Rayleigh Quotient

$$
\begin{aligned}
-\nabla \cdot (p\nabla u) + qu = \lambda \widehat{m} u \qquad & \text{in } D \\
u = 0 \qquad & \text{on } \partial D
\end{aligned}
$$

We know that there are an infinite number of (positive) eigenvalues tending to ∞ . Denote them by

$$0 < \lambda_1 \le \lambda_2 \le \ldots$$

Define

$$(f, g) = \iiint \widehat{m}(x) f(x) \overline{g(x)} dx$$

Let

$$Q(w) = \frac{\iiint \left[p(x)|\nabla w|^2 + q(x)|w|^2 \right] dx}{\iiint \widehat{m}(x)|w|^2 dx}$$

Theorem 1 *Define*

$$m = \min Q(w) \qquad w = 0 \text{ on } \partial D \quad w \text{ smooth}$$

Then

$$\lambda_1 = m$$

and the minimizing function $u(x)$is an eigenfunction, i.e.

$$-\nabla \cdot (p\nabla u) + qu = \lambda_1 \widehat{m} u \qquad \text{in } D \quad w = 0 \text{ on } \partial D$$

Proof. Consider the case $p = 1$, $q = 0$ and $\widehat{m} = 1$ (we shall now use m as minimum of Q).

So for an eigenfunction u with eigenvalue λ

$$-\Delta u = \lambda u$$

Multiply by u and integrate we get

$$
\begin{aligned}
-\iiint u \Delta u dx &= \lambda \iiint u^2 dx \\
\text{by Green's identity} \qquad \iiint |\nabla u|^2 dx &= \lambda \iiint u^2 dx \\
\lambda &= \frac{\iiint |\nabla u|^2 dx}{\iiint u^2 dx}
\end{aligned}
$$

So we define

$$Q(w) = \frac{\iiint |\nabla w|^2 dx}{\iiint |w|^2 dx}$$

Then

$$Q(w) = \frac{\iiint |\nabla w|^2 dx}{\iiint |w|^2 dx}$$

Since m is a minimum and $u(x)$ is the minimizer function

$$m \leq f(\varepsilon) = \frac{\iiint |\nabla (u + \varepsilon v)|^2 dx}{\iiint (u + \varepsilon v)^2 \, dx}$$

Expanding

$$f(\varepsilon) = \frac{\iiint |\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2 dx}{\iiint (u^2 + 2\varepsilon uv + v^2) \, dx}$$

Differentiating with respect to ε and setting $\varepsilon = 0$ we get

$$f^{'}(0) = \frac{\iiint u^2 dx \cdot \iiint 2\nabla u \cdot \nabla v dx - \iiint |\nabla u|^2 dx \cdot 2 \iiint uv dx}{\left(\iiint u^2 dx\right)^2}$$

So

$$\iiint \nabla u \cdot \nabla v dx = \frac{\iiint |\nabla u|^2 dx}{\iiint u^2 dx} \iiint uv dx = m \iiint uv dx$$

By the first Green lemma

$$\iiint (\Delta u + mu) \, v dx = 0$$

Since v is arbitarary

$$\Delta u + mu = 0$$

i.e. m is an eigenvalue of $-\Delta u$.

However,

$$\begin{aligned} m &\leq \frac{\iiint |\nabla v_j|^2 dx}{\iiint v_j^2 dx} = \frac{\iiint v_j \Delta v_j dx}{\iiint v_j^2 dx} \\ &= \frac{\iiint (\lambda_j v_j) \, v_j dx}{\iiint v_j^2 dx} = \lambda_j \qquad \text{for all j} \end{aligned}$$

Hence, m is the smallest eigenvalue of $-\Delta u$ ■
Using these ideas (integration by parts) we can prove that

Theorem 2 *The eigenvalues are all non-negative if for every eigenfunction*

- $-pu\frac{du}{dx}|_a^b \geq 0$
- $q \leq 0$

Other eigenvalues

Theorem 3 *Minimum principle for n-th eigenvalue:*
Suppose $\lambda_1,...\lambda_{n-1}$ *are known with their eigenfunctions* $v_1(x),...v_{n-1}(x)$. *Then*

$$\lambda_n = \Big\{ \min \frac{\iiint |\nabla w|^2 dx}{\iiint |w|^2 dx} \quad w \neq 0 \quad w = 0 \textit{ on } \partial D \quad w \textit{ in } C^2$$
$$\textit{and } 0 = (w, v_1) = ... = (w, v_{n-1}) \}$$

We wish to find a formulation where λ_n *doesn't explicitly depend on the previous eigenvalues and eigenfunctions.*

Theorem 4 *Minimax principle*
Given n-1 linearly independent piecewise continuous functions $v_1, ..., v_n$ *we define*

$$\Lambda_n(v_1, ..., v_n) = \inf \iiint |\nabla w|^2 dx \quad (w, v_j) = 0 \quad (w, w) = 1 \quad w = 0 \textit{ on } \partial D$$

Then

$$\lambda_n = \sup \Lambda_n(v_1, ..., v_n) \quad \textit{all possible } v_j$$
$$= \max_{v_1,,,v_n} \Big\{ \min_w \iiint |\nabla w|^2 dx \quad (w, v_j) = 0 \quad (w, w) = 1 \quad w = 0 \textit{ on } \partial D \Big\}$$

example

$$u'' + \lambda u = 0 \qquad 0 \le x \le l$$
$$u(0) = u(l) = 0$$

exact: $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ *so* $\lambda_1 = \left(\frac{\pi}{l}\right)^2$

$$\lambda_1 = \min_w \frac{\int\limits_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx}{\int\limits_0^l w^2 dx} \le \frac{\int\limits_0^l \left(\frac{\partial \widehat{w}}{\partial x}\right)^2 dx}{\int\limits_0^l \widehat{w}^2 dx}$$

choose

$$\widehat{w}(x) = x(l-x) \quad \text{so } \widehat{w}(0) = \widehat{w}(l) = 0$$
$$\widehat{w}'(x) = l - 2x$$
$$\frac{\int\limits_0^l \left(\frac{\partial \widehat{w}}{\partial x}\right)^2 dx}{\int\limits_0^l \widehat{w}^2 dx} = \frac{\int\limits_0^l (l-2x)^2 dx}{\int\limits_0^l x^2(l-x)^2 dx} = \frac{10}{l^2}$$

So we have $\lambda_1 \le \frac{10}{l^2}$ *while the exact answer is* $\lambda_1 = \frac{\pi^2}{l^2} \sim \frac{9.8}{l^2}$