Laplace Equation-Separation of Variables

Consider

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 0 \le x \le a \quad 0 \le y \le b$$

$$\frac{\partial u}{\partial y} + u = 0 \qquad y = 0$$

$$u(0, y) = 0$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = a$$

$$u(x, b) = g(x) \qquad y = b$$

Assume (to be justified later)

$$u = X(x)Y(y)$$

Substituting into the Laplace equation we get

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Hence,

$$X'' + \lambda^2 X = 0 \qquad X(0) = 0 \qquad X'(a) = 0$$

$$Y'' - \lambda^2 Y = 0 \qquad Y'(0) + Y(0) = 0$$

Solving we get

$$X_n(x) = \sin(\lambda_n x) \qquad \lambda_n = \frac{(n + \frac{1}{2})\pi}{a}$$

$$Y_n(y) = A\cosh(\lambda_n y) + B\sinh(\lambda_n y)$$

Using the boundary conditions

$$0 = Y_n'(0) + Y_n(0) = B\beta_n + A$$

So

$$B = -1 \qquad A = \beta_n$$

and

$$Y_n(y) = \lambda_n \cosh(\lambda_n y) - \sinh(\lambda_n y)$$

and

$$u(x, y) = \sum_{n=0}^{\infty} A_n \sin(\lambda_n x) \left[\lambda_n \cosh(\lambda_n y) - \sinh(\lambda_n y)\right]$$

At $y = b$ we have

$$u(x,b) = g(x) = \sum_{n=0}^{\infty} A_n \sin(\lambda_n x)\left[\lambda_n \cosh(\lambda_n b) - \sinh(\lambda_n b)\right]$$

$$g(x) = \sum_{n=0}^{\infty} \widetilde{A}_n \sin(\lambda_n x) \quad \widetilde{A}_n = A_n\left[\lambda_n \cosh(\lambda_n b) - \sinh(\lambda_n b)\right]$$

So

$$\begin{aligned} \widetilde{A}_n &= \frac{2}{a}\int_0^a g(x)\sin(\beta_n x)dx \\ A_n &= \frac{2}{a}\frac{1}{\lambda_n \cosh(\lambda_n b) - \sinh(\lambda_n b)}\int_0^a g(x)\sin(\lambda_n x)dx \end{aligned}$$

Theory

We consider the easier case

$$\begin{aligned} \Delta u = 0 \qquad 0 \le x \le L \quad 0 \le y \le L \\ u(x,0) = 0 \\ u(0,y) = u(L,y) = 0 \\ u(x,L) = g(x) \end{aligned}$$

with $|g(x)| \le L$.

Usingseparation of variables we get

$$u(x,y) = \sum_{n=0}^{\infty} A_n X_n(x) \sinh\left(\frac{n\pi}{L} y\right) = \sum_{n=0}^{\infty} A_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

So

$$|X_n(x)| \le \sqrt{\frac{2}{L}}$$

$$A_n = \frac{1}{\sinh(n\pi)}\int_0^L X_n(x) g(x) dx$$

So

$$\begin{aligned} |A_n| &\le \frac{1}{|\sinh(n\pi)|}\int_0^L |X_n(x)| \cdot |g(x)| dx \\ &\le \frac{K\sqrt{\frac{2}{L}}L}{|\sinh(n\pi)|} = \frac{K\sqrt{2L}}{|\sinh(n\pi)|} \end{aligned}$$

So for $y \leq y_0 < L$

$$\begin{aligned}
|u(x,y)| &\leq \sum_{n=0}^{\infty} |A_n| \cdot |X_n(x)| \cdot \left|\sinh\left(\frac{n\pi}{L} y_0\right)\right| \\
&\leq \sum_{n=0}^{\infty} \frac{K\sqrt{2L}}{|\sinh(n\pi)|} \cdot \sqrt{\frac{2}{L}} \cdot \left|\sinh\left(\frac{n\pi}{L} y_0\right)\right| \\
&= 2K \sum_{n=0}^{\infty} \left| \frac{\sinh\left(\frac{n\pi}{L} y_0\right)}{\sinh(n\pi)} \right| \\
&\leq 2K \sum_{n=0}^{\infty} e^{-\frac{n\pi}{L}(L-y_0)} = 2K \sum_{n=0}^{\infty} \left(e^{-\frac{\pi}{L}(L-y_0)}\right)^n \\
&= 2K \sum_{n=0}^{\infty} r^n = \frac{2K}{1-r} \qquad r = e^{-\frac{\pi}{L}(L-y_0)}
\end{aligned}$$

Note:

$$\begin{aligned}
\frac{\sinh(\alpha z)}{\sinh(z)} &= \frac{e^{az} - e^{-az}}{e^z - e^{-z}} = e^{-(1-\alpha)z} \frac{1 - e^{-2\alpha z}}{1 - e^{-2z}} \\
&\leq e^{-(1-\alpha)z} \qquad \text{if } \alpha < 1
\end{aligned}$$

So the series converges absolutely and uniformly when $y \leq y_0 < L$ and so $u(x,y)$ is continuous even though $g(x)$ is only bounded and need not be continuous ! . However, as we approach the upper boundary the convergence is no longer uniform. If $g(x)$ is more regular then we get terms $\frac{1}{n^k}$ which improves the convergence at the upper boundary itself. However, if g is bounded bit not continuous then the convergence can not be uniform at the upper boundary itself.

Term by term differentiation is valid since $|X_n'(x)|$ is bounded. Since $\sum_{n=0}^{\infty} nr^n \leq \infty$ the sum for the derivative is absolutely and uniformly convergent. Similarly for the second derivative and so $u(x,y)$ satisfies the Laplace equation in the strong sense. By the same argument all derivatives of u exist and are bounded.