Fourier Series in Complex notation

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$
$$= -\frac{i}{2}\left(e^{ix} - e^{-ix}\right)$$

So

$$\begin{aligned}
\varphi(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right) \qquad -l < x < l \\
&= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2} - \frac{i}{2} B_n \left(e^{in\pi x/l} - e^{-in\pi x/l}\right) \\
&= \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{A_n - iB_n}{2} e^{in\pi x/l} + \frac{A_n + iB_n}{2} e^{-in\pi x/l} \\
&= \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \qquad\qquad C_n = \begin{cases} \frac{A_n - iB_n}{2} & n > 0 \\ \frac{A_{-n} + iB_{-n}}{2} & n < 0 \\ \frac{A_0}{2} & n = 0 \end{cases}
\end{aligned}$$

So $\quad C_{-n} = C_n^*$

$$\int_{-l}^{l} e^{\frac{in\pi x}{l}} e^{-\frac{im\pi x}{l}} dx dx = \begin{cases} 0 & m \neq n \\ 2l & m = n \end{cases}$$
$$C_n = \frac{1}{2l} \int_{-l}^{l} \varphi(x) e^{-\frac{in\pi x}{l}} dx$$

Sturm-Liouville

$$(*) \quad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = \lambda m(x)u(x) \qquad a < x < b$$
$$p(x) > 0 \quad q(x) \geq 0 \quad m(x) > 0$$

and all quantities are real.

Definition 1 *λ is the eigenvalue and u is the eigenfunction*

Definition 2 *A homogeneous boundary condition is* ***symmetric*** *if*

$$p\left[fg' - f'g\right]_a^b = 0$$

Examples:

- Dirichlet
- Neumann
- periodic
- Robin (?)

$$a_l u(a) + b_l \frac{du}{dx}(a) = 0$$
$$a_r u(b) + b_r \frac{du}{dx}(b) = 0$$

then

$$p\left[fg' - f'g\right]_a^b = f(a)\left[-\frac{a_l}{b_l}g(a)\right] - \left(-\frac{a_l}{b_l}\right)f(a)g(a) = 0$$

Definition 3 *Inner product*

$$(u, v) = \int_a^b m(x)u(x)v(x)dx \qquad m > 0$$

Example

$$p = 1 \quad q = 0 \quad m = 1$$

Then with Dirichlet conditions we have

$$-\frac{d^2u}{dx^2} = \lambda u(x) \qquad 0 < x < l$$
$$u(0) = u(l) = 0$$

If $\lambda > 0$ then

$$u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Using $u(0) = 0$ we get

$$u(x) = B\sin(\sqrt{\lambda}x)$$

and using $u(l) = 0$

$$\lambda = \left(\frac{n\pi}{l}\right)^2$$
$$u(x) = B\sin(\frac{n\pi}{l}x)$$

So we have an infinite number of eigenvalues/eigenfunctions.

Green's Identities

First Identity:

$$\int_a^b \frac{d}{dx}\left(p(x)\frac{du}{dx}\right)v(x)dx + \int_a^b p(x)\frac{du}{dx}\frac{dv}{dx}dx = \left[p(x)\frac{du}{dx}v\right]_a^b$$

Second Identity:

$$\int_a^b -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)v(x)dx + \int_a^b \frac{d}{dx}\left(p(x)\frac{dv}{dx}\right)u(x)dx = \left[p(x)\left(-\frac{du}{dx}v + \frac{dv}{dx}u\right)\right]_a^b$$

In multi-dimensions this generalizes to $(p = 1)$

$$\iiint_D (\nabla u \cdot \nabla v + v\Delta u)\, dV = \iint_{\partial D} v\frac{\partial u}{\partial n}dS$$
$$\iiint_D (v\Delta u - u\Delta v)\, dV = \iint_{\partial D} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS$$

Proof. From the divergence theorem

$$\begin{aligned}
\iiint \operatorname{div}(F) dV &= \iint F \cdot n dS \\
\operatorname{div}(v \text{ grad } u) &= \nabla u \cdot \nabla v + v \Delta u \\
\text{So} \iiint (\nabla u \cdot \nabla v + v \Delta u)\, dV &= \iiint \operatorname{div}(v \text{ grad } u) dV \\
&= \iint v \text{ grad } u \cdot n dS = \iint v \frac{\partial u}{\partial n} dS
\end{aligned}$$

■

Interchange u and v and subtract to get the second identity.

From (*)

$$
\begin{aligned}
&(1) \quad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = \lambda_1 m(x)u(x) \\
&(2) \quad -\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v(x) = \lambda_2 m(x)v(x)
\end{aligned}
$$

Multiply (1) by v and (2) by u and subtract and integrate

$$
\int_a^b \left[-v(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + u(x)\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right)\right] dx = (\lambda_1 - \lambda_2)\int_a^b m(x)u(x)v(x)dx
$$

Integrate by parts (Green's theorem in multidimensions)

$$
\begin{aligned}
\int_a^b \left[\frac{dv}{dx}\left(p(x)\frac{du}{dx}\right) - \frac{du}{dx}\left(p(x)\frac{dv}{dx}\right)\right] dx + \left[p\left(\frac{du}{dx}v - \frac{dv}{dx}u\right)\right]_a^b &= (\lambda_1 - \lambda_2)\int_a^b m(x)u(x)v(x)dx \\
\left[p\left(\frac{du}{dx}v - \frac{dv}{dx}u\right)\right]_a^b &= (\lambda_1 - \lambda_2)\int_a^b m(x)u(x)v(x)dx
\end{aligned}
$$

If the boundary conditions are symmetric then

$$
(\lambda_1 - \lambda_2)\int_a^b m(x)u(x)v(x)dx = 0
$$

Hence, if $\lambda_1 \neq \lambda_2$

Theorem 4 *For symmetric boundary conditions , if $\lambda_1 \neq \lambda_2$ then $(u, v) = 0$*
If $\lambda_1 = \lambda_2$ then we have a subspace and we can choose an orthogonal basis. This is done by Gram Schmidt

Gram-Schmidt

If $\{\psi_k(x)\}$ is a linearly independent basis then we can construct an orthonormal basis that spans the same space.

- $\varphi_1(x) = \psi_1(x)$
- $\varphi_2(x) = \psi_2(x) - \frac{(\psi_2,\psi_1)}{(\varphi_1,\varphi_1)}\varphi_1$
  Then $(\varphi_2, \varphi_1) = (\psi_2, \varphi_1) - \frac{(\psi_2,\varphi_1)}{(\varphi_1,\varphi_1)}(\varphi_1, \varphi_1) = 0$
- $\varphi_k(x) = \psi_k(x) - \sum\limits_{j=1}^{k-1} \frac{(\psi_k,\varphi_j)}{(\varphi_j,\varphi_j)}\varphi_j$

Hence, we can consider all the solutions of (*) to be orthogonal to each other. So consider a sequence of orthogonal solutions of (*) $\{\varphi_k(x)\}$ .Then if

$$\begin{aligned} f(x) &= \sum_n a_n \varphi_n(x) \\ (f, \varphi_m) &= \sum_n a_n (\varphi_n, \varphi_m) \\ &= a_m (\varphi_m, \varphi_m) \end{aligned}$$

So

$$a_m = \frac{(f, \varphi_m)}{(\varphi_m, \varphi_m)} = \frac{\int_a^b m(x) f(x) \varphi_m(x) dx}{\int_a^b m(x) \varphi_m^2(x) dx}$$

Theorem 5 *If p,q,m are real and the boundary conditions are symmetric then there are no complex eigenvalues*

Proof.

$$(1) \quad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = \lambda m(x)u(x)$$

$$(2) \quad -\frac{d}{dx}\left(p(x)\frac{d\overline{u}}{dx}\right) + q(x)\overline{u}(x) = \overline{\lambda} m(x)\overline{u}(x)$$

As before multiply first equation by $\overline{u}$, the second by u, subtract and integrate. Then

$$\int_a^b \left[-\overline{u}(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + u(x)\frac{d}{dx}\left(p(x)\frac{d\overline{u}}{dx}\right)\right] dx = \left(\lambda - \overline{\lambda}\right) \int_a^b m(x)u(x)\overline{u}(x)dx$$

Again integrate by parts and use the symmetry of the boundary conditions.

$$\left(\lambda - \overline{\lambda}\right) \int_a^b m(x)|u(x)|^2 dx = 0$$

So $\lambda = \overline{\lambda}$ i.e. λ is real

■

If u is complex then its real and imaginary components are solutions.

Negative Eigenvalues

$$(*) \qquad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = \lambda m(x)u(x) \qquad a < x < b$$

By Green's first identity we have for all v

$$\int_a^b \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) v(x)dx = -\int_a^b p(x)\frac{du}{dx}\frac{dv}{dx}dx + \left[p(x)\frac{du}{dx}v\right]_a^b$$

Choose $v = u$ and assume symmetric boundary conditions. Then

$$\int_a^b u(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) dx = -\int_a^b p(x)\left(\frac{du}{dx}\right)^2 dx \leq 0$$

Using the ODE we get

$$\int_a^b u(x)\left[q(x)u(x) - \lambda m(x)u(x)\right] dx = -\int_a^b p(x)\left(\frac{du}{dx}\right)^2 dx$$

So

$$\lambda \int_a^b m(x)u^2(x)dx = \int_a^b q(x)u^2(x)dx + \int_a^b p(x)\left(\frac{du}{dx}\right)^2 dx$$

$$\lambda = \frac{\int_a^b q(x)u^2(x)dx + \int_a^b p(x)\left(\frac{du}{dx}\right)^2 dx}{\int_a^b m(x)u^2(x)dx} \geq 0$$

We can have equality only if $q(x) = 0$ and $\frac{du}{dx} = 0$

Note: All these proofs work equally well in multidimensions using Green's theorem instead of integration by parts.

Completeness

Theorem 6 *There are an infinite number of eigenvalues for (*) and* $\lambda_n \to \infty$. *Furthermore*

$$f(x) = \sum c_n \varphi_n(x)$$
$$c_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}$$

Convergence:

$$\sum_{n=1}^{\infty} \varphi_n(x) \stackrel{?}{\to} f(x)$$

Definition 7 *Pointwise Convergence:*

$$\lim_{N\to\infty} \left| f(x) - \sum_{n=1}^{N} \varphi_n(x) \right| = 0 \qquad \text{for every } x$$

Definition 8 *Uniform Convergence:*

$$\lim_{N\to\infty} \max_{a \le x \le b} \left| f(x) - \sum_{n=1}^{N} \varphi_n(x) \right| = 0 \qquad \text{for every } x$$

Definition 9 L_2 *(root mean square)*

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} \varphi_n(x) \right|^2 dx = 0$$

Uniform convergence implies pointwise convergence.
Uniform convergence implies root mean square convergence.

Examples

- $$f(x) = x^n \qquad 0 \le x \le 1$$

  Then

  $$x_n \to \begin{cases} 0 & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

-

$$f_n(x) = (1-x)x^{n-1} = x^{n-1} - x^n$$
$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} (x^{n-1} - x^n) = 1 - x^N \to 1 \qquad \text{as } N \to \infty$$

so we have pointwise convergence.

However
$$\max_{0\le x\le 1} \left|1 - \left(1 - x^N\right)\right| = \max_{0\le x\le 1} \left|x^N\right| = 1 \neq 0$$

So we don't have uniform convergence.

For L_2 we have
$$\int_0^1 \left|x^N\right|^2 dx = \frac{1}{2N+1} \to 0$$

So we have L_2 convergence.

Theorem 10 *If*

- f, f', f'' *exist and are continuous in* $a \leq x \leq b$ *i.e.* $f \epsilon C^2[a,b]$
- f *satisfies the boundary conditions*

then

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \qquad \textit{converges uniformly}$$

Theorem 11 *If* $\int_a^b f^2(x)dx < \infty$

then $f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$ *converges in* L_2

Theorem 12 *For sine and cosine series only.*

If

- f *is continuous on* $a \leq x \leq b$
- $f^{'}$ *is piecewise continuous*

  Then the series converges pointwise. If f *and* $f^{'}$ *are piecewise continuous then*

$$\sum a_n \varphi_n(x) \rightarrow \frac{f(x+) + f(x-)}{2}$$

Theorem 13 *Integration: If formally*

$$f(x) \leftrightarrow \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right) \qquad \textbf{\textit{not necessarily convergent}}$$

Then

$$\int_{-l}^{x} f(y)dy = \frac{A_0}{2}(x+l) + \frac{l}{\pi} \sum_{n=1}^{\infty} \left[\frac{A_n}{n} \sin\left(\frac{n\pi t}{l}\right) - \frac{B_n}{n} \cos\left(\frac{n\pi t}{l}\right)\right]_{-l}^{x} \qquad \textit{is convergent}$$

We note that for differentiation it is the opposite i.e. the derivative of a convergent series may not converge.
Example: expanding x in a sine series we have

$$x = 2\sum_{n=1}^{\infty} \frac{l}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{l}\right) \quad 0 \leq x \leq l$$

Differentiating we get

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos\left(\frac{n\pi x}{l}\right) \quad 0 \leq x \leq l$$

This is certainly **NOT** the cosine series of 1 which is just 1 . In fact this series does not converge!

We begin with the proof of convergence in least squares. Restating the theorem we have

Theorem 14 *If φ_n are the eigenfunctions of a Sturm-Liouville problem with symmetric boundary conditions and $||f|| < \infty$. Then*

$$||f - \sum_{n=1}^{N} a_n \varphi_n|| \to 0 \qquad a_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}$$

Theorem 15 *Let φ_n be an orthogonal set and $||f|| < \infty$, Then the choice of constants c_n that minimizes $||f - \sum_{n \le N} c_n \varphi_n||$ is $c_n = a_n$*

Proof. Assume for simplicity that all quantities are real

$$\begin{aligned} E_N &= ||f - \sum_{n \le N} c_n \varphi_n||^2 = \int |f(x) - \sum_{n \le N} c_n \varphi_n(x)|^2 dx \\ &= \int |f(x)|^2 - 2 \sum_{n \le N} c_n \int f(x) \varphi_n(x) dx + \sum_n \sum_m c_n c_m \int \varphi_n(x) \varphi_m(x) dx \\ &= ||f||^2 - 2 \sum_{n \le N} c_n (f, \varphi_n) + \sum_{n \le N} c_n^2 ||\varphi_n(x)||^2 \\ &= ||f||^2 + \sum_{n \le N} ||\varphi_n(x)||^2 \left[c_n - \frac{(f, \varphi_n)}{||\varphi_n(x)||^2} \right]^2 - \sum_{n \le N} \frac{(f, \varphi_n)^2}{||\varphi_n(x)||^2} \end{aligned}$$

To minimize we can only "play" with c_n . Since the middle term is positive we minimize E_N if

$$c_n = \frac{(f, \varphi_n)}{||\varphi_n(x)||^2} = a_n$$

Then

$$E_N = ||f||^2 - \sum_{n \le N} \frac{(f, \varphi_n)^2}{||\varphi_n(x)||^2} = ||f||^2 - \sum_{n \le N} A_n^2 ||\varphi_n(x)||^2 \ge 0$$

So we have Bessel's inequality. If $||f||^2 < \infty$ then

$$\sum_{n \le N} \frac{(f, \varphi_n)^2}{||\varphi_n(x)||^2} \le ||f||^2$$

■

Parseval's Equality

Theorem 16 *The Fourier Series converges to $f(x)$ in L_2 if and only if*

$$\sum_{n \le N} \frac{(f, \varphi_n)^2}{||\varphi_n(x)||^2} = \sum_{n \le N} a_n^2 ||\varphi_n(x)||^2 = ||f||^2$$

Definition 17 *A sequence $\{\varphi_n(x)\}$ is complete if Parseval's equality holds whenever $||f||^2 < \infty$*

Riemann-Lebesque Theorem

Theorem 18 *If (a)* $f \in C^1$
or (b) $||f||_{L^2} < \infty$
Then

$$\lim_{n\to\infty} \int_{-l}^{l} f(x) \begin{cases} \sin(\frac{n\pi x}{l}) \\ \cos(\frac{n\pi x}{l}) \end{cases} dx = 0$$

Proof.

1. integration by parts
2. In a Fourier series $B_n = \int_{-l}^{l} f(x)\sin(\frac{n\pi x}{l})dx$ by Bessel's inequality $B_n \to 0$

■

Example

Consider $f(x) = 1$ on $(0, \pi)$. We find that

$$1 = \sum_{\text{n odd}} \frac{4}{n\pi}\sin(nx)$$

So by Parseval's equality

$$\int_0^{\pi} 1^2 dx = \sum_{\text{n odd}} \left(\frac{4}{n\pi}\right)^2 \frac{\pi}{2}$$
$$\sum_{\text{n odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Pointwise Convergence

If

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$
$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy \qquad n = 0, 1, 2, 3...$$
$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy \qquad n = 1, 2, 3...$$

Dirichlet kernel

Consider the partial sum

$$\begin{aligned}
S_N &= \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n=1}^{N} (\cos(nx)\cos(ny) + \sin(nx)\sin(ny)) \right] f(y)dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n=1}^{N} \cos(nx - ny) \right] f(y)dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-y) f(y)dy
\end{aligned}$$

where

$$K_N(\theta) = 1 + 2 \sum_{n=1}^{N} \cos(n\theta) = \frac{\sin(N + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})}$$

Proof. Use $\cos(\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2}$ and get geometric series. ■

Now let $\theta = y - x$. Then

$$\begin{aligned}
S_N &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta) f(x+\theta) d\theta \\
S_N(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta) \left[f(x+\theta) - f(x) \right] d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \sin(N + \frac{1}{2})\theta \, d\theta \qquad \text{where } g(\theta) = \frac{f(x+\theta) - f(x)}{\sin(\frac{\theta}{2})}
\end{aligned}$$

Let $\phi_n(\theta) = \sin(N + \frac{1}{2})\theta$. By Bessel's inequality we have

$$\sum_{n=1}^{\infty} \frac{|(g, \phi_n)|}{||\phi_n||^2} = \frac{1}{\pi} \sum_{n=1}^{\infty} |(g, \phi_n)| \leq ||g||^2 = \int_{-\pi}^{\pi} \frac{[f(x+\theta) - f(x)]^2}{\sin^2(\frac{\theta}{2})} d\theta$$

By L'hopital's rule the integrand is finite at $\theta = 0$. Hence it is bounded everywhere and the integral exists. Since the sum converges each term much approach zero and so

$$|(g, \phi_n)| = \int_{-\pi}^{\pi} g(\theta) \sin(N + \frac{1}{2})\theta \, d\theta \to 0$$

Gibbs Phenomena

If instead we are interested in uniform convergence we need to analyze

$$\lim_{N\to\infty} \max_x |S_N(x) - f(x)|$$

One can show that if the function $f(x)$ has a discontinuity at $x = x_0$ then in fact this limit is nonzero and is about 9% of the size of the jump on either side.