Diffusion - half line

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{k constant} \qquad x > 0$$
$$u(x,0) = \varphi(x) \quad \text{initial condition}$$
$$\text{Dirichlet: } u(0,t) = 0 \qquad \text{boundary conditions}$$
$$\text{Neumann: } \frac{\partial u}{\partial x}(0,t) = 0$$

Def: odd function $\varphi(-x) = -\varphi(x)$
even function $\varphi(-x) = +\varphi(x)$

$$\varphi_{\text{odd}}(x) = \begin{cases} \varphi(x) & x > 0 \\ -\varphi(-x) & x < 0 \\ 0 & x = 0 \end{cases}$$

Important: $\varphi_{\text{odd}}(0) = 0$

$$\varphi_{\text{even}}(x) = \begin{cases} \varphi(x) & x > 0 \\ \varphi(-x) & x < 0 \end{cases}$$

CASE I: Dirichlet $u(0,t) = 0$
Solve

$$\frac{\partial u_{\text{odd}}}{\partial t} = k\frac{\partial^2 u_{\text{odd}}}{\partial x^2} \qquad -\infty < x < \infty$$
$$u_{\text{odd}}(x,0) = \varphi_{\text{odd}}(x)$$

Note: Since $u(x,t)$ is an odd function at $t = 0$ it will remain odd for all t.

$$u(x,t) = \int_0^\infty S(x-y,t)\varphi(y)dy + \int_{-\infty}^0 S(x-y,t)\begin{bmatrix}\text{sign not subtract} \\ -\varphi(-y)\end{bmatrix}dy$$
$$= \int_0^\infty \left[S(x-y,t) - S(x+y,t)\right]\varphi(y)dy$$
$$= \frac{1}{\sqrt{4k\pi t}}\int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right]\varphi(y)dy$$

CASE II: Neumann $\frac{\partial u}{\partial x}(0,t) = 0$

Solve

$$\frac{\partial u_{\text{even}}}{\partial t} = k\frac{\partial^2 u_{\text{even}}}{\partial x^2} \qquad -\infty < x < \infty$$
$$u_{\text{even}}(x,0) = \varphi_{\text{even}}(x)$$

Note: Since $u(x,t)$ is an even function at $t=0$ it will remain even for all t.
Also since $u(x,t)$ is an even function $\frac{\partial u}{\partial x}$ is an odd function

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\varphi(y)dy + \int_{-\infty}^0 S(x-y,t)\varphi(-y)dy \\ &= \int_0^\infty \left[S(x-y,t) + S(x+y,t)\right]\varphi(y)dy \\ &= \frac{1}{\sqrt{4k\pi t}}\int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}\right]\varphi(y)dy \end{aligned}$$

Wave Equation - half line

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{c constant} \qquad x > 0 \\
u(x,0) &= \varphi(x) \\
u_t(x,0) &= \psi(x) \qquad\qquad \text{initial condition} \\
\text{Dirichlet: } u(0,t) &= 0 \qquad \text{boundary conditions} \\
\text{Neumann: } \frac{\partial u}{\partial x}(0,t) &= 0
\end{aligned}$$

Consider the Dirichlet problem: As before we wish $u(x,t)$ to be an odd function

$$u(x,t) = \frac{\varphi_{\text{odd}}(x+ct) + \varphi_{\text{odd}}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(z)dz$$

Note: for $x > c|t|$ $\qquad \varphi_{\text{odd}} = \varphi$

$0 < x < c|t| \qquad \varphi_{\text{odd}}(x-ct) = -\varphi(ct-x)$

So

$$u(x,t) = \begin{cases} \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)dz & x > c|t| \\ \frac{\varphi(ct+x)-\varphi(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(z)dz & x < c|t| \end{cases}$$

For small times we don't feel the boundary condition. For large times there is an odd reflection off the boundary

$$\begin{aligned}
\int_{x-ct}^{x+ct} \psi_{\text{odd}}(z)dz &= -\int_{x-ct}^{0} \psi(z)dz + \int_0^{x+ct} \psi(z)dz \\
&\overset{z\to -z}{=} \int_{ct-x}^{0} \psi(z)dz + \int_0^{x+ct} \psi(z)dz \\
&= \int_{ct-x}^{ct+x} \psi(z)dz
\end{aligned}$$

Duhamel

$$
\begin{aligned}
u_t &= ku_{xx} + q(x,t) \qquad -\infty < x < \infty \quad t > 0 \\
u(x,0) &= \varphi(x)
\end{aligned}
$$

Split u into two parts. One satisfies the inhomogenous RHS and the other the inhomeogenous initial conditions, Hence,
$u = v + w$ with

$$
\begin{aligned}
v_t &= kv_{xx} + q(x,t) \\
v(x,0) &= 0 \\
w_t &= kw_{xx} \\
u(x,0) &= \varphi(x)
\end{aligned}
$$

So

$$
w(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)dy
$$

For v we first consider the ODE case

$$
(*) \qquad \begin{aligned}
\frac{du}{dt} + Au &= q \\
u(0) &= \varphi
\end{aligned}
$$

Define $S(t) = e^{-At}$. Then use the integrating factor e^{At} i.e.

$$
\begin{aligned}
e^{At}\frac{du}{dt} + e^{At}Au &= e^{At}q \\
\frac{d}{dt}\left(e^{At}u\right) &= e^{At}q \\
e^{At}u &= \int_0^t e^{As}q(s)ds + \varphi \\
u(t) &= e^{-At}\varphi + \int_0^t e^{-A(t-s)}q(s)ds \\
u(t) &= S(t)\varphi + \int_0^t S(t-s)q(s)ds
\end{aligned}
$$

Returning to the diffusion, the solution to the homogenous problem (w) is given by

$$(\mathcal{S}(t)\varphi)(x) = \int\limits_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy$$

i.e. $\mathcal{S}$ denotes an operator that turns the function φ into the above integral. Hence, we guess that the solution to (*) is given by

$$\begin{aligned}
u(x,t) &= \mathcal{S}(t)\varphi + \int\limits_0^t \mathcal{S}(t-s)q(s)ds \\
&= \int\limits_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy + \int\limits_0^t \int\limits_{-\infty}^{\infty} S(x-y,t-s)q(y,s)dyds \\
&= \frac{1}{\sqrt{4k\pi t}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)dy + \int\limits_0^t \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi\,(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} q(y,s)dyds
\end{aligned}$$

We are only going to verify that this is indeed the solution. By linearity we need only consider the case with zero initial conditions. Then

$$u(x,t) = \int\limits_0^t \int\limits_{-\infty}^{\infty} S(x-y,t-s)q(y,s)dyds$$

Therefore

$$\frac{\partial u}{\partial t} = \int\limits_0^t \int\limits_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t-s)q(y,s)dyds + \lim_{s\to t} \int\limits_{-\infty}^{\infty} S(x-y,t-s)q(y,s)dy$$

However, $\quad S_t = kS_{xx}. \quad$ Therefore,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \int\limits_0^t \int\limits_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,t-s)q(y,s)dyds + \lim_{\varepsilon\to 0} \int\limits_{-\infty}^{\infty} S(x-y,\varepsilon)q(y,t)dy \\
&= k\frac{\partial^2}{\partial x^2} \int\limits_0^t \int\limits_{-\infty}^{\infty} S(x-y,t-s)q(y,s)dyds + q(x,t) \\
&= k\frac{\partial^2 u}{\partial x^2} + q(x,t)
\end{aligned}$$

Nonhomogenous boundary condition:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + q(x,t) \qquad \text{k constant} \qquad x>0 \quad t>0\\
u(x,0) &= \varphi(x)\\
u(0,t) &= h(t) \qquad \text{boundary condition}
\end{aligned}$$

Define $V(x,t) = u(x,t) - h(t)$, $u(x,t) = V(x,t) + h(t)$. Then

$$\frac{\partial V}{\partial t} + h^{'}(t) = k\frac{\partial^2 V}{\partial x^2} + q(x,t)$$

or we have the following set

$$\begin{aligned}
\frac{\partial V}{\partial t} &= k\frac{\partial^2 V}{\partial x^2} + q(x,t) - h^{'}(t)\\
V(x,0) &= \varphi(x) - h(0)\\
V(0,t) &= 0
\end{aligned}$$

As before we now introduce $\varphi_{odd}(x)$ and $q_{\mathrm{odd}}(x,t)$.

Note: If $\varphi(0) \neq h(0)$ we have a discontinuity at the corner $x=0, t=0$ which disappears immediately, i.e. the solution is analytic in the interior.
Example: consider sticking hot iron bar into a cold bath

Waves with a Source

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} + q(x,t) \qquad -\infty < x < \infty \\
u(x,0) &= \varphi(x) \\
u_t(x,0) &= \psi(x)
\end{aligned}
$$

Theorem:

The unique solution is

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)dz + \frac{1}{2c}\iint_{\Delta} q dA$$

where Δ is the triangle bounded by the intial line and the two characteristics. So

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)dz + \frac{1}{2c}\int_{0}^{t}\int_{x-c(t-s)}^{x+c(t-s)} q(y,s))dyds$$

Note: we again include only the history between the characteristic lines, i.e. causality principle.

If there are two boundaries and homogenous boundary conditions we have waves that bounce off both ends reflecting forever.