Tomography

A Lecture for:
Inverse Problems Seminar

Asaf Zarmi

Dec. 2009

- Introduction
- The Radon Transform
- Projection Slice Theorem
- Inversion of Radon Transform
  - Theory
  - Aplication
- Cone Beam Transform
- Various problems in tomography

What is Tomography?

- **Wikipedia:**

  Tomography is imaging by sections or sectioning, through the use of wave of energy.

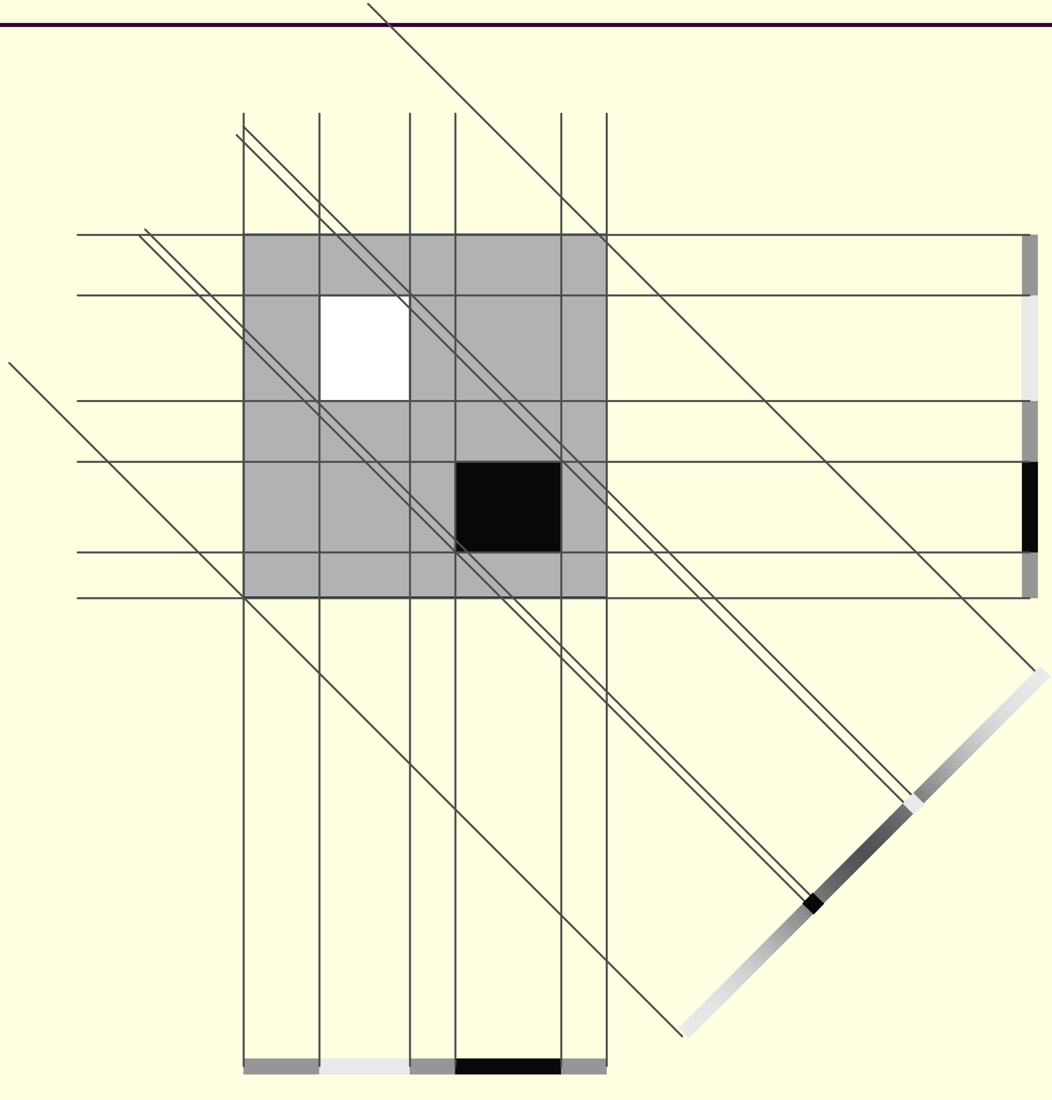
  A device used in tomography is called a **tomograph**, while the image produced is a **tomogram**.

  The method is used in medicine, archaeology, biology, geophysics, oceanography, materials science, astrophysics and other sciences.

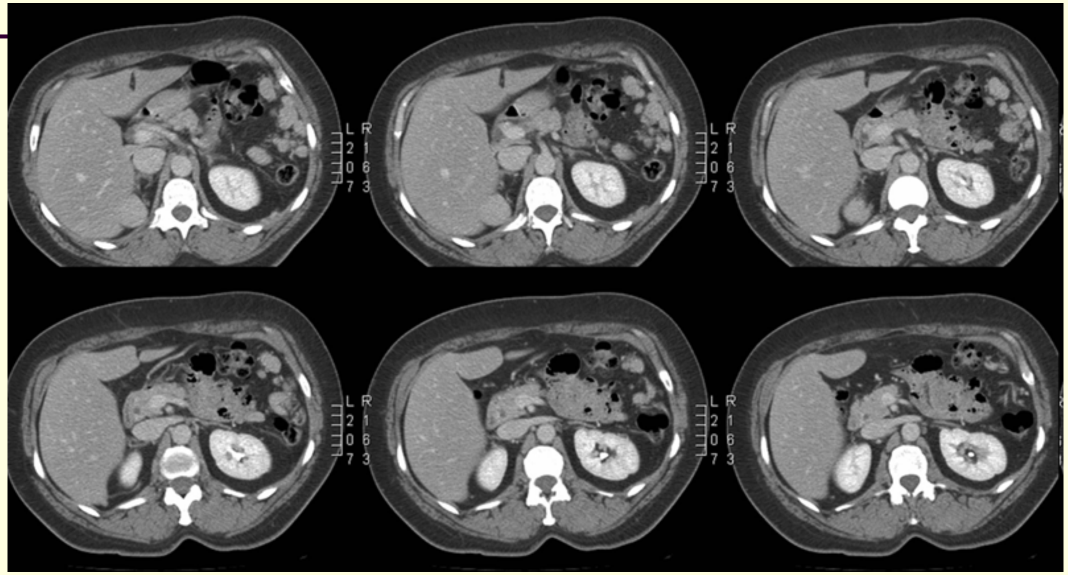
τομοσ /*τόμος* - slice/section/cutting

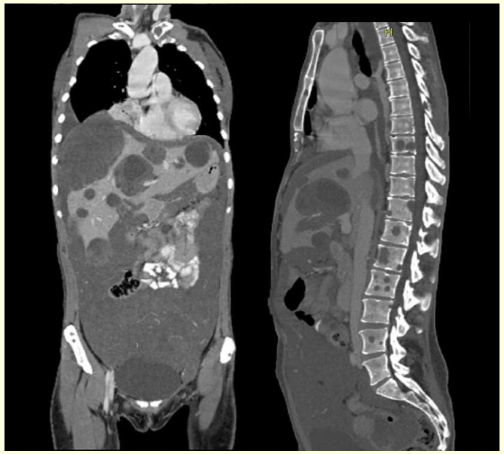
- A. Cormack and G. Hounseld built the first computed tomography scanners in 1960s, won 1979 Nobel prize in medicine.

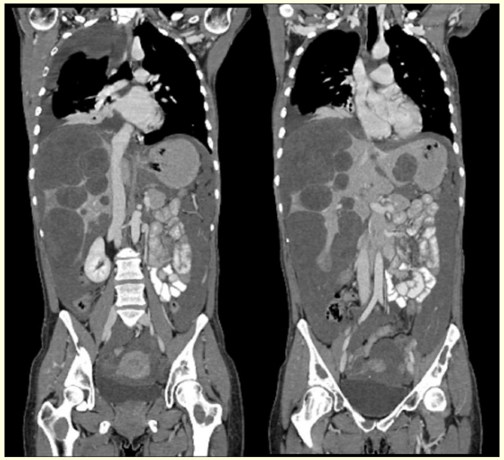
How is it done?



תמונות







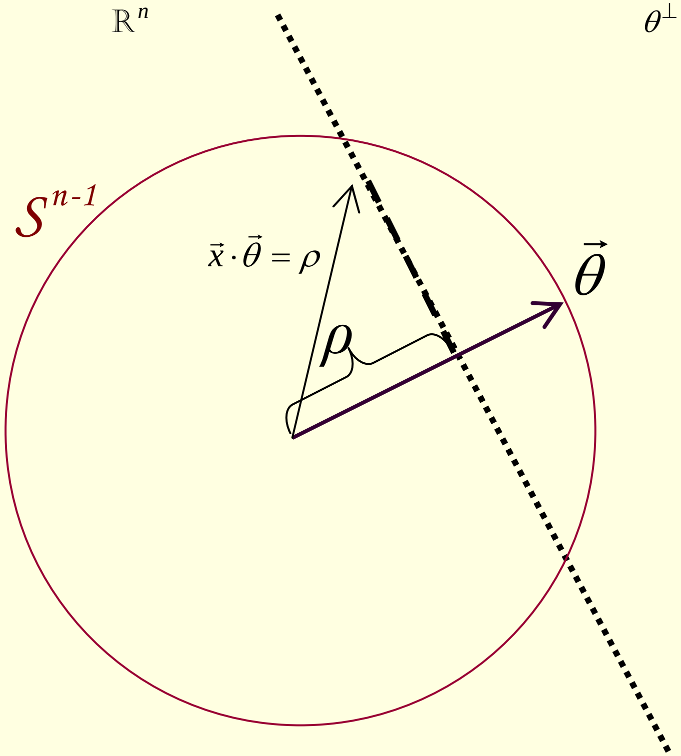
- CT – Computerized Tomography
- CAT – Comp. Axial Tomo.
- SPECT – Single Particle Emission Tomo.
- PET – Positrion Emission Tomo.
- MRI – Magnetic Resonance Tomo.
- Optical Tomo.
- Thermal Tomo.
- Acoustic Tomo.

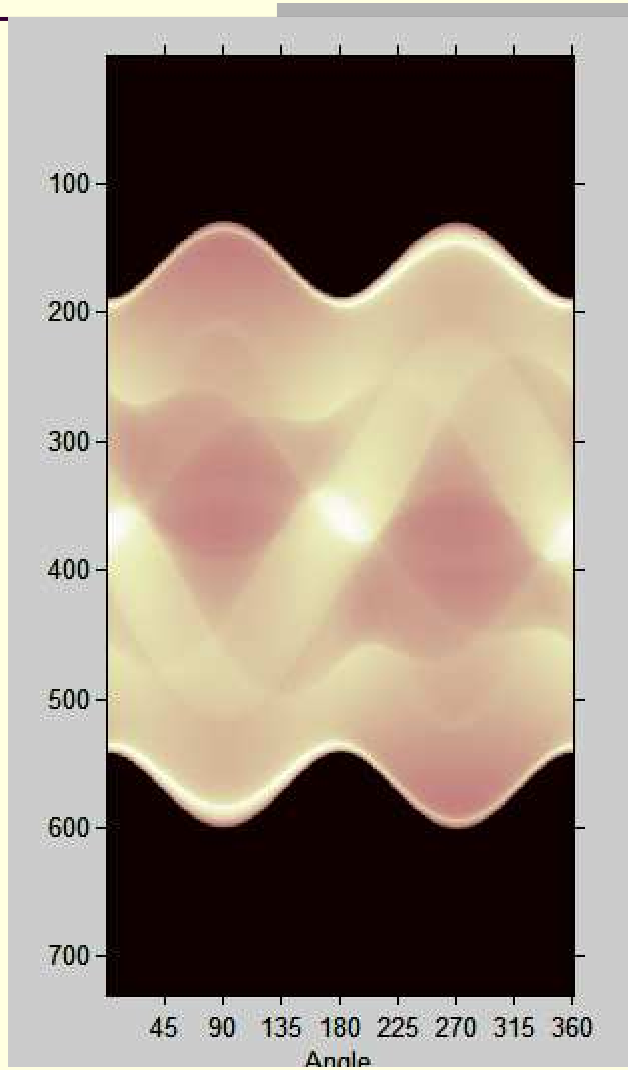
Radon Transform

$f:\mathbb{R}^n \to \mathbb{R}, \quad \vec{\theta} \in S^{n-1}, \quad \rho \in \mathbb{R}$

$\mathcal{R}\left[f(\vec{\cdot})\right]\left(\vec{\theta},\rho\right) = \widetilde{f(\vec{\cdot})}\left(\vec{\theta},\rho\right) = \tilde{f}\left(\vec{\theta},\rho\right)$

$:= \int\limits_{\vec{x}\cdot\vec{\theta}=\rho} f(\vec{x})\,d\vec{x} = \int\limits_{\mathbb{R}^n} \delta\left(\vec{x}\cdot\vec{\theta}-\rho\right) f(\vec{x})\,d\vec{x} = \int\limits_{\theta^{\perp}} f\left(\rho\vec{\theta}+\vec{y}\right)d\vec{y}$




100
200
300
400
500
600
700
45
90
135
180
225
270
315
360
Angle

Radon Transform in Freq. Domain Projection Slice Thm.

$$\mathcal{F}\left[\tilde{f}\left(\vec{\theta},\cdot\right)\right](\omega)=\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=\hat{\tilde{f}}\left(\vec{\theta},\omega\right)$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\rho\omega}\tilde{f}\left(\vec{\theta},\rho\right)d\rho=$$

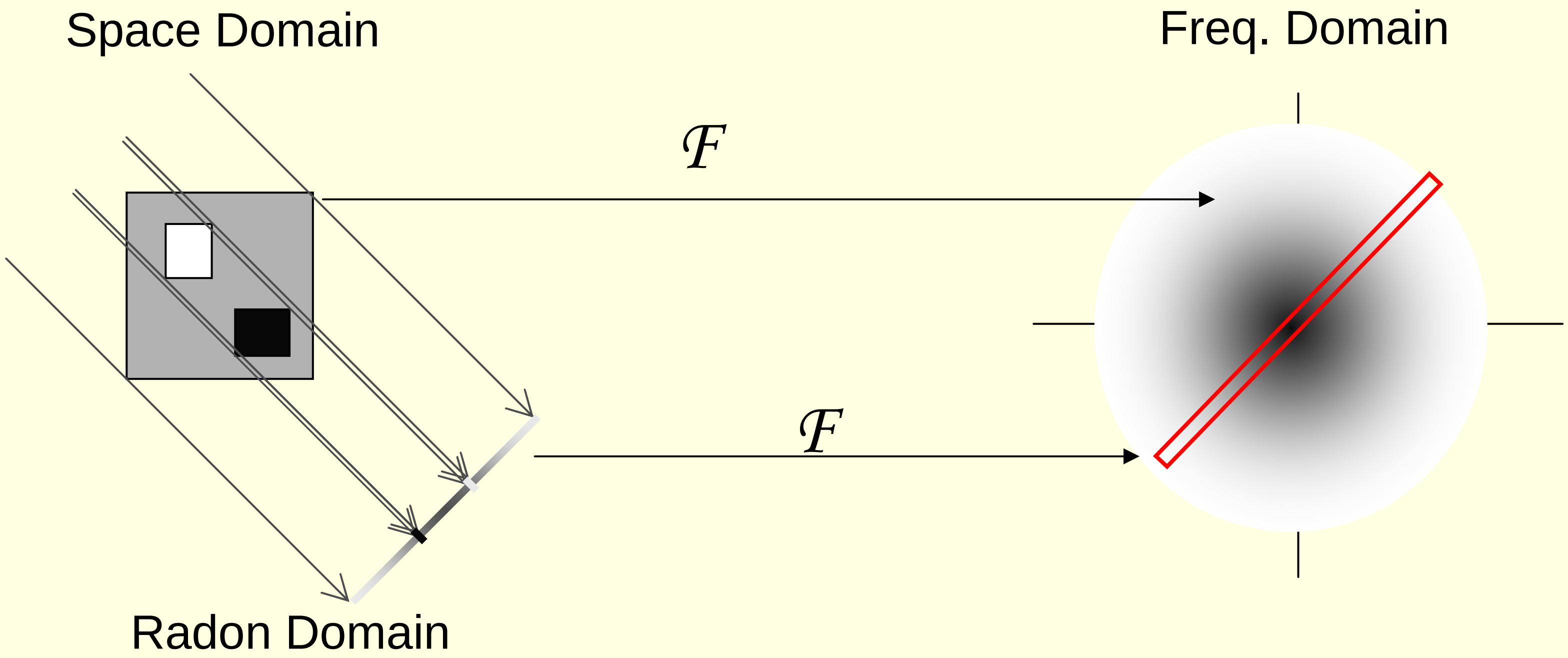
$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\rho\omega}\int_{\vec{x}\cdot\vec{\theta}=\rho} f\left(\vec{x}\right)d\vec{x}\,d\rho=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\vec{\theta}\cdot\vec{x}=\vec{\rho}} e^{-i\omega\rho}f\left(\vec{x}\right)d\vec{x}\,d\rho$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}^n} e^{-i\omega\vec{\theta}\cdot\vec{x}}f\left(\vec{x}\right)d\vec{x}=\left(2\pi\right)^{\frac{n-1}{2}}\widehat{f\left(\vec{\cdot}\right)}\left(\omega\vec{\theta}\right)$$

$$\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=\left(2\pi\right)^{\frac{n-1}{2}}\widehat{f\left(\vec{\cdot}\right)}\left(\omega\vec{\theta}\right)$$

Radon Transform in Freq. Domain
Projection Slice Thm.

$$\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=(2\pi)^{\frac{n-1}{2}}\,\widehat{f\left(\vec{\cdot}\right)}\left(\omega\vec{\theta}\right)$$



The Backprojection Operator

- $\mathcal{R}$ was defined on C^n. It can be extended to L_2 (as a bounded operator). Thus there exist an adjoint:

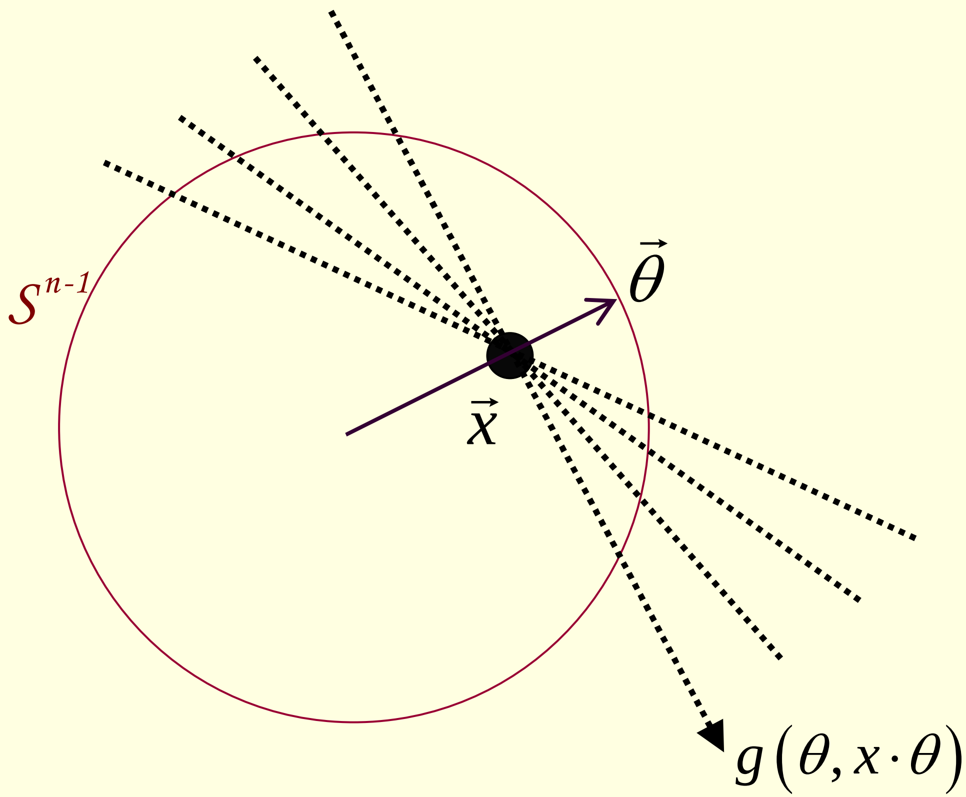
$$\mathcal{R} : L_2\left(\mathbb{R}^n\right) \to L_2\left(S^{n-1} \times \mathbb{R}\right)$$

$$\mathcal{R}^* : L_2\left(S^{n-1} \times \mathbb{R}\right) \to L_2\left(\mathbb{R}^n\right)$$

$$\mathcal{R}^*\left[g\left(\vec{\cdot},\cdot\cdot\right)\right](\vec{x}) = \int_{S^{n-1}} g\left(\vec{\theta}, \vec{x}\cdot\vec{\theta}\right) d\vec{\theta}$$

The Backprojection Operator

$$\mathcal{R}^*\left[g\left(\vec{\cdot},\cdot\right)\right]\left(\vec{x}\right)=\int_{S^{n-1}} g\left(\vec{\theta},\vec{x}\cdot\vec{\theta}\right)d\vec{\theta}$$



So, does it work?

$$\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=\hat{\tilde{f}}\left(\vec{\theta},\omega\right)=(2\pi)^{\frac{n-1}{2}}\hat{f}\left(\omega\vec{\theta}\right)$$

$$\tilde{f}\left(\vec{\theta},\rho\right)=\mathcal{F}^{-1}\left[(2\pi)^{\frac{n-1}{2}}\hat{f}\left(\cdot\vec{\theta}\right)\right](\rho)=(2\pi)^{-\frac{n}{2}}\int_{-\infty}^{\infty}(2\pi)^{\frac{n-1}{2}}e^{i\rho\omega}\hat{f}\left(\omega\vec{\theta}\right)d\omega$$

$$\mathcal{R}^{*}\left[\tilde{f}(\vec{\cdot},\cdot\cdot)\right](\vec{x})=\mathcal{R}^{*}\left[(2\pi)^{-\frac{n}{2}}(2\pi)^{\frac{n-1}{2}}\int_{-\infty}^{\infty}e^{i(\cdot\cdot)\omega}\hat{f}(\omega\vec{\cdot})d\omega\right](\vec{x})=$$

$$\int_{S^{n-1}}\left((2\pi)^{\frac{n-1}{2}}(2\pi)^{-\frac{n}{2}}\int_{-\infty}^{\infty}e^{i\vec{x}\vec{\theta}\omega}\hat{f}\left(\omega\vec{\theta}\right)d\omega\right)d\vec{\theta}=$$

$$2(2\pi)^{\frac{n-1}{2}}(2\pi)^{-\frac{n}{2}}\int_{S^{n-1}}\int_{0}^{\infty}e^{i\vec{x}\vec{\theta}\omega}\frac{1}{\omega^{n-1}}\hat{f}\left(\omega\vec{\theta}\right)\omega^{n-1}d\omega d\vec{\theta}=2(2\pi)^{\frac{n-1}{2}}\mathcal{F}^{-1}\left[\frac{1}{\left|\vec{\cdot}\right|^{n-1}}\hat{f}(\vec{\cdot})\right](\vec{x})=$$

So, does it work?

$$\mathcal{R}^*\left[\tilde{f}(\vec{\cdot},\cdot\cdot)\right](\vec{x}) = 2\sqrt{2\pi}\left(\frac{1}{|\vec{x}|}\otimes f(\vec{x})\right)$$

Blurring!

Inversion Formula

$$f(\vec{x}) = \frac{\pi}{(2\pi)^n} \mathcal{R}^* \left[\mathcal{H}^{n-1} \left[\frac{\partial^{n-1}}{\partial(\cdots)^{n-1}} \tilde{f}(\vec{\cdot}, \cdots) \right] (\vec{\cdot}, \cdot\cdot) \right] (\vec{x})$$

$$\mathcal{H}[f(\cdot)](s) = \frac{1}{\pi} \int \frac{f(t)}{s-t} dt = f(t) \otimes \frac{1}{\pi t} \qquad \widehat{\mathcal{H}[f(\cdot)]}(\omega) = -i \operatorname{sgn}(\omega) \hat{f}(\omega)$$

$$\widehat{\mathcal{H}^{n-1} \left[\frac{\partial^{n-1}}{\partial(\cdot)^{n-1}} \tilde{f}(\vec{\theta}, \cdot) \right]}(\omega) = \left(-i \operatorname{sgn}(\omega)\right)^{n-1} \widehat{\frac{\partial^{n-1}}{\partial(\cdot)^{n-1}} \tilde{f}(\vec{\theta}, \cdot)}(\omega) =$$

$$= \left(-i \operatorname{sgn}(\omega)\right)^{n-1} (2\pi i \omega)^{n-1} \widehat{\tilde{f}(\vec{\theta}, \cdot)}(\omega) = (2\pi)^{n-1} \left(\omega \operatorname{sgn}(\omega)\right)^{n-1} \hat{\tilde{f}}(\vec{\theta}, \omega) = \left(2\pi |\omega|\right)^{n-1} \hat{\tilde{f}}(\vec{\theta}, \omega)$$

$$\mathcal{H}^{n-1} \left[\frac{\partial^{n-1}}{\partial(\cdot)^{n-1}} \tilde{f}(\vec{\theta}, \cdot) \right] (\vec{\theta}, s) = (2\pi)^{n-1} b(s) \otimes \tilde{f}(\vec{\theta}, s)$$

$$b(s) = \mathcal{F}^{-1} \left[|\cdot|^{n-1} \right](s) = \begin{cases} i^n \sqrt{\frac{2}{\pi}} \dfrac{\Gamma(n)}{s^n} & n \text{ even} \\ i^{n-1} \sqrt{2\pi} \delta^{(n-1)}(s) & n \text{ odd} \end{cases}$$

Inversion for 2D and 3D

$$f_{2D}(\vec{x}) = \frac{1}{4\pi^2} \int_{S^1} \int_{\mathbb{R}} \frac{\frac{\partial}{\partial \rho} \tilde{f}(\vec{\theta}, \rho)}{\vec{x} \cdot \vec{\theta} - \rho} \, d\rho \, d\vec{\theta}$$

$$f_{3D}(\vec{x}) = -\frac{1}{8\pi^2} \int_{S^2} \frac{\partial^2}{\partial \rho^2} \tilde{f}(\vec{\theta}, \vec{x} \cdot \vec{\theta}) \, d\vec{\theta}$$

- For 3D inversion, only integrals of plains through x_0 are needed to reconstruct $f(x_0)$
- For 2D inversion, the entire data of the Radon Transform is needed.

Reconstruction algorithms

- Filtered Backprojection
- Fourier methods

Filtered Backprojection

$$f(\vec{x}) = \frac{\pi}{(2\pi)^n} \mathcal{R}^* \left[\mathcal{H}^{n-1} \left[\frac{\partial^{n-1}}{\partial(\cdots)^{n-1}} \tilde{f}(\vec{\cdot}, \cdots) \right] (\vec{\cdot}, \cdot\cdot) \right] (\vec{x})$$

$$\mathcal{H}^{n-1} \left[\frac{\partial^{n-1}}{\partial(\cdot)^{n-1}} \tilde{f}(\vec{\theta}, \cdot) \right] (\vec{\theta}, s) = (2\pi)^{n-1} b(s) \otimes \tilde{f}(\vec{\theta}, s)$$

$$b(s) = \mathcal{F}^{-1} \left[|\cdot|^{n-1} \right] (s) = \begin{cases} i^n \sqrt{\frac{2}{\pi}} \dfrac{\Gamma(n)}{s^n} & n \text{ even} \\ i^{n-1} \sqrt{2\pi} \, \delta^{(n-1)}(s) & n \text{ odd} \end{cases}$$

Filtered Backprojection

$$\mathcal{R}^*\left[\tilde{g}(\vec{\cdot},\cdot\cdot)\right](\vec{x})\otimes f(\vec{x})=\mathcal{R}^*\left[\tilde{g}(\vec{\cdot},\cdot\cdot)\otimes_{\cdot\cdot}\tilde{f}(\vec{\cdot},\cdot\cdot)\right](\vec{x})$$

$$\mathcal{R}^*\left[\tilde{g}(\vec{\cdot},\cdot\cdot)\right](\vec{x})=2(2\pi)^{\frac{n-1}{2}}\mathcal{F}^{-1}\left[\frac{1}{|\vec{\cdot}|^{n-1}}\hat{g}(\vec{\cdot})\right](\vec{x})$$

$$\mathcal{R}^*\left[\tilde{g}(\vec{\cdot},\cdot\cdot)\right](\vec{x})\approx\delta(\vec{x})\;\Rightarrow$$

$$\Rightarrow\;2(2\pi)^{\frac{n-1}{2}}\frac{1}{|\vec{\omega}|^{n-1}}\hat{g}(\vec{\omega})\approx(2\pi)^{-\frac{n}{2}}\;\Rightarrow$$

$$\Rightarrow\;\hat{g}(\vec{\omega})\approx\frac{(2\pi)^{\frac{1}{2}-n}}{2}|\vec{\omega}|^{n-1}\Rightarrow\hat{g}(\vec{\omega})=\frac{(2\pi)^{\frac{1}{2}-n}}{2}|\vec{\omega}|^{n-1}\cdot low\_pass\_filter(\vec{\omega})$$

Fourier Methods

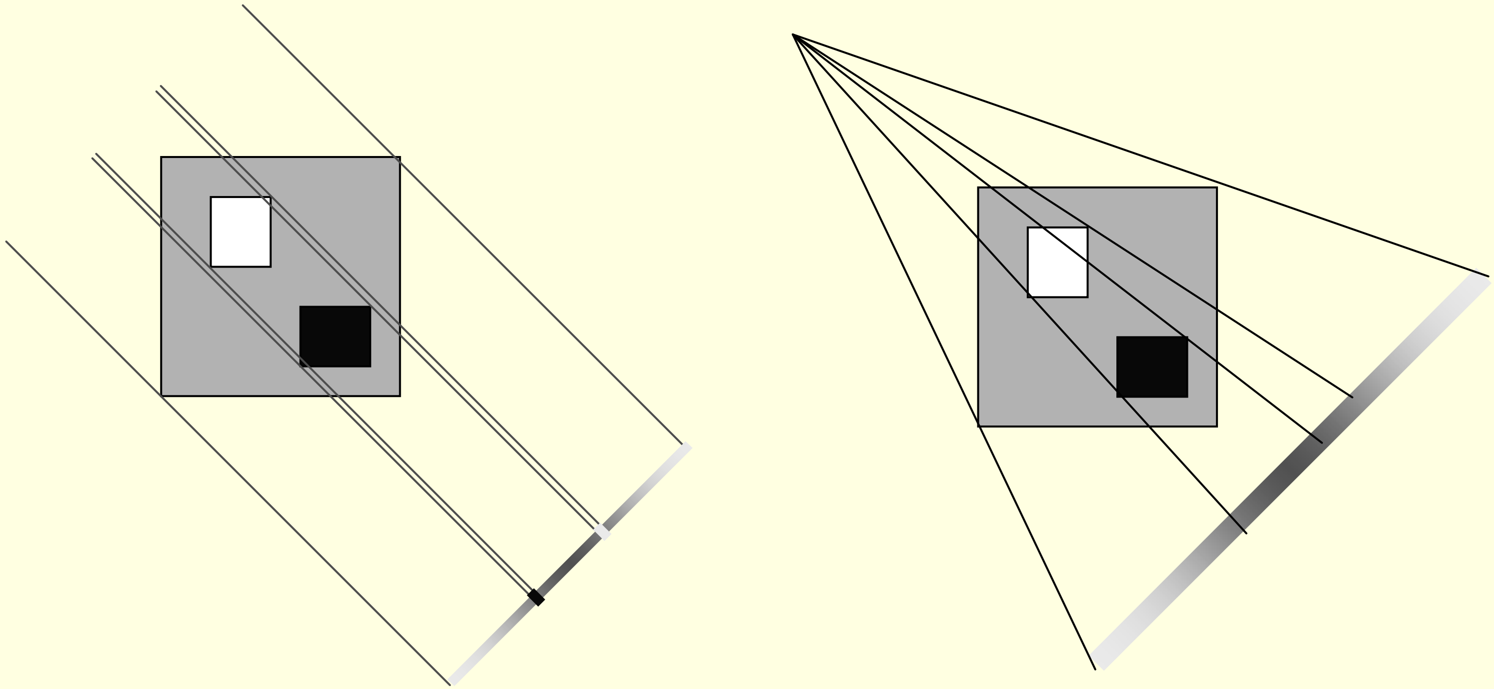
$$\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=\left(2\pi\right)^{\frac{n-1}{2}}\widehat{f\left(\vec{\cdot}\right)}\left(\omega\vec{\theta}\right)$$

$$\tilde{f}\left(\vec{\theta},\rho\right)\xrightarrow{\text{fft}}\widehat{\tilde{f}\left(\vec{\theta},\cdot\right)}(\omega)=\left(2\pi\right)^{\frac{n-1}{2}}\widehat{f\left(\vec{\cdot}\right)}\left(\omega\vec{\theta}\right)\xrightarrow{\text{ifft}}f\left(\vec{x}\right)$$

$$\downarrow \text{interpolation}$$

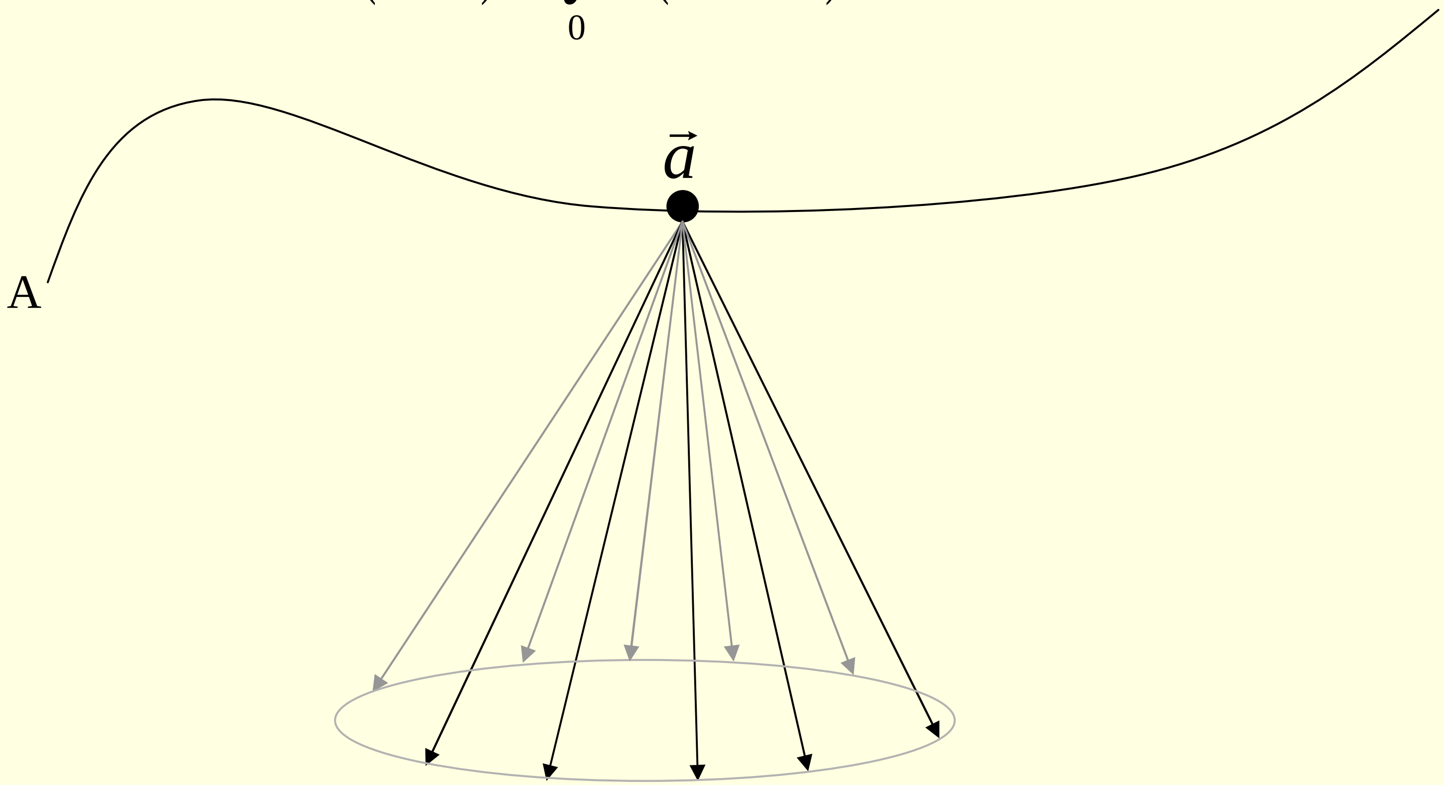
$$\widehat{f\left(\vec{\cdot}\right)}\left(u,v\right)\xrightarrow{\text{ifft}}f\left(\vec{x}\right)$$

But this is not how it is realy done!



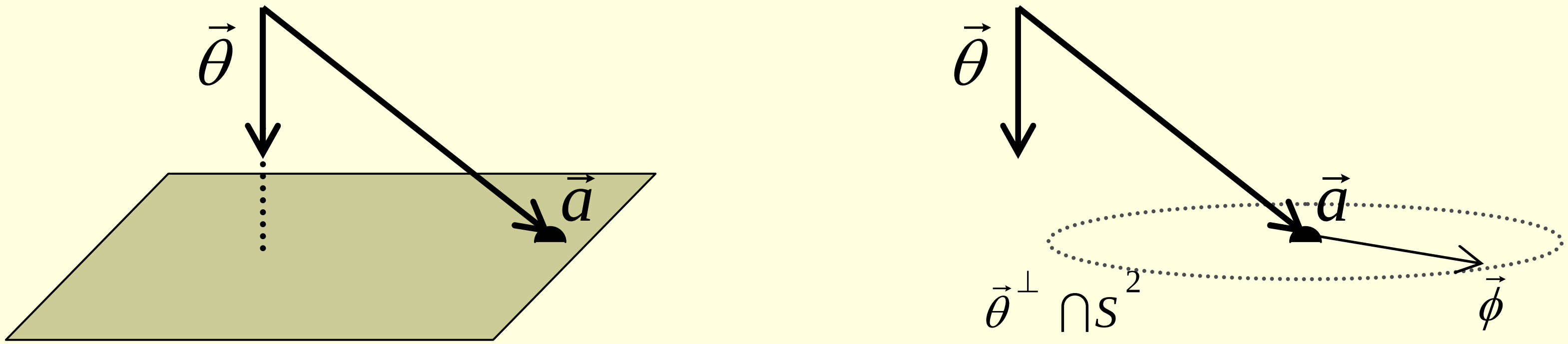
Cone-Beam Transform

$$C\left[f\left(\vec{x}\right)\right]\left(\vec{a},\vec{\theta}\right)=\int_{0}^{\infty}f\left(\vec{a}+t\vec{\theta}\right)dt \qquad \vec{\theta}\in S^{2}$$



Our work was not in vain…

$$\frac{\partial}{\partial \rho}\tilde{f}\left(\vec{\theta}, \vec{a}\cdot\vec{\theta}\right) = \int_{\vec{\theta}^{\perp}\cap S^{2}} \frac{\partial}{\partial \vec{\theta}} C[f]\left(\vec{a}, \vec{\phi}\right) d\vec{\phi}$$

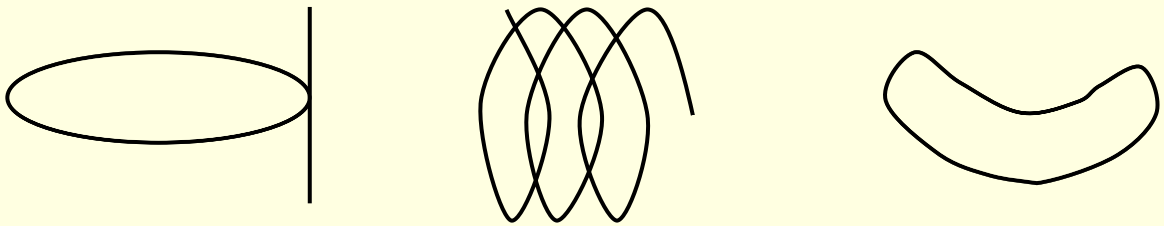


Kirilov-Tuy Condition

$$\frac{\partial}{\partial \rho}\tilde{f}\left(\vec{\theta},\vec{a}\cdot\vec{\theta}\right)=\int_{\vec{\theta}^{\perp}\cap S^2}\frac{\partial}{\partial \vec{\theta}}C[f]\left(\vec{a},\vec{\phi}\right)d\vec{\phi}$$

$$f_{3D}\left(\vec{x}\right)=-\frac{1}{8\pi^2}\int_{S^2}\frac{\partial^2}{\partial \rho^2}\tilde{f}\left(\theta,\vec{x}\cdot\vec{\theta}\right)d\vec{\theta}$$

- If each plane through supp(f) intersects the source curve transversally, then f is uniquely (and stably) determined by the Cone-Beam Transform.

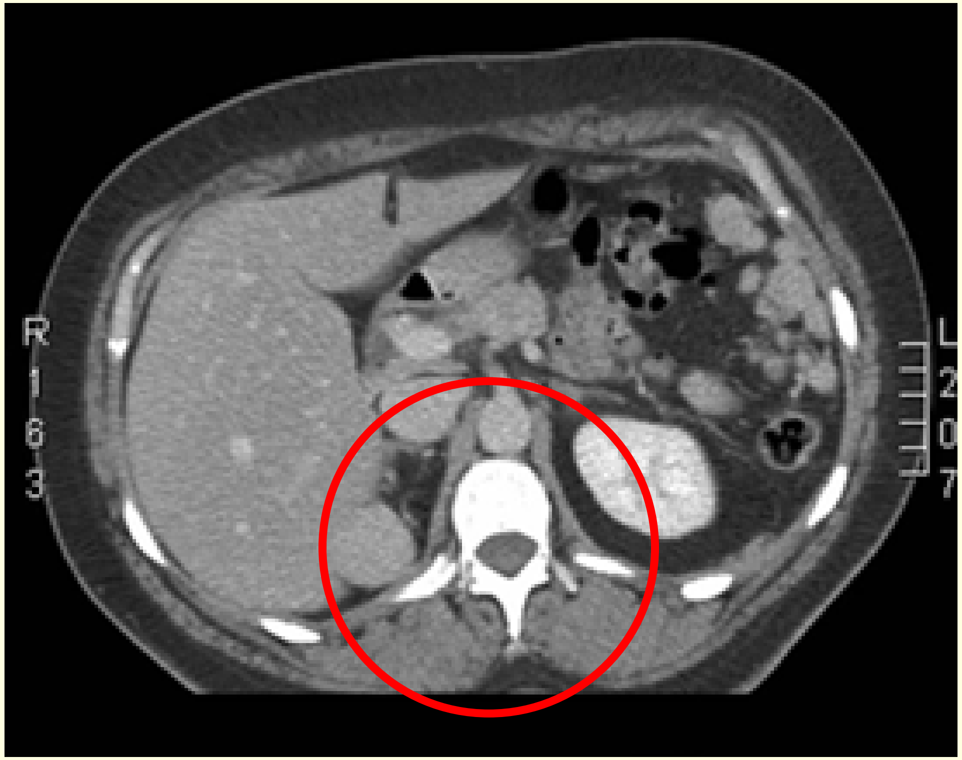


Various Problems in Tomography

- Limited Data
  - Angular
  - Radial
  - Local Tomography
    - Inner
    - Outer
- Noise
  - Motion
  - reflection
- Other technologies
  - MRI
  - PET
  - Thermal \ Acoustic Tomo.

Inner Tomography

- Reducing radiation dosage
  - And other costs (money, time) etc.



Outer Tomogrpahy

- Metal blocks X-Rays