Traffic flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0$$

ρ = density of cars = cars/km
u = speed of cars
$f = \rho u$ = flux = cars/hours past a given point

Assumptions:

- flux is low at low speeds (bumper to bumper)
  and high speed (distance between cars)
- length of car = 5m
- highest density is $u_b = 200$ cars/km
- $f_{\max} = 1500$ cars/hour
- $f_{\max}$ is achieved at 35 km / hour
- optimal density is about 43 cars /km

So f(u) is a concave function.
Assume we have a red light at $x = x_j$. We have cars stretched from x_s to x_j. So cars approach the traffic jam from $x < x_s$ and they slow down.

$$s = \frac{[f]}{[u]} = \frac{\text{negative}}{\text{positive}} < 0$$

i.e. the shock wave travels backward. As drivers enter the shock wave they have to decelerate rapidly. When the cars pass the traffic light $x > x_s$ we have an expansion wave.

If there is a sequence of red lights separated by a distance L then the time for the shock to reach the previous light is

$$T = \frac{L(u_b - u_{\max})}{f_{\max}} \simeq \frac{L \text{ km (200 cars/km - 43 cars/km)}}{\text{1500 cars/hr}} \simeq \frac{L}{10} \text{ hr}$$

If the light is red longer than T hours the traffic will come to a halt!

Another example

Consider $u = 1 - \rho$ (ρ is relative density). So as the density increases the velocity decreases. Then $f = \rho(1-\rho) = \rho - \rho^2$. $f' = 1 - 2\rho$.
So the characteristics are straight lines with slope $1 - 2\rho$.
From the Rankine Hugoniot relationship we have

$$s = \frac{[\rho(1-\rho)]}{\rho} = \frac{\rho_r(1-\rho_r) - \rho_l(1-\rho_l)}{\rho_r - \rho_l}$$

Assume we have a stop light at $x = 1$ at $t = 0$. Then the initial condition is

$$\rho(x,0) = \begin{cases} 0 & x \le 0 \quad \text{and} \quad x \ge 1 \\ x & 0 < x < 1 \end{cases}$$

The characteristics are given by

$$\begin{cases} x \le t & x - s = t & \rho = 0 \\ t < x < 1 - t & x - s = (1-2s)t & \rho = s \\ 1 - t < x < 1 + t & x - 1 = (1-2s)t & \rho = \frac{t-x+1}{2t} \\ x > t + 1 & x - s = t & \rho = 0 \end{cases}$$

They first "meet" at $x = \frac{1}{2}$ $t = \frac{1}{2}$. The speed of the shock is then given by

$$S = \frac{s+t-1}{2t}$$