Laplace Equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Also Cauchy-Riemann equations (2D)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Maximum Principle

Theorem 1 *Let $\Delta u = 0$ in D. Then the maximum and minimum occur on the boundary and not inside unless u=constant.*

Proof. proof #1 (weak version)

same as for heat equation consider $v(x,y) = u(x,y) + \varepsilon(x^2+y^2)$.

Then $\Delta v > 0$.

However, at a maximum $\Delta v \leq 0$ and so we can't have an interior maximum for v

and then one can estimate the same for u

Proof. #2

By the first Green formula

$$\iint_D v\Delta u dV + \iint_D \nabla u \cdot \nabla v dV = \oint_{\partial D} v\frac{\partial u}{\partial n} dS$$

First choose $v = 1$. Then

$$(*) \qquad 0 = \iint_D \Delta u dV = \oint_{\partial D} \frac{\partial u}{\partial n} dS$$

Next choose $v = u$. Then

$$\iint_D u\Delta u dV + \iint_D |\nabla u|^2 dV = \oint_{\partial D} u\frac{\partial u}{\partial n} dS$$

Assume that there is a maximum at an interior point (x_0, y_0). Then we can draw a circle about this point so that u is constant on this circle. Then using we have $\Delta u = 0$ we have

$$\iint_D |\nabla u|^2 dV = u \oint_{\partial D} \frac{\partial u}{\partial n} dS \overset{by\ (*)}{=} 0$$

Since $|\nabla u|^2 \geq 0$ it follows that $|\nabla u|^2 = 0$ and so u is constant ■

Neumann problem

$$\Delta u = 0 \qquad \text{in } \text{ D}$$
$$\frac{\partial u}{\partial n} = h \qquad \text{on } \partial D$$

By (*)

$$0 = \iint\limits_{D} \Delta u dV = \oint\limits_{\partial D} \frac{\partial u}{\partial n} dS = \oint\limits_{\partial D} h dS$$

Fredholm alternative:
Hence, a necessary condition for a solution to exist is that

$$\oint\limits_{\partial D} h dS = 0$$

If this is true then there are an infinite number of solutions since we can add an arbitrary constant to any solution.

Uniqueness

Theorem 2

$$\Delta u = f \qquad \textit{in } \; D$$
$$u = h \qquad \textit{on } \partial D$$

Then the solution is unique

Proof. Follows from maximum principle ■
Existence is hard to prove.

Dirichlet Principle

$$E(w) = \frac{1}{2} \iint\limits_{D} |\nabla w|^2 dV$$

Consider all functions w that satisfy $w = h$ on ∂D . Then the minimum of w is reached for $w = u$ and

$$\Delta u = 0 \qquad \text{in } \text{ D}$$
$$u = h \qquad \text{on } \partial D$$

Problem with the proof is that it is not clear the minimum is achieved !

Energy

$$\iint\limits_D u\Delta u dV + \iint\limits_D |\nabla u|^2 dV = \oint\limits_{\partial D} u\frac{\partial u}{\partial n} dS$$

So if $\Delta u = 0$ in D and either $u = 0$ or $\frac{\partial u}{\partial n} = 0$ on ∂D then $\iint\limits_D |\nabla u|^2 dV = 0$. For the Dirichlet boundary condition this implies that $u = 0$ everywhere and for the Neumann boundary condition that u is a constant in D.