Inhomogeneous data

$$
\begin{aligned}
(*) \qquad u_t &= k u_{xx} \qquad 0 < x < l, \quad t > 0 \\
u(x,0) &= 0 \qquad \text{initial condition} \\
u(0,t) &= h(t) \qquad \text{boundary condition} \\
u(l,t) &= j(t) \qquad \text{boundary condition}
\end{aligned}
$$

Assume:

$$
\begin{aligned}
(**) \qquad u(x,t) &= \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi x}{l}) \\
u_n &= \frac{2}{l} \int_0^l u(x,t) \sin(\frac{n\pi x}{l}) dx
\end{aligned}
$$

Note: Since $u(x,t)$ is nonzero at the boundaries the convergence **cannot** be uniform but either pointwise or else least square !!
Substitute (**) into (*) giving

$$
0 = u_t - k u_{xx} = \sum_{n=1}^{\infty} \left[\frac{du_n}{dt} + k u_n(t) \left(\frac{n\pi x}{l} \right)^2 \right] \sin(\frac{n\pi x}{l})
$$

This **can't** match the boundary conditions:

Lesson: Differentiating (**) is illegal !!

Method #1
Instead we avoid differentiation os a nonuniform converging series. Assume

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi x}{l}) \\
\frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} v_n(t) \sin(\frac{n\pi x}{l}) \\
v_n(t) &= \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \sin(\frac{n\pi x}{l}) dx = \frac{du_n}{dt} \qquad \text{since new integrand is continuous} \\
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} w_n(t) \sin(\frac{n\pi x}{l}) \\
w_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \sin(\frac{n\pi x}{l}) dx \\
&= -\frac{2}{l} \int_0^l \left(\frac{n\pi}{l} \right)^2 u(x,t) \sin(\frac{n\pi x}{l}) dx + \frac{2}{l} \left[\frac{\partial u}{\partial x} \sin(\frac{n\pi x}{l}) - \frac{n\pi}{l} \cos(\frac{n\pi x}{l}) \right]_0^l \\
&= -\left(\frac{n\pi}{l} \right)^2 u_n(t) - \frac{2n\pi}{l^2} \left[(-1)^n j(t) - h(t) \right] \qquad \text{let } \lambda_n = \left(\frac{n\pi}{l} \right)^2
\end{aligned}
$$

So

$$
\begin{aligned}
v_n(t) - kw_n(t) &= 0 \\
\frac{du_n}{dt} &= kw_n(t)
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{du_n}{dt} &= -k\lambda_n u_n(t) + \frac{2n\pi}{l^2}\left[(-1)^n j(t) - h(t)\right] \\
u_n(0) &= 0
\end{aligned}
$$

Solving the ODE we get

$$
u_n(t) = Ce^{-\lambda_0 kt} - \frac{2n\pi}{l^2} k \int_0^t e^{-\lambda_n k(t-s)}\left[(-1)^n j(s) - h(s)\right] ds
$$

Consider the PDE

$$
\begin{aligned}
(*) \qquad u_{tt} &= c^2 u_{xx} + f(x,t) \qquad 0 < x < l, \quad t > 0 \\
u(x,0) &= \varphi(x) \qquad \text{initial condition} \\
\frac{\partial u}{\partial t}(x,0) &= \psi(x) \qquad \text{initial condition} \\
u(0,t) &= h(t) \qquad \text{boundary condition} \\
u(l,t) &= k(t) \qquad \text{boundary condition}
\end{aligned}
$$

Method #2: Define a new variable v using $g(x,t)$that satisfies the homogenous boundary conditions

$$
v(x,t) = u(x,t) - \frac{(l-x)\,h(t) + xk(t)}{l}
$$

Now

$$
\begin{aligned}
v_{tt} &= c^2 v_{xx} + f(x,t) - \frac{(l-x)\,h''(t) + xk''(t)}{l} = c^2 v_{xx} + F(x,t) \\
v(x,0) &= \varphi(x) - \frac{(l-x)\,h(0) + xk(0)}{l} \\
\frac{dv}{dt}(x,0) &= \psi(x) - \frac{(l-x)\,h'(0) + xk'(0)}{l} \\
v(0,t) &= v(l,t) = 0
\end{aligned}
$$

More generally

$$
v(x,t) = u(x,t) - U(x,t)
$$

then

$$
\begin{aligned}
v_{tt} &= c^2 v_{xx} + f(x,t) - \big(U_{tt} - c^2 U_{xx}\big) \\
v(x,0) &= \varphi(x) - U(x,0) \\
\frac{dv}{dt}(x,0) &= \psi(x) - U_t(x,0) \\
v(0,t) &= h(t) - U(0,t) \\
v(l,t) &= k(t) - U(l,t)
\end{aligned}
$$

So we choose U so that

$$
\begin{aligned}
U_{tt} &= c^2 U_{xx} + f(x,t) \\
U(0,t) &= h(t) \\
U(l,t) &= k(t)
\end{aligned}
$$

e.g. if f,h,k are independent of t then choose U also independent of t and

$$
\begin{aligned}
U_{xx} &= -\frac{1}{c^2} f(x) \\
U(0) &= h \\
U(l) &= k
\end{aligned}
$$

Inhomogeneous right hand side

Consider

$$(*) \qquad -\nabla\cdot(p(x)\nabla u)+q(x)u(x)=a\,m(x)u(x)+f(x) \qquad x\in D$$

with homogeneous Dirichlet or Neumann (or more generally symmetric) boundary conditions.

Theorem 1 *(a) If* ***a*** *is not an eigenvalue (f=0) then there exists a unique solution for all f with $||f||^2<\infty$*

(b) if ***a*** *is an eigenvalue then either there is no solution or else an infinite number of solutions (Fredholm alternative) depending on $f(x)$*

Proof. (a) Denote the eigenvalues by $\lambda_n$and the eigenfunctions by $v_n(x), n=1,2,3...$

Let δ be the distance between **a** and the nearest eigenvalue. By completeness we have

$$(**) \qquad u(x)=\sum_{n=1}^{\infty}\frac{(u,v_n)}{(v_n,v_n)}v_n(x) \quad (u,v)\equiv\iiint_D muv dV$$

and the sum converges in L^2.

Multiply (*) by v_n and integrate to get

$$-\int\nabla\cdot(p\nabla u)v_n dV+\int quv_n dV=a\int muv_n dV+\int fv_n dV$$

By Green's second identity we replace the first integral by (using the symmetric homogenous boundary conditions on u and v)

$$-\int\nabla\cdot(p\nabla v_n)u dV+\int quv_n dV=a\int muv_n dV+\int fv_n dV$$

But v_n satisfies the homogenous version of (*)
and so

$$-\nabla\cdot(p(x)\nabla v_n)+q(x)v_n(x)=\lambda_n m(x)v_n(x)$$

Hence, subtracting we get

$$(\lambda_n-a)\int muv_n dV=\int fv_n$$

or

$$(u,v_n)=\frac{\int fv_n}{(\lambda_n-a)}$$

Putting into (**) we finally arrive at

$$u(x)=\sum_{n=1}^{\infty}\frac{\int fv_n}{(\lambda_n-a)\,(v_n,v_n)}v_n(x)$$

To show convergence we have

$$|u(x)| \le \sum_{n=1}^{\infty} \frac{|\int f v_n|}{\delta(v_n, v_n)} v_n(x)$$

For simplicity assume the eigenfunctions are orthonormal so $(v_n, v_n) = 1$. Then

$$||u||^2 \stackrel{Schwarz}{\le} \frac{1}{\delta^2} |\sum_{n=1}^{\infty} | \int f v_n(x)|^2 \stackrel{Bessel}{\le} \frac{1}{\delta^2} \int f^2 \frac{1}{m} dV < \infty$$

So the series converges

(b) Suppose $a = \lambda_n$. Then

$$(\lambda_n - a) \int m u v_n dV = \int f v_n$$

implies that

$$\int f v_n = 0$$

If this is not true then there are no solutions. If it is true then we can add Cv_n to any solution to get an infinite set of solutions. ■