First Order Equations

$$(*) \qquad a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

Cauchy Data: Given curve Γ in (x,y) and given u on Γ

Introduce new parametric coordinates:

- s goes out from curve Γ. So the initial curve is $s = 0$
- t runs along the curve Γ.

s and t need to be independent. For convenience we will usually choose them to be orthogonal.

We can rewrite $(*)$ as a directional derivative in the direction s away from Γ . We then get

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

Similarly differentiating the initial condition along Γ (denoted by $'$) we get

$$(**) \qquad u_0' = \frac{\partial u}{\partial x}x_0' + \frac{\partial u}{\partial y}y_0'$$

Combining (*) and (**) we have two equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ on Γ. A solution exists and

is unique if the determinant is nonzero, i.e.

Non-characteristic condition:

$$\begin{vmatrix} a & b \\ x_0' & y_0' \end{vmatrix} = ay_0' - bx_0' \neq 0 \qquad \text{all} \quad s_1 \leq s \leq s_2$$

- We therefore solve (*) and (**) with the following procedure
- paramertize initial curve in terms of t.
- We write the PDE and intial conditions as the characteristic equations

$$\begin{aligned} \frac{\partial x}{\partial s} &= a \\ \frac{\partial y}{\partial s} &= b \\ \frac{\partial u}{\partial s} &= c \end{aligned}$$

and solve for $x = x(s,t), \quad y = y(s,t), \quad u = u(s,t)$.

- invert to find $s = s(x,y), \quad t = t(x,y)$ and so $u = u(x,y)$.

example - characteristic IC

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \qquad u = 1 \quad \text{on } y = x$$

We parameterize the initial conditions by $x = t$, $y = t$, $u = 1$. The characteristic equations become

$$\begin{aligned}\frac{\partial x}{\partial s} &= 1\\ \frac{\partial y}{\partial s} &= 1\\ \frac{\partial u}{\partial s} &= u\end{aligned}$$

The solution for the initial data is then $x = s + t$, $y = s + t$, Hence, we cannot invert to find (x,y). Thus, there is NO solution.
Checking directly the characteristic condition

$$ay_0' - bx_0' = 1 \cdot 1 - 1 \cdot 1 = 0$$

So the initial curve is characteristic

example

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

The characteristic curve is $\frac{dy}{dx} = \frac{y}{1}$.
So $y = Ce^x$
Hence
$\frac{d}{dx}\left[u(x, Ce^x)\right] = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$
So

$$u(x, Ce^x) = u(0, Ce^0) = u(0, C)$$

Let $y = Ce^x \qquad C = ye^{-x}$

$$\begin{aligned}u(x,y) &= u(0, e^{-x}y) = f(e^{-x}y)\\ u(0,y) &= f(y)\end{aligned}$$

If $u(0,y) = y^3$ initially then

$$u(x,y) = (e^{-x}y)^3 = y^3e^{-3x}$$

example

$$\begin{aligned}\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= 1 \qquad -\infty < y < \infty \quad x > 0 \\ u(0,y) &= y^2\end{aligned}$$

characteristic equations

$$\begin{aligned}\frac{\partial x}{\partial s} &= 1 \\ \frac{\partial y}{\partial s} &= y \\ \frac{\partial u}{\partial s} &= 1\end{aligned}$$

So $\frac{dy}{dx} = \frac{\frac{\partial y}{\partial s}}{\frac{\partial x}{\partial s}} = y$. This has a solution $x = s$ and $y = te^s$. Reversing we have

$$\begin{aligned}s &= x \\ t &= ye^{-s} = ye^{-x}\end{aligned}$$

The initial condition is

$$x = 0 \quad y = t \quad u = y^2 = t^2$$

So the characteristic is $y = te^s = y_0 e^x$. So $y_0 = ye^{-x}$.

On the characteristic $\frac{\partial u}{\partial s} = 1$ and so $u = s + c(t)$.
So

$$c = u - s = u(0,y) - 0 = y_0^2$$

In conclusion

$$u(x,y) = s + c = x + y_0^2 = x + (ye^{-x})^2 = x + y^2 e^{-2x}$$