the entire region with 1’s

• All non-boundary (background) points are labeled 0

• Assign a value of 1 to $p$ to begin...

• The following procedure fills the region with 1’s,

\[ X_k = (X_{k-1} \oplus B) \cap A^c, \quad k = 1, 2, 3, \ldots, \]

where $X_0 = p$, and $B$ is the symmetric structuring element in figure 9.15 (c)

• The algorithm terminates at iteration step $k$ if $X_k = X_{k-1}$

• The set union of $X_k$ and $A$ contains the filled set and its boundary

Note that the intersection at each step with $A^c$ limits the dilation result to inside the region of interest
Example 9.6: Morphological region filling

Figure 9.16 (a) Binary image (the white dot inside one of the regions is the starting point for the region-filling algorithm), (b) Result of filling that region (c) Result of filling all regions.
9.5.3 Extraction of connected components

Let $Y$ represent a connected component contained in a set $A$ and assume that a point $p$ of $Y$ is known. Then the following iterative expression yields all the elements of $Y$:

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots,$$

where $X_0 = p$, and $B$ is a suitable structuring element. If $X_k = X_{k-1}$, the algorithm has converged and we let $Y = X_k$.

This algorithm is applicable to any finite number of sets of connected components contained in $A$, assuming that a point is known in each connected component.

![Diagram of connected components extraction](image)

**FIGURE 9.17** (a) Set $A$ showing initial point $p$ (all shaded points are valued 1, but are shown different from $p$ to indicate that they have not yet been found by the algorithm). (b) Structuring element. (c) Result of first iterative step. (d) Result of second step. (e) Final result.
Example 9.7:

9.5.4 Convex hull

Morphological algorithm for obtaining the convex hull, $C(A)$, of a set $A$...

Let $B_1$, $B_2$, $B_3$ and $B_4$ represent the four structuring elements in figure 9.19 (a), and then implement the equation ...
\[ X^i_k = (X_{k-1} \ominus B^i) \cup A, \quad i = 1, 2, 3, 4, \quad k = 1, 2, \ldots, \quad X^i_0 = A \]

Now let \( D^i = X^i_{\text{conv}} \), where “conv” indicates convergence in the sense that \( X^i_k = X^i_{k-1} \). Then the convex hull of \( A \) is

\[ C(A) = \bigcup_{i=1}^4 D^i \]
Shortcoming of above algorithm: convex hull can grow beyond the minimum dimensions required to guarantee convexity
Possible solution: Limit growth so that it does not extend past the vertical and horizontal dimensions of the original set of points

Boundaries of greater complexity can be used to limit growth even further in images with more detail

9.5.5 Thinning

The thinning of a set $A$ by a structuring element $B$:

$$A \ominus B = A - (A \ominus B) = A \cap (A \ominus B)^c$$

Symmetric thinning: sequence of structuring elements,

$$\{B\} = \{B^1, B^2, B^3, \ldots, B^n\},$$

where $B^i$ is a rotated version of $B^{i-1}$
$$A \otimes \{B\} = ((\ldots ((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$$

Illustration: Note that figure 9.21 (in the handbook) has many errors — this one is correct...
9.5.6 Thickening

Thickening is the morphological dual of thinning and is defined by

\[ A \ominus B = A \cup (A \ominus B), \]

where \( B \) is a structuring element

Similar to thinning...

\[ A \ominus \{B\} = (((A \ominus B^1) \ominus B^2) \ldots) \ominus B^n \]

Structuring elements for thickening are similar to those of figure 9.21 (a), but with all 1’s and 0’s interchanged

A separate algorithm for thickening is seldom used in practice — we thin the background instead, and then complement the result

**FIGURE 9.22** (a) Set \( A \). (b) Complement of \( A \). (c) Result of thinning the complement of \( A \). (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.
9.5.7 Skeletons

The algorithm proposed in this section is similar to the medial axis transformation (MAT). The MAT transformation is discussed in section 11.1.5 and is far inferior to the skeletonization algorithm introduced in section 11.1.5. The skeletonization algorithm proposed in this section also does not guarantee connectivity. We therefore do not discuss this algorithm.

Illustration of the above comments...

**FIGURE 9.23**
(a) Set $A$.
(b) Various positions of maximum disks with centers on the skeleton of $A$.
(c) Another maximum disk on a different segment of the skeleton of $A$.
(d) Complete skeleton.
A further illustration...

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A \ominus kB$</th>
<th>$(A \ominus kB) \cdot B$</th>
<th>$S_k(A)$</th>
<th>$\bigcup_{k=0}^K S_k(A)$</th>
<th>$S_k(A) \ominus kB$</th>
<th>$\bigcup_{k=0}^K S_k(A) \ominus kB$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
<td><img src="image4" alt="Image" /></td>
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<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
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<td><img src="image8" alt="Image" /></td>
<td><img src="image9" alt="Image" /></td>
<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
<td><img src="image12" alt="Image" /></td>
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<tr>
<td>2</td>
<td><img src="image13" alt="Image" /></td>
<td><img src="image14" alt="Image" /></td>
<td><img src="image15" alt="Image" /></td>
<td><img src="image16" alt="Image" /></td>
<td><img src="image17" alt="Image" /></td>
<td><img src="image18" alt="Image" /></td>
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<td></td>
<td><img src="image19" alt="Image" /></td>
<td><img src="image20" alt="Image" /></td>
<td><img src="image21" alt="Image" /></td>
<td><img src="image22" alt="Image" /></td>
<td><img src="image23" alt="Image" /></td>
<td><img src="image24" alt="Image" /></td>
</tr>
</tbody>
</table>

**Figure 9.24** Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

### 9.5.8 Pruning

- Cleans up “parasitic” components left by thinning and skeletonization
- Use combination of morphological techniques
Illustrative problem: hand-printed character recognition

- Analyze shape of skeleton of character
- Skeletons characterized by spurs ("parasitic" components)
- Spurs caused during erosion of non-uniformities in strokes
- We assume that the length of a parasitic component does not exceed a specified number of pixels

**FIGURE 9.25**
(a) Original image, (b) and (c) Structuring elements used for deleting end points, (d) Result of three cycles of thinning, (e) End points of (d), (f) Dilation of end points conditioned on (a), (g) Pruned image.
Any branch with three or less pixels is to be eliminated

(1) Three iterations of:

\[ X_1 = A \otimes \{B\} \]

(2) Find all the end points in \(X_1\):

\[ X_2 = \bigcup_{k=1}^{8} (X_1 \oplus B^k) \]

(3) Dilate end points three times, using \(A\) as a delimiter:

\[ X_3 = (X_2 \oplus H) \cap A, \quad H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

(4) Finally:

\[ X_4 = X_1 \cup X_3 \]
<table>
<thead>
<tr>
<th>Operation</th>
<th>Equation</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translation</td>
<td>$(A)_z = {w \mid w = a + z, \text{ for } a \in A}$</td>
<td>Translates the origin of $A$ to point $z$.</td>
</tr>
<tr>
<td>Reflection</td>
<td>$\hat{B} = {w \mid w = -b, \text{ for } b \in B}$</td>
<td>Reflects all elements of $B$ about the origin of this set.</td>
</tr>
<tr>
<td>Complement</td>
<td>$A^c = {w \mid w \not\in A}$</td>
<td>Set of points not in $A$.</td>
</tr>
<tr>
<td>Difference</td>
<td>$A - B = {w \mid w \in A, w \not\in B}$</td>
<td>Set of points that belong to $A$ but not to $B$.</td>
</tr>
<tr>
<td>Dilation</td>
<td>$A \oplus B = {z \mid (\hat{B}_z \cap A \neq \emptyset}$</td>
<td>“Expands” the boundary of $A$.</td>
</tr>
<tr>
<td>Erosion</td>
<td>$A \ominus B = {z \mid (B_z \subseteq A}$</td>
<td>“Contracts” the boundary of $A$.</td>
</tr>
<tr>
<td>Opening</td>
<td>$A \circ B = (A \ominus B) \oplus B$</td>
<td>Smooths contours, breaks narrow isthmuses, and eliminates small islands</td>
</tr>
<tr>
<td>Closing</td>
<td>$A \bullet B = (A \ominus B) \ominus B$</td>
<td>and sharp peaks. (I)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hit-or-miss transform</th>
<th>$A \circ B = (A \ominus B) \cap (A^c \ominus B)$</th>
<th>The set of points (coordinates) at which, simultaneously, $B_i$ found a match (“hit”) in $A$ and $B_i$ found a match in $A^c$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary extraction</td>
<td>$\beta(A) = A - (A \ominus B)$</td>
<td>Set of points on the boundary of set $A$. (I)</td>
</tr>
<tr>
<td>Region filling</td>
<td>$X_k = (X_{k-1} \oplus B) \cap A; X_0 = p$ and $k = 1, 2, 3, \ldots$</td>
<td>Fills a region in $A$, given a point $p$ in the region. (II)</td>
</tr>
<tr>
<td>Connected components</td>
<td>$X_k = (X_{k-1} \oplus B) \cap A; X_0 = p$ and $k = 1, 2, 3, \ldots$</td>
<td>Finds a connected component $Y$ in $A$, given a point $p$ in $Y$. (I)</td>
</tr>
<tr>
<td>Convex hull</td>
<td>$X^i_k \cap (X^i_{k-1} \oplus B) \cup A; i = 1, 2, 3, 4; k = 1, 2, 3, \ldots; X^i_0 = A$ and $D^i = X^i_{\text{conv}}$</td>
<td>Finds the convex hull $C(A)$ of set $A$, where “conv” indicates convergence in the sense that $X^i_k = X^i_{k-1}$. (III)</td>
</tr>
<tr>
<td>Operation</td>
<td>Equation</td>
<td>Comments</td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td><strong>Thinning</strong></td>
<td>$A \ominus B = A - (A \ominus B) = A \cap (A \ominus B)^c$</td>
<td>Thins set $A$. The first two equations give the basic definition of thinning. The last two equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)</td>
</tr>
<tr>
<td><strong>Thickening</strong></td>
<td>$A \oplus B = A \cup (A \oplus B)$</td>
<td>Thickens set $A$. (See preceding comments on sequences of structuring elements.) Uses IV with 0's and 1's reversed.</td>
</tr>
<tr>
<td><strong>Skeletons</strong></td>
<td>$S(A) = \bigcup_{k=0}^{K} S_k(A)$</td>
<td>Finds the skeleton $S(A)$ of set $A$. The last equation indicates that $A$ can be reconstructed from its skeleton subsets $S_k(A)$. In all three equations, $K$ is the value of the iterative step after which the set $A$ erodes to the empty set. The notation $(A \ominus kB)$ denotes the $k$th iteration of successive erosion of $A$ by $B$, (I)</td>
</tr>
<tr>
<td><strong>Pruning</strong></td>
<td>$X_1 = A \ominus {B}$, $X_2 = \bigcup_{k=1}^{8} (X_1 \ominus B^k)$, $X_3 = (X_2 \oplus H) \cap A$, $X_4 = X_1 \cup X_3$</td>
<td>$X_4$ is the result of pruning set $A$. The number of times that the first equation is applied to obtain $X_1$ must be specified. Structuring elements $V$ are used for the first two equations. In the third equation $H$ denotes structuring element I.</td>
</tr>
</tbody>
</table>
9.6 Extensions to grey-scale images

\[ f(x, y) : \] Input image  
\[ b(x, y) : \] Structuring element image

9.6.1 Dilation

Grey-scale dilation of \( f \) by \( b \), is defined as

\[
(f \oplus b)(s, t) = \max \{ f(s - x, t - y) + b(x, y) \mid (s - x), (t - y) \in D_f; (x, y) \in D_b \},
\]

where \( D_f \) and \( D_b \) are the domains of \( f \) and \( b \), respectively.
Simple 1D example. For functions of one variable:

\[(f \oplus b)(s) = \max \{f(s - x) + b(x) \mid (s - x) \in D_f; \ x \in D_b\}\]

General effect of dilation of a grey-scale image:

1. If all values of \(b(x, y)\) are positive \(\Rightarrow\) output image brighter
2. Dark details are reduced or eliminated, depending on size
9.6.2 Erosion

Grey-scale erosion of $f$ by $b$, is defined as

$$(f \ominus b)(s,t) = \max \{f(s + x, t + y) - b(x, y) \mid (s + x), (t + y) \in D_f; (x, y) \in D_b\},$$

where $D_f$ and $D_b$ are the domains of $f$ and $b$, respectively.

**Simple 1D example.** For functions of one variable:

$$(f \ominus b)(s) = \max \{f(s + x) - b(x) \mid (s + x) \in D_f; x \in D_b\}$$

**FIGURE 9.28**
Erosion of the function shown in Fig. 9.27(a) by the structuring element shown in Fig. 9.27(b).
General effect of erosion of a grey-scale image:

(1) If all values of $b(x, y)$ are positive $\implies$ output image darker
(2) Bright details are reduced or eliminated, depending on size

**Example 9.9:** Dilation and erosion on grey-scale image

$$f(x, y): 512 \times 512$$

$$b(x, y): \text{“flat top”, unit height, size of } 5 \times 5$$

*Figure 9.29*

(a) Original image, (b) Result of dilation, (c) Result of erosion.

(Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)
9.6.3 Opening and closing

The opening of image $f$ by $b$, is defined as

$$f \circ b = (f \ominus b) \oplus b$$

The closing of image $f$ by $b$, is defined as

$$f \circ b = (f \oplus b) \ominus b$$

Explanation using “rolling ball”:

![Diagram showing the process of opening and closing using a rolling ball.](image)
Opening and closing of “horse image”:

![Horse Image](image1)

9.6.4 Some applications of grey-scale morphology

Morphological smoothing

Opening followed by closing $\Rightarrow$ remove or attenuate bright and dark artifacts or noise

![Morphological Smoothing](image2)
Morphological Image Processing  Lecture 22 (p. 25)

Morphological gradient

Definition: \( g = (f \oplus b) - (f \ominus b) \)

![Image](image_url)

**FIGURE 9.33** Morphological gradient of the image in Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

Top-hat transformation

Definition: \( h = f - (f \circ b) \)

![Image](image_url)

**FIGURE 9.34** Result of performing a top-hat transformation on the image of Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)
Input image (left): Two texture regions

Output image (right): Boundary between the two regions

Algorithm:

(1) Close input image using succ larger struct elements. When size(struct element) \(\approx\) size(small blobs), blobs are removed

(2) Single opening with struct element that is large in relation to separation between large blobs \(\rightarrow\) light patches between blobs removed \(\rightarrow\) light region on left, dark region on right

(3) Thresholding \(\rightarrow\) boundary
Granulometry

Granulometry: Field that deals with determining the size distribution of particles in an image

Algorithm:

1. Opening with struct elements of increasing size
2. Difference between original image and its opening computed after each pass
3. Differences are normalized and used to construct histogram

The histogram indicates the presence of three predominant particle sizes