Inverse Problems

Fredholm Integral of the first kind

$$g(x,y) = \int_0^1 \int_0^1 k(x-x', y-y') f(x',y') dx'dy' = Kf(x,y)$$

Fourier Transform

$$\widehat{g} = \widehat{K}\widehat{f}$$
$$\widehat{f} = \widehat{K}^{-1}\widehat{g}$$

So

$$f = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}(g)}{F(K)}\right\}$$

example: atmospheric turbulence

$$k(x,y) = Ce^{-\frac{x^2+y^2}{2\gamma^2}}$$

telescope

$$k(x,y) = \left|\mathcal{F}^{-1}\left\{Ae^{i\varphi}\right\}\right|^2$$

A=aperture φ is the phase

Given k and g find x
Problems:

- K is ill=conditioned
- Noise

Definition 1 *Well posed:*

$$(*) \qquad Kf = g \qquad L : H_1 \to H_2$$

1. *for each $g \in H_2$ there exists $f \in H_1$ where $(*)$ holds*
2. *f is unique*
3. *Stability*
   If $Kf_* = g_*$
   $Kf = g$
   and $g \to g_*$ then $f \to f_*$

Tikhonov Regularization

- replace (∗ by least square minimization. This is equivalent to

$$K^* K f = K^* g$$

- Regularize

$$(K^* K + \alpha I)\, f = K^* g$$

  So

$$f_\alpha = (K^* K + \alpha I)^{-1} K^* g$$

- iterate regularization

More generally we consider

$$f_\alpha = \arg\min ||Kf - g||^2 + \underbrace{\alpha ||Lf||^2}_{\text{penalty function}}$$

Even more general

$$f_\alpha = \arg\min ||Kf - g||^2 + \rho\,(f)$$

So the penalty function is given by

$$\rho(f)$$

For example

$$\rho(f) = ||Lf||^2 \qquad \text{in some norm}$$

e.g.

$$\rho(f) = \frac{1}{2} \iint \sum_{i=1}^{d} \left(\frac{\partial f}{\partial x_i} \right)^2 dx_i$$

This penalizes non-smooth functions. Integrate by parts

$$\int \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} dx = -\int f \frac{\partial^2 f}{\partial x^2} dx + \text{boundary terms}$$

Let $Lf = -\Delta f$. Then

$$\rho(f) = \frac{1}{2} \iint f\, Lf\, dx$$