Heat/Diffusion Equation

$$\frac{\partial w}{\partial t} - k\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{k constant}$$
$$w(x,0) = \varphi(x) \qquad \text{initial condition}$$
$$w(0,t) = w(l,t) = 0 \qquad \text{boundary conditions}$$

Energy estimate:

$$0 = w(w_t - kw_{xx}) = (\tfrac{w^2}{2})_t - (kww_x)_x + k(w_x)^2$$

So

$$0 = \int_0^l \left[(\frac{w^2}{2})_t - (kww_x)_x + k(w_x)^2\right] dx$$
$$= \int_0^l (\frac{w^2}{2})_t dx - kww_x|_0^l + k\int_0^l (w_x)^2 dx$$

or

$$\frac{d}{dt}\int_0^l \frac{w^2}{2} dx = -k\int_0^l (w_x)^2 dx \leq 0$$

and therefore $E(t) = \int_0^l \frac{w^2}{2} dx$ is a non-increasing function of time or equivalently

$$0 \leq E(t) \leq E_0$$

For the initial value problem if $\varphi(x) = 0$ then $E(0) = 0$ and $E(t) = 0$ i.e. uniqueness.

Stability

Consider $\quad u_t = ku_{xx} \qquad u(x,0) = \varphi(x)$
$\qquad v_t = kv_{xx} \qquad v(x,0) = \psi(x)$

Let $w(x,t) = u(x,t) - v(x,t)$
$\qquad w(x,0) = u(x,0) - v(x,0) = \varphi(x) - \psi(x)$

From the energy estimate we get

$$\int_0^l [u(x,t) - v(x,t)]^2 \, dx \leq \int_0^l [\varphi(x) - \psi(x,t)]^2 \, dx$$

So small changes in φ causes small changes in v in the L^2 norm.
The maximum principle will give us stability in the maximum norm.

Maximum Principle

Let

$$Lu = -\frac{\partial u}{\partial t} + \sum_{i,j} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u = f$$

Strong version: Assume $Lu \geq 0$, $M = \max(u)$. Assume $u = M$ at an interior point (x_0, t_0) and one of the following is true

1. $c = 0$ and M arbitrary
2. $c \leq 0$ and $M \geq 0$
3. $M = 0$ and c arbitrary

Then $u = M$ everywhere in the domain (i.e. u is constant).
Weak Version: If $Lu \geq 0$ and $c(x,t) = 0$ then the maxiumum occurs initially or on the boundary.
So if $|u| < M$ at $t = 0, x = 0, x = l$ then $u < M$ everywhere

For a minimum principle consider $\frac{-\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$
Proof (weak version in one dimension)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Intuitively at a maximum the first derivatives are zero and $\frac{\partial^2 u}{\partial x^2} < 0$ so we have a contradiction. The problem is that we may have a saddle point with $\frac{\partial^2 u}{\partial x^2} = 0$. So instead we supply a perturbation argument.

Define:

$$v(x,t) = u(x,t) + \epsilon x^2 \qquad \epsilon > 0$$

Wish: $v(x,t) < M + \epsilon l^2$ with $M = \max |u(x,t)|$ so $u(x,t) < M + \epsilon(l^2 - x^2)$

Formally:

$$v(x,t) < M + \epsilon l^2$$
$$\text{on } t = 0 \text{ and } x = 0 \text{ and } x = l$$

Also

$$(*) \quad v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - ku_{xx} - 2\epsilon k < 0$$

Assume there is a maximum at (x_0, t_0) in the interior.
At such a maximum $v_t = 0$ and $v_{xx} \le 0$.
Hence, $v_t - kv_{xx} \ge 0$ at (x_0, t_0).
This contradicts (*). Hence, $v(x,t) < M + \epsilon l^2$.
Note that low order terms destroy this property. Consider

$$u_t = u_{xx} + au \quad a > 0$$
$$u(x,0) = \sin \pi x \quad u(0,t) = u(1,t) = 0$$

has a solution $u(x,t) = e^{(a-\pi^2)t} \sin(\pi x)$ which grows in time if $a > \pi^2$.

Lemma:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

proof:
One proof is by a complex integral and Cauchy's Theorem.
Another proof is

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dx dy \qquad r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y \sin\theta \\ &= \int_{0}^{2\pi} \int_{-\infty}^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_{-\infty}^{\infty} e^{-r^2} r dr \\ &= -2\pi \int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dr} \left(e^{-r^2}\right) dr = -\pi e^{-r^2} |_0^\infty = \pi \end{aligned}$$

Properties of heat equation

Let $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

1. $u(x-y,t)$ is a solution (shift)
2. Any derivative of u is a solution
3. Any linear combinations of solutions is a solution
4. The integral $\int S(x-y,t)g(y)dy$ is a solution
5. $u(\sqrt{a}x, at)$ $\quad a>0$ is a solution
6. u=u(p) $\qquad p = \frac{1}{\sqrt{4k}}\frac{x}{t}$. Then $u(\sqrt{a}x, at) = \frac{1}{\sqrt{4k}}\frac{\sqrt{a}x}{\sqrt{at}} = u(x,t)$.

$$(*) \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = \phi(x)$$

proof of #4: If S(x,t) is a solution of (*) then by linearity so is

$$v(x,t) = \int\limits_{-\infty}^{\infty} S(x-y,t)g(y)dy$$

Then

$$v_t = \int\limits_{-\infty}^{\infty} S_t(x-y,t)g(y)dy$$
$$v_{xx} = \int\limits_{-\infty}^{\infty} S_{xx}(x-y,t)g(y)dy$$
$$v_t - kv_{xx} = \int\limits_{-\infty}^{\infty} \left[S_t(x-y,t) - kS_{xx}(x-y,t)\right] g(y)dy = 0$$

proof of (5)

$$v = u(\sqrt{a}x, at)$$
$$v_t = au_t(\sqrt{a}x, at)$$
$$v_{xx} = au_{xx}(\sqrt{a}x, at)$$
$$v_t - kv_{xx} = a(u_t - ku_{xx}) = 0$$

Define

$$Q_t = kQ_{xx}$$
$$\begin{cases} Q(x,0) = 1 & x > 0 \\ Q(x,0) = 0 & x < 0 \end{cases}$$

Assume $\quad Q(x,t) = g(p) \qquad p = \frac{x}{\sqrt{4kt}}$
then:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) = -\frac{p}{2t}g'(p) \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{1}{4kt}g''(p) \end{aligned}$$

So

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right]$$

Therefore

$$\begin{aligned} g''(p) + 2pg'(p) &= 0 \\ g'(p) &= C_1 e^{-p^2} \\ Q(x,t) = g(p) &= C_1 \int_0^{p=\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + C_2 \end{aligned}$$

We now take the limit as $t \downarrow 0$. Then
Define

$$\begin{cases} x > 0 \quad 1 = Q(x,0) = C_1 \int_0^{\infty} e^{-p^2}\,dp + C_2 = \frac{C_1\sqrt{\pi}}{2} + C_2 \\ x < 0 \quad 0 = Q(x,0) = C_1 \int_0^{-\infty} e^{-p^2}\,dp + C_2 = -\frac{C_1\sqrt{\pi}}{2} + C_2 \end{cases}$$

So

$$C_1 = \frac{1}{\sqrt{\pi}} \qquad C_2 = \frac{1}{2}$$

Therefore

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp$$

Error Function

Remember

$$\int_0^\infty e^{-p^2}\,dp = \int_{-\infty}^0 e^{-p^2}\,dp = \frac{1}{2}\int_{-\infty}^\infty e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}$$

Define

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp \qquad \operatorname{erf}c(x) = 1 - \operatorname{erf}(x)$$

$$\operatorname{erf}(0) = 0 \qquad \operatorname{erf}(\infty) = 1 \qquad \operatorname{erf}(-x) = -\operatorname{erf}(x)$$

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^x e^{-p^2}\,dp = \frac{2}{\sqrt{\pi}}\int_{-\infty}^0 e^{-p^2}\,dp + \frac{2}{\sqrt{\pi}}\int_{-0}^x e^{-p^2}\,dp = 1 + \operatorname{erf}(x)$$

Then

$$Q(x,t) = \frac{1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)}{2}$$

Define

$$S(x,t) = \frac{\partial Q}{\partial x}$$

Then S is also a solution and

$$u(x,t) = \int\limits_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy$$

is also a solution. Does it satisfy the initial condition $u(x,0) = \varphi(x)$?

$$\begin{aligned}
u(x,t) &= \int\limits_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy = \int\limits_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\varphi(y)dy \\
&= -\int\limits_{-\infty}^{\infty} \frac{\partial Q}{\partial y}(x-y,t)\varphi(y)dy \qquad \text{since} \frac{\partial Q}{\partial x} = -\frac{\partial Q}{\partial y} \\
&= \int\limits_{-\infty}^{\infty} Q(x-y,t)\frac{\partial \varphi}{\partial y}(y)dy - Q(x-y,t)\varphi(y)|_{-\infty}^{\infty}
\end{aligned}$$

Assume φ is zero at infinity then

$$u(x,t) = \int\limits_{-\infty}^{\infty} Q(x-y,t)\frac{\partial \varphi}{\partial y}(y)dy$$

Using

$$\begin{cases} Q(s,0) = 1 & s > 0 \\ Q(s,0) = 0 & s < 0 \end{cases}$$

We have

$$\begin{aligned}
u(x,0) &= \int\limits_{-\infty}^{\infty} Q(x-y,0)\frac{\partial \varphi}{\partial y}(y)dy \\
&= \int\limits_{-\infty}^{x} \frac{\partial \varphi}{\partial y}(y)dy = \varphi|_{-\infty}^{x} = \varphi(x)
\end{aligned}$$

Conclusion

$$S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy$$

$$\int\limits_{-\infty}^{\infty} S(x,t) dx = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-p^2} dp = 1$$

$$\lim_{t=0} S(x,t) = \delta(x)$$

Importance of Limiting Process

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = 0$$

Then a possible solution is

$$u(x,t) = \frac{1}{4\sqrt{\pi t^3}} e^{-\frac{x^2}{4t}}$$

For x fixed and $t \to 0$ then $u(x,t) \to 0$.

However, for $\frac{x^2}{4t}$ constant and $t \to 0$ the solution is not bounded !! Hence, this is not a legitimate solution

Main Theorem

Theorem: Assume

$$(*) \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = \varphi(x) \qquad |\phi(x)| < \infty$$

Then

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)dy$$

- $u \epsilon C^\infty \qquad -\infty < x < \infty \quad , \qquad 0 < t < \infty$
- each derivative of $u(x,t)$ satisfies $(*)$
- $\lim_{t \downarrow 0} \; u(x,t) = \varphi(x)$

proof

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy = \int_{-\infty}^{\infty} S(z,t)\varphi(x-z)dz$$
$$S(z,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{z^2}{4kt}}$$

Let $p = \frac{z}{\sqrt{kt}} \quad z = \sqrt{kt}p$. Then

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{0}^{x} e^{-\frac{p^2}{4}} \varphi(x - p\sqrt{kt})dp$$

Assume $|\varphi(x)| \leq M$. Then

$$|u(x,t)| \leq \frac{M}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} dp = M$$

Differentiating we get

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\varphi(y)dy = -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{z}{2kt} e^{-\frac{z^2}{4kt}} \varphi(x-z)dz$$
$$= \frac{\text{const}}{\sqrt{t}} \int_{-\infty}^{\infty} pe^{-\frac{p^2}{4}} \varphi(x - p\sqrt{kt})dp$$
$$\leq \frac{\text{const}}{\sqrt{t}} M \int_{-\infty}^{\infty} pe^{-\frac{p^2}{4}} dp \leq \frac{\text{const}}{\sqrt{t}} M$$

Similarly

$$\frac{\partial^n u}{\partial x^n} \leq \frac{\text{const}}{\sqrt{t}} M \int_{-\infty}^{\infty} p^n e^{-\frac{p^2}{4}} dp \leq \frac{\text{const}}{\sqrt{t}} M$$

Remembering that $\int_{-\infty}^{\infty} S(x,t)dx = 1$ we get

$$u(x,t) - \varphi(x) = \int_{-\infty}^{\infty} S(x-y,t)\left[\varphi(y) - \varphi(x)\right] dy$$
$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4}} \left[\varphi(x - p\sqrt{kt}) - \varphi(x)\right] dp$$

We assume that $\varphi(x)$ is continuous. Hence,
$|\varphi(y) - \varphi(x)| \leq \varepsilon$ when $|x-y| \leq \delta$. So

$$|u(x,t) - \varphi(x)| = \underbrace{\int_{-\frac{\delta}{\sqrt{kt}}}^{\frac{\delta}{\sqrt{kt}}} ...dp}_{} + \underbrace{\int_{|p|>\frac{\delta}{\sqrt{kt}}} ...dp}_{}$$

bounded by	$\frac{\varepsilon}{2}$	$\frac{\varepsilon}{2}$
because	δ is small	t is small

So $\quad |\varphi(y) - \varphi(x)| \leq \frac{\varepsilon}{2} \quad p \to \infty$

Backward heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2}$$

This gives growth in time and is not well posed. Equivalently we are trying to solve the heat equation backward in time, i.e. given distribution of temperature at time T find original temperature.

Consider

$$\begin{aligned} u_n(x,0) &= \frac{1}{n}\sin(nx) \to 0 \quad \text{as} \quad n \to \infty \\ u_n(x,t) &= \frac{1}{n}\sin(nx)e^{n^2kt} \quad \text{is unbounded} \end{aligned}$$

Black Scholes

$$\frac{\sigma^2 s^2}{2}V_{ss} + rsV_s - rV + V_t = 0$$

s=market value of asset being optioned
t=time
σ=constant volatility
r=constant interest rate

Given the final value we wish to determine how to price it initially. So we have backward problem ! - but we have additional minus sign $\implies$ well posed

Exercise

$$\begin{cases} \varphi(x,0) = 1 & |x| < l \\ \varphi(x,0) = 0 & |x| > l \end{cases}$$

Then

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy \\
&= \frac{1}{\sqrt{4k\pi t}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy \\
&= \frac{1}{\sqrt{4k\pi t}} \sqrt{4kt} \int_{\frac{x+l}{\sqrt{4kt}}}^{\frac{x-l}{\sqrt{4kt}}} e^{-q^2} dq \\
&\overset{q=\frac{x-y}{\sqrt{4kt}}}{=} -\frac{1}{\sqrt{4\pi}} \int_{\frac{x+l}{\sqrt{4kt}}}^{\frac{x-l}{\sqrt{4kt}}} e^{-q^2} dq = \frac{1}{\sqrt{4\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-q^2} dq \\
&= \operatorname{erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-l}{\sqrt{4kt}}\right)
\end{aligned}$$

Note: Solution is different from zero for **all** x when $t > 0$.

Lower Order Terms

$$
\begin{aligned}
u_t &= k u_{xx} - \gamma u \\
u(x,0) &= \varphi(x)
\end{aligned}
$$

Let $v(x,t) = e^{\gamma t} u(x,t)$ so $u(x,t) = e^{-\gamma t} v(x,t) \qquad v(x,0) = \varphi(x)$

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= e^{-\gamma t} \frac{\partial v}{\partial t} - \gamma e^{-\gamma t} v \\
\frac{\partial^2 u}{\partial x^2} &= e^{-\gamma t} \frac{\partial^2 v}{\partial x^2}
\end{aligned}
$$

So

$$
e^{-\gamma t} \frac{\partial v}{\partial t} - \gamma e^{-\gamma t} v = e^{-\gamma t} \frac{\partial^2 v}{\partial x^2} - \gamma e^{-\gamma t} v
$$

or

$$
\begin{aligned}
v_t &= k v_{xx} \\
v(x,0) &= \varphi(x)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy \\
u(x,t) &= \frac{e^{-\gamma t}}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy
\end{aligned}
$$

Convection - Diffusion

$$u_t + cu_x = ku_{xx}$$
$$u(x,0) = \varphi(x)$$

Let $v(x,t) = u(x+ct,t)$ or $u(x,t) = v(\underbrace{x-ct}_{y},t) = v(y,\tau)$

Then using the chain rule

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial v}{\partial \tau} - c\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial v}{\partial y}
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial v}{\partial \tau} - c\frac{\partial v}{\partial y} + c\frac{\partial v}{\partial y} &= k\frac{\partial^2 v}{\partial y^2} \\
\frac{\partial v}{\partial \tau} &= k\frac{\partial^2 v}{\partial y^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)dy \\
u(x,t) &= \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-ct-y)^2}{4kt}}\varphi(y)dy
\end{aligned}$$

For example choose

$$\varphi(x) = \begin{cases} \alpha & x < 0 \\ \beta & |x| > 0 \end{cases}$$

Then

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}}\left\{\alpha\int_{-\infty}^{0} e^{-\frac{(x-ct-y)^2}{4kt}}\varphi(y)dy + \beta\int_{0}^{\infty} e^{-\frac{(x-ct-y)^2}{4kt}}\varphi(y)dy\right\}$$

Define $p = \frac{x-ct-y}{\sqrt{4kt}}$. Then $y = \sqrt{4kt}p + x - ct$.

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{\pi}} \left\{ \alpha \int_{-\infty}^{\frac{x-ct}{\sqrt{4kt}}} e^{-p^2} dy + \beta \int_{\frac{x-ct}{\sqrt{4kt}}}^{0} e^{-p^2} dy + \right\} \\
&= \frac{1}{\sqrt{\pi}} \left\{ \alpha \left[\left(\int_{-\infty}^{0} + \int_{0}^{\frac{x-ct}{\sqrt{4kt}}} \right) e^{-p^2} dy \right] + \beta \left[\left(\int_{0}^{\infty} - \int_{0}^{\frac{x-ct}{\sqrt{4kt}}} \right) e^{-p^2} dy \right] \right\} \\
&= \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} \operatorname{erf}\left(\frac{x-ct}{\sqrt{4kt}}\right)
\end{aligned}$$

Comparison: Wave Equation – Diffusion Equation

Property	Wave	Diffusion
speed	finite	infinite
singularity	along characteristics	disappear
well posed $t > 0$	✓	✓
well posed $t < 0$	✓	X
maximum principle	X	✓
Energy	conserved	decays
information	transported	decays