horizontal displacements of particles of the string are negligible compared with vertical displacements; that is, displacements may be taken as purely transverse, representable in the form $y(x, t)$.

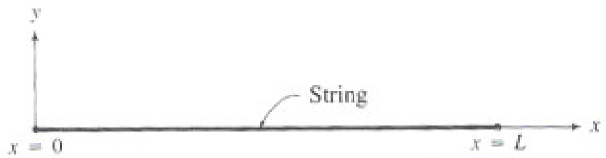


Figure 1.11

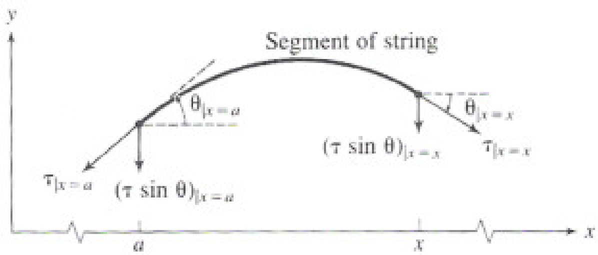


Figure 1.12

To find a PDE for $y(x, t)$, we analyze the forces on a segment of the string from a fixed position $x = a$ to an arbitrary position x (Figure 1.12). We denote by $\tau(x, t)$ the magnitude of the tension in the string at position x and time t. Because the string is perfectly flexible, tension in the string is always in the tangential direction of the string. This means that the y-component of the resulting force due to tension at the ends of the segment is $(\tau \sin \theta)_{|x=x} - (\tau \sin \theta)_{|x=a}$. We group all other forces acting on the segment into one function by letting $F(x, t)$ be the y-component of the sum of all external forces acting on the string per unit length in the x-direction. The total of all external forces acting on the segment then has y-component

$$\int_a^x F(\zeta, t)\, d\zeta.$$

Newton's second law states that the time rate of change of the momentum of the segment of the string must be equal to the resultant force thereon:

$$\frac{\partial}{\partial t}\left(\int_a^x \frac{\partial y(\zeta, t)}{\partial t} \rho(\zeta, t) \sqrt{1 + \left(\frac{\partial y(\zeta, t)}{\partial x}\right)^2}\, d\zeta\right)$$
$$= (\tau \sin \theta)_{|x=x} - (\tau \sin \theta)_{|x=a} + \int_a^x F(\zeta, t)\, d\zeta, \tag{34}$$

where $\rho(x, t)$ is the density of the string (mass per unit length). The quantity $\sqrt{1 + [\partial y(\zeta, t)/\partial x]^2}\, d\zeta$ is the length of string that projects onto a length $d\zeta$ along the x-axis. Multiplication by $\rho(\zeta, t)\partial y(\zeta, t)/\partial t$ gives the momentum of this infinitesimal length of string, and integration yields the momentum of that segment of the string from $x = a$ to an arbitrary position x. If we differentiate this equation with respect to x, we obtain

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\right)=\frac{\partial}{\partial x}(\tau\sin\theta)+F(x,t). \tag{35}$$

When vibrations of the string are such that the slope of the displaced string, $\partial y/\partial x$, is very much less than unity (and this is the only case that we consider), the radical may be dropped from the equation and $\sin\theta$ approximated by $\tan\theta = \partial y/\partial x$. The resulting PDE for $y(x, t)$ is

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right)=\frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right)+F(x,t). \tag{36}$$

For most applications, both the density of and the tension in the string may be taken as constant, in which case (36) reduces to

$$\boxed{\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}+\frac{F}{\rho}, \qquad c^2=\frac{\tau}{\rho}.} \tag{37}$$

This is the mathematical model for small transverse vibrations of a taut string; it is called the *one-dimensional wave equation*. In its derivation we have assumed that the slope of the string at every point is always very much less than 1 and that tension and density are constant.

When the only external force acting on the string is gravity, $F(x, t)$ takes the form

$$F=\rho g, \qquad g<0. \tag{38}$$

Other possibilities include a damping force proportional to velocity,

$$F=-\beta\frac{\partial y}{\partial t}, \qquad \beta>0; \tag{39}$$

and a restoring force proportional to displacement,

$$F=-ky, \qquad k>0. \tag{40}$$

Accompanying the wave equation will be initial and/or boundary conditions. Initial conditions describe the displacement and velocity of the string at some initial time (usually $t = 0$):

$$y(x,0)=f(x), \qquad x \text{ in } I, \tag{41a}$$

$$\frac{\partial y(x,0)}{\partial t}=y_t(x,0)^\dagger=g(x), \qquad x \text{ in } I, \tag{41b}$$

where I is the interval over which the string is stretched. In Figure 1.11, I is $0 < x < L$, but other intervals are also possible. Interval I also dictates the number of boundary conditions. There are three possibilities, depending upon whether the string is of finite

† Subscripts are often used to denote partial derivatives. In (41b), y_t denotes $\partial y/\partial t$. In a similar way, we may use the notation y_{tt} in place of $\partial^2 y/\partial t^2$.

length, of "semi-infinite" length, or of "infinite" length. If the string is of finite length, the interval I is customarily taken as $0 < x < L$ and two boundary conditions result, one at each end. The string is said to be of semi-infinite length, or the problem is semi-infinite, if the string has only one end that satisfies some prescribed condition. The interval I in this case is always chosen as $0 < x < \infty$, and the one boundary condition is at $x = 0$. The string is said to be of infinite length, or the problem is infinite, if the string has no ends. In this case interval I becomes $-\infty < x < \infty$ and there are no boundary conditions.

It might be argued that there is no such thing as a semi-infinitely long or infinitely long string, and we must agree. There are, however, situations in which the model of a semi-infinite or infinite string is definitely advantageous. For example, suppose a fairly long string (with ends at $x = 0$ and $x = L$) is initially at rest along the x-axis. Suddenly, something disturbs the string at its midpoint, $x = L/2$ (perhaps it is struck by an object). The effect of this disturbance travels along the string in both directions toward $x = 0$ and $x = L$. Before the disturbance reaches $x = 0$ and $x = L$, the string reacts exactly as if it had no ends whatsoever. If we are interested only in these initial disturbances, and consideration of the "infinite" problem provides straightforward explanations, it is an advantage to analyze the "infinite" problem rather than the finite one.

We consider only three types of boundary conditions at an end of the string—Dirichlet, Neumann, and Robin. When the string has an end at $x = 0$, a Dirichlet boundary condition takes the form

$$y(0, t) = f_1(t), \qquad t > 0. \tag{42a}$$

It states that the end $x = 0$ of the string is caused by some external mechanism to perform the vertical motion described by $f_1(t)$. Similarly, if the string has an end at $x = L$, a Dirichlet condition

$$y(L, t) = f_2(t), \qquad t > 0 \tag{42b}$$

indicates that this end has a vertical displacement described by $f_2(t)$. For the string in Figure 1.11, $f_1(t) = f_2(t) = 0$.

Instead of prescribing the motion of the end $x = 0$ of the string, suppose that this end is attached to a mass m (Figure 1.13) and, furthermore, that motion of the mass is restricted to be vertical by a containing tube. The vertical component of the tension of the string acting on m at $x = 0$ is $\tau(0, t)\sin\theta$, which for small slopes can be approximated by

$$\tau(0, t)\sin\theta \approx \tau(0, t)\tan\theta = \tau(0, t)\frac{\partial y(0, t)}{\partial x}. \tag{43}$$

Consequently, when Newton's second law is applied to the motion of m,

$$m\frac{\partial^2 y(0, t)}{\partial t^2} = \tau(0, t)\frac{\partial y(0, t)}{\partial x} + f_1(t), \qquad t > 0, \tag{44}$$

where $f_1(t)$ represents the y-component of all other forces acting on m.

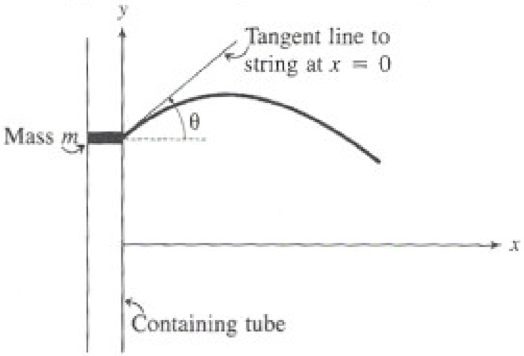
If m is sufficiently small that it may be taken as negligible (for instance, as with a very light loop around a vertical rod), this equation takes the form

$$\frac{\partial y(0,t)}{\partial x} = -\frac{1}{\tau(0,t)} f_1(t), \qquad t > 0, \tag{45}$$

a Neumann boundary condition. In particular, if the massless end of the string is free to slide vertically with no forces acting on it except tension in the string, it satisfies a homogeneous Neumann condition

$$\frac{\partial y(0,t)}{\partial x} = 0. \tag{46}$$

What this equation says is that when the end of a taut string is free of external forces, the slope of the string there will always be zero.



Similarly, if the string has a massless end at $x = L$ that is subjected to a vertical force with component $f_2(t)$, the boundary condition there is once again Neumann:

$$\frac{\partial y(L,t)}{\partial x} = \frac{1}{\tau(L,t)} f_2(t), \qquad t > 0. \tag{47}$$

What we have shown, then, is that Neumann boundary conditions result when the ends of the string, taken as massless, move vertically under the influence of forces that are specified as functions of time.

Robin boundary conditions, which are linear combinations of Dirichlet and Neumann conditions, arise when the ends of the string are attached to springs that are unstretched on the x-axis (Figure 1.14). When this is the case at $x = 0$, equation (44) becomes

$$m\frac{\partial^2 y(0,t)}{\partial t^2} = \tau(0,t)\frac{\partial y(0,t)}{\partial x} - ky(0,t) + f_1(t), \tag{48}$$

where $f_1(t)$ now represents all external forces acting on m other than the spring and tension in the string. For a massless end ($m = 0$) and constant tension τ, (48) takes the form

$$-\tau\frac{\partial y}{\partial x} + ky = f_1(t), \qquad x = 0, \qquad t > 0. \tag{49a}$$

Similarly, attaching the end $x = L$ to a spring gives the Robin condition

$$\tau\frac{\partial y}{\partial x} + ky = f_2(t), \qquad x = L, \qquad t > 0. \tag{49b}$$