**Wronskian**

Given two functions $u(x)$ and $v(x)$

$$W[u,v] = \begin{vmatrix} u(x) & v(x) \\ u^{'}(x) & v^{'}(x) \end{vmatrix} = uv^{'} - u^{'}v$$

Theorem

- $u$ and $v$ are linearly independent iff $W[u,v] \neq 0$ for all x
- If $W[u,v] = 0$ then $v(x) = cu(x)$

**Jacobian**

Given two functions $u(x,y)$ and $v(x,y)$

$$J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Then $dudv = J(u,v)dxdy$

Theorem:
Given $u(x,y)$ and $v(x,y)$

- If $J(u,v) \neq 0$ in $D$ then this can be inverted in D

$$x = x(u,v)$$
$$y = y(u,v)$$

- If $J(u,v) \equiv= 0$ in $D$ then

$$v = F(u)$$

*Examples*

1.

$$x = s + t$$
$$y = (s+t)^2$$

then $\frac{\partial(x,y)}{\partial(s,t)} = 0$

2.

$$x = s + t$$
$$y = s - t$$

then $\frac{\partial(x,y)}{\partial(s,t)} = -2$ so

$$s = \frac{x+y}{2}$$
$$t = \frac{x-y}{2}$$

3.

$$x = s + t$$
$$y = s^2 - t^2$$

then $\frac{\partial(x,y)}{\partial(s,t)} = -2(s+t)$. So $J$ vanishes on the line $s = -t$. A global inverse may not exist. However, in this case

$$s = \frac{x^2 + y}{2x}$$
$$t = \frac{x^2 - y}{2x}$$

4.

$$x = s + t$$
$$y = s^2 + t^2$$

then $\frac{\partial(x,y)}{\partial(s,t)} = 2(t-s)$ so $J$ vanishes on the line $s = t$. Only a double-valued inverse exists in any domain containing $s = t$.

**Inhomogenous equations - Duhamel**

$$\frac{du}{dt} + Au(t) = f(t) \qquad u(0) = \phi$$

multiply by $e^{At}$

$$e^{At}\frac{du}{dt} + e^{At}Au(t) = e^{At}f(t)$$

$$\frac{d}{dt}(e^{At}u) = e^{At}f(t)$$

integrate

$$e^{At}u(t) = \int_0^t e^{As}f(s)ds + C$$

set t=0

$$u(0) = C = \phi$$

So

$$u(t) = e^{-At}\int_0^t e^{As}f(s)ds + \phi e^{-At}$$

$$u(t) = \phi e^{-At} + \int_0^t e^{A(s-t)}f(s)ds$$

General case

$$calL(u) = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, uyy) = 0 \tag{1}$$

Notation:
d = number of dimesnions
p=order of equation, i.e. highest derivative
We shall concentrate on first and center order equations

Strong Solution

A strong solution satisfies (1) in some region subject to other constraints (initial & boundary conditions).

*Linear Equations*

An equation is linear if when $u$ and $v$ are solutions so is $a\ u(x, y) + b\ v(x, y)$. i.e.

$$calL(au + bv) = a\ calL(u) + b\ calL(v)$$

An equation is homogenous if $calL(u) = 0$.

**Fredholm Alternative**

Given $A \epsilon R^{n \times n}$
Then exactly one of the following alternatives is true

- The null space of A is trivial. So for every $b \epsilon R^n$ there exists a unique solution $x \epsilon R^n$.
- The null space of A is non-trivial. Then $Ax = b$ has a solution iff whenever $y$ has the property that

  $$y \epsilon R^n \qquad A^T y = 0$$

  So

  $$y \epsilon N(A^T) \qquad (y, b) = 0$$

  If this is true then there are infinitely many solutions of the form

  $$x = x_0 + y \qquad y \epsilon N(A^T)$$

  Note: If $y^T A = 0$ then

  $$(y, b) = y^T b = y^T A x = (y^T A) x = 0$$

  so the condition is necessary.

*example*

Consider

$$-\frac{d^2u}{dx^2} = f \qquad 0 \leq x \leq l$$
$$\frac{du}{dx}(0) = 0 \qquad \frac{du}{dx}(l) = 0$$

Then $u =$ constant is a solution of the homogenous equation. Hence, the second alternatve exists.

*Compatibility Condition*

$$\int_0^l f(x)dx = -\int_0^l \frac{d^2u}{dx^2}dx = -\frac{du}{dx}|_0^l = \frac{du}{dx}(0) - \frac{du}{dx}(l) = 0$$

So a solution exists only if $\int_0^l f(x)dx = 0$.
If this is true then u + constant is also a solution.