Vibrating membrane

Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad 0 \le x \le L \quad 0 \le y \le H$$
$$u(x,0) = \alpha(x,y)$$
$$\frac{\partial u}{\partial t}(x,0) = \beta(x,y)$$
$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,H,t) = 0$$

We can use two approaches

(a) $u(x,y,t) = h(t)\phi(x,y)$

(b) $u(x,y,t) = f(x)g(y)h(t)$

We shall use the second approach. Putting this into the wave equation we get

$$fg\frac{d^2h}{dt^2} = hc^2\left(g\frac{d^2f}{dx^2} + f\frac{d^2g}{dy^2}\right)$$
$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\lambda$$

So

$$\frac{d^2h}{dt^2} + \lambda c^2 h = 0$$

Now

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2g}{dy^2} = -\mu$$

or

$$\frac{d^2f}{dx^2} + \mu f = 0 \qquad f(0) = f(L) = 0$$
$$\frac{d^2g}{dy^2} + (\lambda - \mu)\,g = 0 \qquad g(0) = g(H) = 0$$

To satisfy the boundary conditions we need to choose

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad f_n(x) = \sin\frac{n\pi}{L}x$$
$$\lambda_{nm} - \mu_m = \left(\frac{m\pi}{H}\right)^2 \qquad g_{nm}(y) = \sin\frac{m\pi}{H}y$$
$$\text{So} \quad \lambda_{nm} = \mu_m + \left(\frac{m\pi}{H}\right)^2 = \frac{\left(n^2+m^2\right)\pi^2}{H^2}$$

with eigenfunctions

$$u_{nm}(x) = \sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}y$$

Therefore

$$\begin{aligned}u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\cos(c\sqrt{\lambda_{nm}}t)\\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\sin(c\sqrt{\lambda_{nm}}t)\end{aligned}$$

with

$$\begin{aligned}A_{nm} &= \frac{2}{L}\frac{2}{H}\int_0^L\int_0^H \alpha(x,y)\sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}ydydx\\ c\sqrt{\lambda_{nm}}B_{nm} &= \frac{2}{L}\frac{2}{H}\int_0^L\int_0^H \beta(x,y)\sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}ydydx\end{aligned}$$