Definition 1 *Fourier Transform*

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

examples

Delta function	$\delta(x)$	1
Square Pulse	$H(a-\|x\|)$	$\frac{2}{k}\sin(ak)$
Exponential	$e^{-a\|x\|}$	$\frac{2a}{a^2+k^2}$
Heaviside function	$H(x)$	$\pi\delta(k)+\frac{1}{ik}$
Sign	$H(x)-H(-x)$	$\frac{2}{ik}$
Constant	1	$2\pi\delta(k)$
Gaussian	$e^{-\frac{x^2}{2}}$	$\sqrt{2\pi}e^{-\frac{k^2}{2}}$

Function	Transform
$\frac{df}{dx}$	$ikF(k)$
$xf(x)$	$i\frac{dF}{dk}$
$f(x-a)$	$e^{-iak}F(k)$
$e^{iak}f(x)$	$F(k-a)$
$af(x)+bg(x)$	$aF(k)+bG(k)$
$f(ax)$	$\frac{1}{a}F(\frac{k}{a})$

Heisenberg Inequality

Let $f(x)$ represent a probability density. Then

$$\int\limits_{-\infty}^{\infty} |f(x)|^2 dx = 1$$

Definition 2

$$\begin{aligned}\overline{x}^2 &= \int_{-\infty}^{\infty} |xf(x)|^2 dx \\ \overline{k}^2 &= \int_{-\infty}^{\infty} |kF(k)|^2 dk\end{aligned}$$

Theorem 3

$$\overline{x}\,\overline{k} \geq \frac{1}{2}$$

Proof. By the Schwarz inequality

$$\begin{aligned}\left|\int_{-\infty}^{\infty} xf(x)f'(x)dx\right| &\leq \sqrt{\int_{-\infty}^{\infty} |xf(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |f'(x)|^2 dx} \\ &= \overline{x}\sqrt{\int_{-\infty}^{\infty} |f'(x)|^2 dx} \\ &= \overline{x}\overline{k} \qquad \text{by Parseval}\end{aligned}$$

but using integration by parts

$$\begin{aligned}\left|\int_{-\infty}^{\infty} xf(x)f'(x)dx\right| &= \left|\frac{1}{2}xf^2(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}f^2(x)dx\right| \\ &= \left|0 - \frac{1}{2}\right| = \frac{1}{2}\end{aligned}$$

■

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt$$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds$$

γ is a vertical contour chosen so that all the singularities of $F(s)$ are to the left of it.

The Laplace transform is linear

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$$

Definition 4 *f is of exponential order if there exist positive numbers a and M such that*

$$|f(t)| \le Ms^{at}$$

Theorem 5 *If f is piecewise continuous on $0 \le t < \infty$ and is of exponential order then $\mathcal{L}(f)$ exists for all $s > a$.*

Proof.

$$\begin{aligned}
|\int_0^\infty f(t)e^{-st}dt| &\le \int_0^\infty |f(t)|e^{-st}dt \\
&\le M\int_0^\infty e^{at}e^{-st}dt \\
&= M\int_0^\infty e^{-(s-a)t}dt \\
&= \frac{M}{s-a} < \infty
\end{aligned}$$

■

Note: $\sqrt{\frac{1}{t}}$ is not of exponential order because of its behavior near the origin. However, its Laplace transform exists. Hence the above condition is sufficient but not necessary.

Examples

$$\mathcal{L}(\mathrm{e}^{at}) = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a} < \infty$$

$$\mathcal{L}\left(t^a\right) = \int_0^\infty t^a e^{-st}dt$$

change variables $p = st$. Then

$$\mathcal{L}\left(t^a\right) = \frac{1}{s^{a+1}} \int_0^\infty p^a e^{-p}dp = \frac{1}{s^{a+1}}\Gamma(a+1)$$

So

$$\mathcal{L}\left(t^n\right) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\begin{aligned}
\mathcal{L}\left(\cos(kt) + i(\sin kt)\right) &= \int_0^\infty \left(\cos(kt) + i(\sin kt)\right) e^{-st}dt \\
&= \int_0^\infty e^{-t(s-ik)}dt = \frac{e^{-(s-ik)t}}{s-ik}|_0^\infty = \frac{1}{s-ik} \\
&= \frac{s}{s^2+k^2} + \frac{ik}{s^2+k^2}
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}\left(\cos(kt)\right) &= \frac{s}{s^2+k^2} \\
\mathcal{L}\left(\sin(kt)\right) &= \frac{k}{s^2+k^2}
\end{aligned}$$

Theorem 6 *Suppose f is continuous on $0 \le t < \infty$ and is of exponential order then $\mathcal{L}(f)$ exists for all $s > a$. Also if f' is piecewise continuous on $0 \le t < \infty$ and is of exponential order then*

$$\begin{aligned}
\mathcal{L}\left(f'\right) &= s\mathcal{L}\left(f\right) - f(0) \\
\mathcal{L}\left(f^{(n)}\right) &= s^n\mathcal{L}\left(f\right) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots - f^{(n-1)}
\end{aligned}$$

Proof. integration by parts ■

Similarly

$$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Theorem 7 *Suppose f is of exponential order. Let α be a real number and a the exponential bound. Then for $s > a + \alpha$*

$$\begin{aligned} F(s) &= \mathcal{L}\left(f(t)\right) \\ \mathcal{L}\left(e^{\alpha t} f(t)\right) &= F(s-\alpha) \end{aligned}$$

Some other Laplace transforms

$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	
t^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$	
$tf(t)$	$-\frac{d}{ds}\mathcal{L}\left(f\right)(s)$	
te^{at}	$-\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$	
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\mathcal{L}\left(f\right)(s)$	
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$	
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$	$s > a$
$e^{at}\sinh(kt)$	$\frac{k}{(s-a)^2-k^2}$	$s > a$
$e^{at}\cosh(kt)$	$\frac{s-a}{(s-a)^2-k^2}$	$s > a$
$H(t-a)$	$\frac{1}{s}e^{-as}$	
$\delta(t-a)$	e^{-as}	
$\frac{a}{\sqrt{4\pi t^3}}e^{-\frac{a^2}{4t}}$	$e^{-a\sqrt{s}}$	
$\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}$	$\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}$	
$1 - erf\left(\frac{a}{\sqrt{4t}}\right)$	$\frac{1}{s}e^{-a\sqrt{s}}$	

Inverse Laplace Transform

Instead of using the exact formula we "guess" the inverse transform.

Examples:

1.
$$\mathcal{L}^{-1}\left(\frac{2}{4+(s-1)^2}\right)$$

Comparing with formula for $e^{at}\sin(kt)$ we see that

$$\mathcal{L}^{-1}\left(\frac{2}{4+(s-1)^2}\right) = e^{t}\sin(2t)$$

2.
$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+3}\right)$$

We know that

$$\frac{1}{s^2+2s+3} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2}$$

So

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+3}\right) = \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$

Rational functions

- First approach

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right)$$

Split into its partial fractions

$$\begin{aligned}\frac{1}{s^2+2s-3} &= \frac{1}{(s+3)(s-1)}\\ &= \frac{A}{s+3}+\frac{B}{s-1}\\ &= \frac{(A+B)s+3B-A}{(s+3)(s-1)}\end{aligned}$$

So

$$\frac{1}{s^2+2s-3} = -\frac{1}{4}\frac{1}{s+3}+\frac{1}{4}\frac{1}{s-1}$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right) = -\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}$$

- Second approach: Using above formulae

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right) &= \frac{1}{2}e^{-t}\sinh(2t)\\ &= \frac{1}{2}e^{-t}\frac{e^{2t}-e^{-2t}}{2}\\ &= -\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}\end{aligned}$$

3.

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s+1)^2}\right)$$

Split into its partial fractions

$$\frac{1}{(s^2+1)(s+1)^2} = \frac{A_1}{s+1}+\frac{A_2}{(s+1)^2}+\frac{Bs+C}{s^2+1}$$

To find constants we have two ways

- find common denominator

$$\begin{aligned}\frac{A_1}{s+1}+\frac{A_2}{(s+1)^2}+\frac{Bx+C}{s^2+1} &= \frac{[A_1(s+1)+A_2]\left(s^2+1\right)+(Bx+C)(s+1)^2}{(s^2+1)(s+1)^2}\\ &= \frac{(A_1+B)s^3+(A_1+A_2+C+2B)s^2+(A_1+B+2C)s+A_1+A_2+C}{(s^2+1)(s+1)^2}\end{aligned}$$

So

$$\begin{aligned}A_1+A_2+C&=1\\ A_1+B+2C&=0\\ A_1+A_2+C+2B&=0\\ A_1+B&=0\end{aligned}$$

Therefore

$$\begin{aligned}&A_1=A_2=\frac{1}{2}\\ &B=-\frac{1}{2}\\ &C=0\end{aligned}$$

So

$$\frac{1}{(s^2+1)(s+1)^2} = \frac{1}{2(s+1)}+\frac{1}{2(s+1)^2}-\frac{s}{2(s^2+1)}$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s+1)^2}\right) = \frac{1}{2}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{2}\cos(t)$$

$$\frac{1}{(s^2+1)(s+1)^2} = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{Bs+C}{s^2+1}$$

multiple both sides by $(s+1)^2$ and evaluate at $s=-1 \implies A_2 = \frac{1}{2}$.
multiple both sides by $(s+1)^2$and differentiate with respect to s and evaluate at $s=-1 \implies A_1 = \frac{1}{2}$
Substitute the values for A_1 and A_2 and evaluate at $s=0 \implies C=0$
multiply both sides by s^2 and evaulate at $s=\infty \implies B = -\frac{1}{2}$

$$\mathcal{L}^{-1}\left(\frac{4s}{(s^2+1)^2(s-1)}\right)$$

As before

$$\frac{1}{(s^2+1)(s+1)^2} = \frac{A}{s-1} + \frac{B_1s+C_1}{s^2+1} + \frac{B_2s+C_2}{(s^2+1)^2}$$

Again comparing coefficients we find

$$\begin{aligned}
\frac{1}{(s^2+1)(s+1)^2} &= \frac{1}{s-1} - \frac{s+1}{s^2+1} + \frac{-2s+2}{(s^2+1)^2} \\
&= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1} - 2\frac{s}{(s^2+1)^2} + 2\frac{1}{(s^2+1)^2} \\
&= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{s^2+4s-1}{(s^2+1)^2} \\
&= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{s^2-1}{(s^2+1)^2} - \frac{4s}{(s^2+1)^2}
\end{aligned}$$

So

$$\mathcal{L}^{-1}\left(\frac{4s}{(s^2+1)^2(s-1)}\right) = e^t - \cos(t) - t\cos(t) - 2t\sin(t)$$

Other properties of the Laplace Transform

Definition 8 *Convolution*

$$f * g = \int_0^t f(t-\tau)g(\tau)d\tau$$

Theorem 9 *If f and g are piecewise continuous and of exponential order, then*

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

Definition 10 *Heaviside step function*

$$H(t-a) = \begin{cases} 1 & \textit{if } t \geq a \\ 0 & \textit{if } t < a \end{cases}$$

So

$$H(t-a)f(t-a) = \begin{cases} f(t-a) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

Theorem 11 *Shift*

$$\mathcal{L}\left(H(t-a)f(t-a)\right)(s) = \int_0^\infty H(t-a)f(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

$$\textit{let } p = t-a \textit{ then } \mathcal{L}\left(F(t-a)\right) = \int_0^\infty f(t)e^{-s(p+a)}dp$$

$$= e^{-as}F(s)$$

Definition 12 *delta function*

Suppose f is continuous in $[a,b]$ then

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & \textit{if } t_0 \textit{ is in } [a,b] \\ 0 & \textit{otherwise} \end{cases}$$

Hence

$$\mathcal{L}(\delta(t-t_0)) = \int_0^\infty \delta(t-a)e^{-st}dt = e^{-as} \qquad \text{if } 0 \leq a < \infty$$

Applications to ODE/PDE

Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \infty \quad 0 < t < \infty$$
$$u(x,0) = 0$$
$$u(0,t) = f(t)$$

We Laplace transform in time to get

$$sU(x,s) - u(x,0) = k\frac{\partial^2 U}{\partial x^2}$$
$$\text{or} \quad k\frac{\partial^2 U}{\partial x^2} - sU(x,s) = 0$$

This has the solution

$$U(x,s) = A(s)e^{\sqrt{\frac{s}{k}}x} + B(s)e^{-\sqrt{\frac{s}{k}}x}$$

Since the Laplace transform is bounded at infinity we have

$$U(x,s) = B(s)e^{-\sqrt{\frac{s}{k}}x}$$

Using the boundary condition at $x = 0$ and denoting the Laplace transform of $f(t)$ by $F(s)$

$$U(0,s) = F(s) = B(s)$$

So

$$U(x,s) = F(s)e^{-\sqrt{\frac{s}{k}}x}$$

Using the convolution theorem we have

$$u(x,t) = f(t) * \mathcal{L}^{-1}\left(e^{-\sqrt{\frac{s}{k}}x}\right)$$
$$= f(t) * \frac{x}{\sqrt{4k\pi}t^{\frac{3}{2}}}e^{-\frac{x^2}{4kt}}$$
$$\text{or} \quad u(x,t) = \frac{x}{\sqrt{4k\pi}}\int_0^t \frac{f(\tau)}{(t-\tau)^{\frac{3}{2}}}e^{-\frac{x^2}{4k(t-\tau)}}d\tau$$

If $f(t) = T_0$ then

$$u(x,t) = \frac{x}{\sqrt{4k\pi}}T_0\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}}}e^{-\frac{x^2}{4k(t-\tau)}}d\tau$$

Define $z = \frac{x}{2\sqrt{k(t-\tau)}}$ and $dz = \frac{x}{4\sqrt{k}(t-\tau)}d\tau$ and

$$u(x,t) = \frac{2}{\sqrt{\pi}}T_0\int_{\frac{x}{2\sqrt{kt}}}^{t} e^{-z^2}dz = T_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

Now consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(t) \qquad 0 < x < \infty \quad 0 < t < \infty$$
$$u(x,0) = 0$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$
$$u(0,t) = 0$$

Again Laplace transform in time yielding

$$s^2 U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0) = c^2 \frac{\partial^2 U}{\partial x^2} + F(s)$$

using the IC we get $\quad -c^2 \dfrac{\partial^2 U}{\partial x^2} + s^2 U(x,s) = F(s)$

We now have an inhomogenous second order ODE with constant coefficients. The general solution is the general solution to the homogenous part plus a particular solution to the inhomogenous equation.

So the homogenous part gives:

$$U_{\text{homo}} = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x}$$

However, $F(s)$ is independent of x. so a particular solution is $U = \frac{F(s)}{s^2}$.So the general solution is

$$U(x,s) = A(s)e^{-\frac{s}{c}x} + B(s)e^{\frac{s}{c}x} + \frac{F(s)}{s^2}$$

As before the transform is bounded at infinity and so $B(s) = 0$

$$U(x,s) = A(s)e^{-\frac{s}{c}x} + \frac{F(s)}{s^2}$$

Using the BC at $x = 0$

$$0 = U(0,s) = A(s) + \frac{F(s)}{s^2} \implies A(s) = -\frac{F(s)}{s^2}$$

So

$$U(x,s) = \frac{F(s)}{s^2}\left(1 - e^{-\frac{s}{c}x}\right)$$

Again using the convolution theorem

$$\begin{aligned} u(x,t) &= f(t) * \mathcal{L}^{-1}\left(\frac{1 - e^{-\frac{s}{c}x}}{s^2}\right) \\ &= f(t) * \left[t - (t - \frac{x}{c})H(t - \frac{x}{c})\right] \\ &= \int_0^t f(t-\tau)\left[\tau - (\tau - \frac{x}{c})H(\tau - \frac{x}{c})\right] d\tau \end{aligned}$$

where H is the Heaviside step function

Vibrations of a string subject to gravity

If the only force on a string is gravity we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} - g \qquad 0 < x < \infty \quad 0 < t < \infty \\
u(x,0) &= 0 \\
\frac{\partial u}{\partial t}(x,0) &= 0 \\
u(0,t) &= 0
\end{aligned}$$

So

$$\begin{aligned}
u(x,t) &= -g \int_0^t \left[\tau - (\tau - \frac{x}{c}) H(\tau - \frac{x}{c}) \right] d\tau \\
&= -g \frac{t^2}{2} + g \int_0^t (\tau - \frac{x}{c}) H(\tau - \frac{x}{c}) d\tau
\end{aligned}$$

If $x > ct$ then $\tau - \frac{x}{c}$ is always negative and the Heaviside function is zero. So we only need consider the case $x < ct$. Then

$$\int_0^t (\tau - \frac{x}{c}) H(\tau - \frac{x}{c}) d\tau = \int_{\frac{x}{c}}^t (\tau - \frac{x}{c}) d\tau = \frac{1}{2}(t - \frac{x}{c})^2$$

So

$$u(x,t) = \begin{cases} -\frac{g}{2}\left(t^2 - (t - \frac{x}{c})^2\right) & \text{if } 0 < x < ct \\ -\frac{g}{2} t^2 & \text{if } x > ct \end{cases}$$

Hankel Transform

$$H_v(f)(s) = \int_0^\infty f(r) J_v(sr) r dr \qquad s \geq 0$$

If

$$\int_0^\infty |f(r)| \sqrt{r} dr < \infty$$

Then

$$H_v(H_v(f))(s) = f(x)$$

Thus the Hankel transform is its own inverse !

Properties

$$-sH_0(f)(s) = H_1(f')(s)$$
$$H_0\left(f' + \frac{f}{r}\right) = sH_1(f)(s)$$
$$H_0\left(f'' + \frac{f'}{r}\right) = -s^2 H_0(f)(s)$$

example:
Consider an infinitely long hanging chain fastened at ∞.
We label the vertical axis as x pointing up.
The free motion of the chain is described by

$$\frac{\partial^2 u}{\partial t^2} = g\left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right] \qquad 0 < x < \infty \quad 0 < t < \infty$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = v(x)$$

Change variables

$$z^2 = x \qquad \text{so} \quad 2zdz = dx$$

Then

$$\frac{\partial^2 u}{\partial t^2} = \frac{g}{4}\left[\frac{\partial^2 u}{\partial z^2} + \frac{1}{z}\frac{\partial u}{\partial z}\right] \qquad 0 < x < \infty \quad 0 < t < \infty$$
$$u(z^2,0) = f(z^2)$$
$$\frac{\partial u}{\partial t}(z^2,0) = v(z^2)$$

Denote the zeroeth order Hankel transform of $u(z^2,t)$ by $U(s,t)$.
Then we transform the equations to get

$$\frac{\partial^2 U}{\partial t^2} = -\frac{s^2 g}{4}\frac{\partial^2 u}{\partial x^2}$$
$$U(s,0) = H_0(f(z^2))$$
$$\frac{\partial U}{\partial t}(s,0) = H_0(v(z^2))$$

As in the previous cases we solve the second order ODE to get

$$U(s,t) = A(s)\cos(\frac{\sqrt{g}}{2}st) + B(s)\sin(\frac{\sqrt{g}}{2}st)$$
$$A(s) = H_0(f(z^2)) \qquad B(s) = \frac{2}{\sqrt{g}s}H_0(v(z^2))$$

Using the inverse transform we get

$$u(x,t) = \int_0^\infty \left[A(s)\cos(\frac{\sqrt{g}}{2}st) + B(s)\sin(\frac{\sqrt{g}}{2}st)\right] J_0(s\sqrt{x})sds$$