Second Order equations
Type

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x,y)$$

Definition 1 *Elliptic:* $B^2 < AC$ $u_{xx}+u_{yy}$
Hyperbolic: $B^2 > AC$ $u_{xx}-u_{yy}$
Parabolic: $B^2 = AC$ $u_{xx}-u_y$

Canonical Forms

We can divide the equation by A or equivalently assume $A = 1$. Since low order terms don't affect the type we choose $D = E = F = G = 0$. Then

$$u_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

$$(\frac{\partial}{\partial x} + B\frac{\partial}{\partial y})^2 u + (C - B^2)\frac{\partial^2 u}{\partial y^2} = 0$$

CASE I: elliptic $B^2 < AC$
let $b = \sqrt{C - B^2} > 0$
we now change variables

$$x = \xi \qquad y = B\xi + b\eta$$

$$\xi = x \qquad \eta = \frac{y - Bx}{b}$$

Then

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} = \partial_x + B\partial_y \qquad \frac{\partial^2}{\partial \xi^2} = \partial_x^2 + 2B\partial_x\partial_y + B^2\partial_y^2$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} = 0 + b\partial_y \qquad \frac{\partial^2}{\partial \eta^2} = b^2\partial_y^2$$

and so

$$\begin{aligned}\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} &= \left[\partial_x^2 + 2B\partial_x\partial_y + \left(B^2 + b^2\right)\partial_y^2\right] u \\ &= \left(\partial_x^2 + 2B\partial_x\partial_y + C\partial_y^2\right) u \\ &= 0\end{aligned}$$

CASE II: hyperbolic $B^2 > C$

$$u_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

$$(\frac{\partial}{\partial x} + B\frac{\partial}{\partial y})^2 u - (B^2 - C)\frac{\partial^2 u}{\partial y^2} = 0$$

let $b = \sqrt{B^2 - C} > 0$ we now change variables

$$\begin{aligned} x &= \xi \qquad y = B\xi + b\eta \\ \xi &= x \qquad \eta = \frac{y - Bx}{b} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \partial_x + B\partial_y \qquad \frac{\partial^2}{\partial \xi^2} = \partial_x^2 + 2B\partial_x\partial_y + B^2\partial_y^2 \\ \frac{\partial}{\partial \eta} &= b\partial_y \qquad \frac{\partial^2}{\partial \eta^2} = b^2\partial_y^2 \\ \frac{\partial^2}{\partial \xi \partial \eta} &= b\partial_x\partial_y + B\partial_y^2 \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} &= \left[\partial_x^2 + 2B\partial_x\partial_y + \left(B^2 - b^2\right)\partial_y^2\right] u \\ &= \left(\partial_x^2 + 2B\partial_x\partial_y + C\partial_y^2\right) u \\ &= 0 \end{aligned}$$

Similarly for systems. Consider

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + a_0 u = 0 \qquad a_{ij} = a_{ji}$$

Let $A = (a_{ij})$

Definition 2 *elliptic: The eigenvalues of A are all positive (or all negative)*
hyperbolic: one is negative and the others positive (or the opposite)
ultrahyperbolic : 2 are negative and the others positive
parabolic: one eigenvalue is zero and the others have the same sign

$$(*) \qquad L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y)$$

Consider a general change of variables $\xi = \xi(x,y) \qquad \eta = \eta(x,y)$ Then

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu = G$$

$$\begin{aligned}
A(\xi,\eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
B(\xi,\eta) &= a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y \\
C(\xi,\eta) &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2
\end{aligned}$$

and $AC - B^2 = J^2\left(ac - b^2\right). \qquad J = \xi_x\eta_y - \xi_y\eta_x.$

Hence, the type of equation is invariant under nonsingular transformations.
This leads to a second canonical form for hyperbolic equations given by $\xi_{xy} = 0$
i.e. we choose ξ and η so that

$$\begin{aligned}
a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0 \\
a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0
\end{aligned}$$

by factoring we have two solutions.

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = \frac{\left[a\xi_x + \left(b + \sqrt{b^2 - ac}\right)\xi_y\right]\left[a\xi_x + \left(b - \sqrt{b^2 - ac}\right)\xi_y\right]}{a}$$

We designate one as ξ and the other as η

$$\begin{aligned}
a\xi_x + \left(b + \sqrt{b^2 - ac}\right)\xi_y &= 0 \\
a\eta_x + \left(b - \sqrt{b^2 - ac}\right)\eta_y &= 0
\end{aligned}$$

These are again called characteristic curves. So these obey

$$\begin{aligned}
\frac{dx}{ds} &= a \\
\frac{dy}{ds} &= \left(b + \sqrt{b^2 - ac}\right) \\
\frac{d\xi}{ds} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\xi \text{ is constant on the characteristic } \frac{dy}{dx} &= \frac{\left(b + \sqrt{b^2 - ac}\right)}{a} \\
\eta \text{ is constant on the characteristic } \frac{dy}{dx} &= \frac{\left(b - \sqrt{b^2 - ac}\right)}{a}
\end{aligned}$$

CASE III: parabolic $B^2 = C$

$$u_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

$$(\frac{\partial}{\partial x} + B\frac{\partial}{\partial y})^2 u - (B^2 - C)\frac{\partial^2 u}{\partial y^2} = (\frac{\partial}{\partial x} + B\frac{\partial}{\partial y})^2 u = 0$$

choose

$$x = \xi \qquad y = B\xi + \eta$$
$$\xi = x \qquad \eta = y - Bx$$

Then

$$\frac{\partial^2 u}{\partial^2 \xi} = 0$$