Laplace Equation-Polar Coordinates

Consider

$$
\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0
\end{aligned}
$$

Consider a solution only of r. So

$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = 0 \qquad \frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) &= 0 \\
r\frac{\partial u}{\partial r} &= C_1 \\
u(r) &= C_1 \log(r) + C_2
\end{aligned}
$$

Spherical Coordinates

$$
\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + \cot(\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2 u}{\partial \varphi^2}\right] = 0 \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + \cot(\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2 u}{\partial \varphi^2}\right] = 0
\end{aligned}
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Consider a solution only of r. So

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\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = 0 \qquad \frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) &= 0 \\
r^2\frac{\partial u}{\partial r} &= C_1 \\
u(r) &= \frac{C_1}{r} + C_2
\end{aligned}
$$

Poisson Formula

Consider the Laplace equation inside a circle of radius a and on the boundary $u(a, \theta) = h(\theta)$.
Using separation of variables

$$u(r, \theta) = R(r)\Theta(\theta)$$

Substituting into the Laplace equation

$$\begin{aligned} R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' &= 0 \\ r^2\frac{R'' + \frac{1}{r}R'}{R} &= -\frac{\Theta''}{\Theta} = \lambda_n^2 \end{aligned}$$

So

$$\begin{aligned} \Theta'' + \lambda_n^2\Theta &= 0 \\ \Theta(\theta) &= A\cos(\lambda_n\theta) + B\sin(\lambda_n\theta) \end{aligned}$$

In order for the functions to be periodic with period 2π we require that $\lambda_n = n$.
So

$$\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

Then

$$r^2R'' + rR' - nR = 0$$

guess

$$\begin{aligned} R(r) &= r^\alpha \\ \left(\alpha(\alpha-1) + \alpha - n^2\right)r^\alpha &= 0 \\ \alpha &= \pm n \end{aligned}$$

However r^{-n} is not bounded at $r = 0$ and so we are left with

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

Using the boundary condition at $r = a$ we get

$$\begin{aligned} h(\theta) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n(A_n\cos(n\theta) + B_n\sin(n\theta)) \\ A_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\varphi)\cos(n\varphi)d\varphi \\ B_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\varphi)\sin(n\varphi)d\varphi \end{aligned}$$

Substituting the formula for A and B we get

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)d\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\int_0^{2\pi} h(\varphi)\cos(n\varphi)d\varphi\cos(n\theta) + \int_0^{2\pi} h(\varphi)\sin(n\varphi)d\varphi\sin(n\theta)\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)d\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi} h(\varphi)\left[\cos(n\varphi)\cos(n\theta)+\sin(n\varphi)\sin(n\theta)\right]d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)d\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi} h(\varphi)\cos(n(\varphi-\theta))d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\varphi-\theta))\right]d\varphi
\end{aligned}
$$

but

$$
\begin{aligned}
1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\psi) &= 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{in\psi}+e^{-in\psi}\right) \\
&= 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\psi}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in\psi} \\
&= 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i\psi}\right)^n+\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{-i\psi}\right)^n
\end{aligned}
$$

This is a geometric series. So

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\left[1+\frac{re^{i(\varphi-\theta)}}{a-re^{i(\varphi-\theta)}}+\frac{re^{-i(\varphi-\theta)}}{a-re^{-i(\varphi-\theta)}}\right]d\varphi \\
&= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{a^2-2ar\cos(\varphi-\theta)+r^2}d\varphi \\
&= \frac{a^2-|x|^2}{2\pi a}\oint\frac{u(x')}{|x-x'|^2}ds' \qquad ds' = ad\varphi
\end{aligned}
$$

Mean Value

We choose $r = 0$ in the Poisson formula. We then get

$$u(0,\theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\varphi) d\varphi = \text{average over circumference}$$

This is another proof of the maximum principle!

Solution in annulus

We repeat the above derivation but now in the annulus .

$$\begin{aligned} \Delta u &= 0 \qquad a \le r \le b \\ u(a,\theta) &= g(\theta) \\ u(b,\theta) &= h(\theta) \end{aligned}$$

As before we do separation of variables and get

$$\begin{aligned} u(r,\theta) = {} & \frac{1}{2}(C_0 + D_0) \log(r) \\ & + \sum_{n=1}^{\infty} \left(C_n r^n + \frac{D_n}{r^n} \right) (A_n \cos(n\theta) + B_n \sin(n\theta)) \end{aligned}$$

Imposing boundary conditions we get

$$\begin{aligned} g(\theta) = u(a,\theta) = {} & \frac{1}{2}(C_0 + D_0) \log(a) \\ & + \sum_{n=1}^{\infty} \left(C_n a^n + \frac{D_n}{a^n} \right) (A_n \cos(n\theta) + B_n \sin(n\theta)) \end{aligned}$$

So

$$\begin{aligned} (C_0 + D_0) \log(a) &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) d\theta \\ \left(C_n a^n + \frac{D_n}{a^n} \right) A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta \\ \left(C_n a^n + \frac{D_n}{a^n} \right) B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta \end{aligned}$$

Similarly

$$\begin{aligned} (C_0 + D_0) \log(b) &= \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) d\theta \\ \left(C_n b^n + \frac{D_n}{b^n} \right) A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \cos(n\theta) d\theta \\ \left(C_n b^n + \frac{D_n}{b^n} \right) B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin(n\theta) d\theta \end{aligned}$$

Using the first equation from each set this yields

$$C_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) d\theta - D_0 \log(a)$$

$$D_0 = \frac{1}{\pi \log(\frac{a}{b})} \int_{-\pi}^{\pi} [g(\theta) - h(\theta)]\, d\theta$$

Similarly

$$\frac{A_n}{B_n} = \frac{\int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta}{\int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta}$$

$$\frac{C_n a^n + \frac{D_n}{a^n}}{C_n b^n + \frac{D_n}{b^n}} = \frac{\int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta}{\int_{-\pi}^{\pi} h(\theta) \cos(n\theta) d\theta}$$