7 Wave Equation in Higher Dimensions

We now consider the initial-value problem for the wave equation in n dimensions,

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{7.1}$$

where $\Delta u \equiv \sum_{i=1}^n u_{x_i x_i}$.

7.1 Method of Spherical Means

Ref: Evans, Sec. 2.4.1; Strauss, Sec. 9.2

We begin by introducing a method to solve (7.1) in odd dimensions. First, we introduce some notation. For $x \in \mathbb{R}^n$, let

- $B(x,r)$ = Ball of radius r about x
- $\partial B(x,r)$ = Boundary of ball of radius r about x
- $\alpha(n)$ = Volume of unit ball in $\mathbb{R}^n$
- $n\alpha(n)$ = Surface Area of unit ball in $\mathbb{R}^n$.

With this notation, the volume of the ball of radius r about $x \in \mathbb{R}^n$, written as $\text{Vol}(B(x,r))$, is given by $\alpha(n)r^n$ and the surface area of the ball of radius r about $x \in \mathbb{R}^n$, written as S.A.$(B(x,r))$, is given by $n\alpha(n)r^{n-1}$.

For $f : \mathbb{R}^n \to \mathbb{R}$, we define the **average of f over B(x, r)** as

$$⨍_{B(x,r)} f(y)\,dy \equiv \frac{1}{\text{Vol}(B(x,r))} \int_{B(x,r)} f(y)\,dy = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} f(y)\,dy.$$

We define the **average of f over ∂B(x, r)** as

$$⨍_{\partial B(x,r)} f(y)\,dS(y) \equiv \frac{1}{\text{S.A.}(B(x,r))} \int_{\partial B(x,r)} f(y)\,dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} f(y)\,dS(y),$$

where $dS(y)$ denotes the surface measure of $B(x,r)$ in $\mathbb{R}^n$.

Example 1. For $n = 3$, $\text{Vol}(B(x,r)) = \frac{4}{3}\pi r^3$. Therefore, for $f : \mathbb{R}^3 \to \mathbb{R}$, the average of f over $B(0,r)$ is given by

$$⨍_{B(0,r)} f(y)\,dy = \frac{3}{4\pi r^3} \int_0^{\pi} \int_0^{2\pi} \int_0^r f(\rho,\theta,\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

For $n = 3$, S.A.$(B(x,r)) = 4\pi r^2$. Therefore, for $f : \mathbb{R}^3 \to \mathbb{R}$, the average of f over $\partial B(0,r)$ is given by

$$⨍_{\partial B(0,r)} f(y)\,dS(y) = \frac{1}{4\pi r^2} \int_0^{\pi} \int_0^{2\pi} f(r,\theta,\phi) r^2 \sin\phi\,d\theta\,d\phi = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(r,\theta,\phi) \sin\phi\,d\theta\,d\phi.$$

$\diamond$

Our plan to solve (7.1) is the following. Fix a point $x \in \mathbb{R}^n$. For $r > 0$, we define

$$\overline{u}(x;r,t) \equiv ⨍_{\partial B(x,r)} u(y,t)\,dS(y),$$

the average of $u(\cdot,t)$ over $\partial B(x,r)$. For $r = 0$, we define $\overline{u}(x;0,t) = u(x,t)$. For $r < 0$, we define $\overline{u}(x;r,t) = \overline{u}(x;-r,t)$. We claim that for u smooth, $\overline{u}$ is a continuous function of r, and, therefore,

$$\lim_{r \to 0+} \overline{u}(x;r,t) = u(x,t).$$

In order to solve (7.1), we will assume u is a solution of (7.1) and look for an equation $\overline{u}$ solves. *Note:* We will assume $c = 1$. For $c \neq 1$, we can make a change of variables to derive the solution from the solution in the case $c = 1$.

Lemma 2. *If u solves*

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n, t \geq 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x), \end{cases}$$

then $\overline{u}(x;r,t)$ solves

$$\begin{cases} \overline{u}_{tt} - \overline{u}_{rr} - \dfrac{(n-1)}{r}\overline{u}_r = 0, & 0 < r < \infty, t \geq 0 \\ \overline{u}(x;r,0) = \overline{\phi}(x;r) \equiv ⨍_{\partial B(x,r)} \phi(y)\,dS(y) \\ \overline{u}_t(x;r,0) = \overline{\psi}(x;r) \equiv ⨍_{\partial B(x,r)} \psi(y)\,dS(y) \end{cases}$$

for every $x \in \mathbb{R}^n$.

Proof.

$$\begin{aligned} \overline{u}(x;r,t) &= ⨍_{\partial B(x,r)} u(y,t)\,dS(y) \\ &= ⨍_{\partial B(0,1)} u(x+rz,t)\,dS(z). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\overline{u}_r(x;r,t) &= ⨍_{\partial B(0,1)} \nabla u(x+rz,t)\cdot z\, dS(z) \\
&= ⨍_{\partial B(x,r)} \nabla u(y,t)\cdot \frac{y-x}{r}\, dS(y) \\
&= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y,t)\, dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y,t)\, dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)\, dy \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy
\end{aligned}
$$

by the Divergence Theorem, and using the fact that u solves the wave equation, $u_{tt}-\Delta u = 0$. Therefore,

$$
\overline{u}_r(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy
$$

which implies

$$
r^{n-1}\overline{u}_r(x;r,t) = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}(y,t)\, dy.
$$

Therefore,

$$
\begin{aligned}
(r^{n-1}\overline{u}_r(x;r,t))_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \\
&= \frac{r^{n-1}}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS \\
&= r^{n-1} ⨍_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \\
&= r^{n-1}\overline{u}_{tt}(x;r,t).
\end{aligned}
$$

Therefore,

$$
(r^{n-1}\overline{u}_r(x;r,t))_r = r^{n-1}\overline{u}_{tt}(x;r,t),
$$

which implies

$$
(n-1)r^{n-2}\overline{u}_r + r^{n-1}\overline{u}_{rr} = r^{n-1}\overline{u}_{tt}.
$$

Therefore,

$$
\overline{u}_{tt} - \overline{u}_{rr} - \frac{(n-1)}{r}\overline{u}_r = 0
$$

and

$$\overline{u}(x;r,0) = \fint_{\partial B(x,r)} u(y,0)\, dS = \fint_{\partial B(x,r)} \phi(y)\, dS = \overline{\phi}(x;r).$$

Similarly,

$$\overline{u}_t(x;r,0) = \overline{\psi}(x;r)$$

as claimed. □

Solution for $\mathbf{n} = 3$.

We now consider the case of the wave equation in three dimensions. Assume u is a solution of (7.1) for $n = 3$. As before define the function $\overline{u}(x;r,t)$ such that

$$\overline{u}(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\, dS(y).$$

Next introduce a function $v(x;r,t)$ such that

$$v(x;r,t) = r\overline{u}(x;r,t)$$

and new functions $g(x;r)$ and $h(x;r)$ such that

$$\begin{aligned} g(x;r) &= r\overline{\phi}(x;r) = r \fint_{\partial B(x,r)} \phi(r)\, dS(y) \\ h(x;r) &= r\overline{\psi}(x;r) = r \fint_{\partial B(x,r)} \psi(r)\, dS(y). \end{aligned}$$

Lemma 3. *For each $x \in \mathbb{R}^n$, the function $v(x;r,t)$ solves the one-dimensional wave equation on the half-line with Dirichlet boundary conditions,*

$$\begin{cases} v_{tt} - v_{rr} = 0 & 0 < r < \infty, t \geq 0 \\ v(x;r,0) = g(x;r) & 0 < r < \infty \\ v_t(x;r,0) = h(x;r) & 0 < r < \infty \\ v(x;0,t) = 0 & t \geq 0. \end{cases}$$

Proof.

$$\begin{aligned} v_{tt} &= r\overline{u}_{tt} \\ &= r\left[\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right] \\ &= r\overline{u}_{rr} + 2\overline{u}_r \\ &= (r\overline{u}_r + \overline{u})_r \\ &= (r\overline{u})_{rr} \\ &= v_{rr}. \end{aligned}$$

Next,

$$\begin{aligned}
v(x;r,0) &= r\overline{u}(x;r,0) \\
&= r \fint_{\partial B(x,r)} u(y,0)\,dS(y) \\
&= r \fint_{\partial B(x,r)} \phi(y)\,dS(y) \\
&= r\overline{\phi}(x,r) \\
&= g(x;r)
\end{aligned}$$

Similarly,

$$v_t(x;r,0) = h(x;r).$$

Now,

$$v(x;0,t) = 0 \cdot \overline{u}(x;0,t) = 0.$$

Therefore, $v(x;r,t)$ solves the one-dimensional wave equation on a half-line with Dirichlet boundary conditions, as claimed. □

Now we use this fact to construct the solution of (7.1). By d'Alembert's formula, we know that for $0 \le r \le t$, the solution $v(x;r,t)$ is given by

$$v(x;r,t) = \frac{1}{2}[g(x;r+t) - g(x;t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} h(x;y)\,dy.$$

Now

$$u(x,t) = \lim_{r\to 0^+} \overline{u}(x;r,t)$$

and

$$v(x;r,t) = r\overline{u}(x;r,t).$$

Therefore,

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0^+} \frac{v(x;r,t)}{r} \\
&= \lim_{r\to 0^+} \left\{ \frac{1}{2r}[g(x;t+r) - g(x;t-r)] + \frac{1}{2r}\int_{-r+t}^{r+t} h(x;y)\,dy \right\} \\
&= \frac{d}{dt} g(x;t) + h(x;t).
\end{aligned}$$

Now

$$g(x;r) = r\overline{\phi}(x;r)$$

implies

$$g(x;t) = t\overline{\phi}(x;t) = t \fint_{\partial B(x,t)} \phi(y)\,dS(y).$$

Similarly,

$$h(x;t) = t\overline{\psi}(x;t) = t \fint_{\partial B(x,t)} \psi(y)\,dS(y).$$

Therefore, the solution of the wave equation in $\mathbb{R}^3$ (with $c = 1$) is given by

$$\boxed{u(x,t) = \frac{\partial}{\partial t}\left[t \fint_{\partial B(x,t)} \phi(y)\,dS(y)\right] + t \fint_{\partial B(x,t)} \psi(y)\,dS(y).}$$

If ϕ is smooth, the solution can be simplified further. In particular, for ϕ smooth, we have

$$\begin{aligned}
\frac{d}{dt}g(x;t) &= \frac{d}{dt}\left(t \fint_{\partial B(x,t)} \phi(y)\,dS(y)\right) \\
&= \frac{d}{dt}\left(t \fint_{\partial B(0,1)} \phi(x+tz)\,dS(z)\right) \\
&= \fint_{\partial B(0,1)} \phi(x+tz)\,dS(z) + t \int_{\partial B(0,1)} \nabla\phi(x+tz)\cdot z\,dS(z) \\
&= \fint_{\partial B(x,t)} \phi(y)\,dS(y) + t \fint_{\partial B(x,t)} \nabla\phi(y)\cdot\left(\frac{y-x}{t}\right)\,dS(y) \\
&= \fint_{\partial B(x,t)} \phi(y)\,dS(y) + \fint_{\partial B(x,t)} \nabla\phi(y)\cdot(y-x)\,dS(y).
\end{aligned}$$

And,

$$h(x;t) = t\overline{\psi}(x;t) = t \fint_{\partial B(x,t)} \psi(y)\,dS(y).$$

Therefore, we have

$$u(x,t) = \fint_{\partial B(x,t)} [\phi(y) + \nabla\phi(y)\cdot(y-x) + t\psi(y)]\,dS(y).$$

We note that in $\mathbb{R}^3$,

$$\fint_{\partial B(x,t)} = \frac{1}{n\alpha(n)t^{n-1}} \int_{\partial B(x,t)} = \frac{1}{4\pi t^2} \int_{\partial B(x,t)}.$$

Therefore, the solution of the IVP for the wave equation in $\mathbb{R}^3$ (with $c = 1$ and ϕ smooth) is given by

$$\boxed{u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y)\cdot(y-x) + t\psi(y)]\,dS(y).} \tag{7.2}$$

This is known as **Kirchoff's formula** for the solution of the initial value problem for the wave equation in $\mathbb{R}^3$.

Remark. Above we found the solution for the wave equation in $\mathbb{R}^3$ in the case when $c = 1$. If $c \neq 1$, we can simply use the above formula making a change of variables. In particular, consider the initial-value problem

$$\begin{cases} v_{tt} - c^2\Delta v = 0 & x \in \mathbb{R}^n \\ v(x,0) = \phi(x) \\ v_t(x,0) = \psi(x). \end{cases} \tag{7.3}$$

though a point, its information is immediately forgotten. This property, the *strong version of Huygens' principle*, is valid not only for $n = 3$, but in all odd-dimensional spaces. It does not apply, however, to the wave equation in spaces of even dimensionality. There, even though information still propagates at speed one, it does not do it through sharp fronts, leaving instead a trace behind as it passes through a point. Hence, when a tsunami shakes the (2D) ocean, it leaves significant wave action behind its leading front. We shall see now that this is the case in two dimensions, through an application of Hadamard's *method of descent*. The same methodology applies in all even dimensions $n = 2d$, once we have the general solution to the initial value problem in the odd-dimensional space $n = 2d + 1$.

3.5 The method of descent

The method of descent, also due to Hadamard, consists simply in thinking of any solution to the wave equation in even ($n = 2d$) dimensions as a solution in one more dimension which does not depend on one of the space variables. In two dimensions, in particular, we can write

$$u(x, y, t) = \tilde{u}(x, y, z, t)\,,$$

where $\tilde{u}$ is a solution to the three–dimensional wave equation with initial data that do not depend on z:

$$\tilde{u}(x, y, z, 0) = \tilde{g}(x, y, z) = g(x, y)\,, \quad \tilde{u}_t(x, y, z, 0) = \tilde{h}(x, y, z) = h(x, y)\,.$$

For $\tilde{u}$ we have the exact formula (53), so the same applies to u. However, by definition, the corresponding $\tilde{G}(x, r)$ and $\tilde{H}(x, r)$ are the spherical means over three–dimensional balls of functions $\tilde{g}(x)$ and $\tilde{h}(x)$ that do not depend on z. Then we have

$$\tilde{G}(x, r) = \int_{B(x,r)} \tilde{g}(s)\, dS_r = \int_{S(x,r)} g(s)\, J\, dA\,,$$

where B is the surface of a three–dimensional sphere, S is the surface of a two–dimensional circle, and J is the Jacobian

$$J = \frac{r}{|s - x|}$$

that projects one area element onto the other. For our purposes, it is enough to notice that now the formula for u involves integrals over the *interior* of circles of radius t, not just their circumference. Hence the strong version of Huygens principle does not apply in two dimensions: the solution to the wave equation at point x and time t depends on all the initial data within a circle of radius t around x, not just on their values and derivatives on the circumference $|y - x| = t$.

Suppose v is a solution of (7.3). Then define $u(x,t) \equiv v(x, \frac{1}{c}t)$. Then

$$u_{tt} - \Delta u = \frac{1}{c^2} v_{tt} - \Delta v = 0$$

implies u is a solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \dfrac{1}{c}\psi(x). \end{cases}$$

Therefore, u is given by Kirchoff's formula above. Now by making the change of variables $\tilde{t} = \frac{1}{c}t$, we see that

$$v(x,\tilde{t}) = u(x, c\tilde{t}),$$

and we arrive at the solution for (7.3),

$$v(x,t) = \frac{1}{4\pi c^2 t^2} \int_{\partial B(x,ct)} [\phi(y) + \nabla\phi(y) \cdot (y-x) + t\psi(y)]\, dS(y).$$

7.2 Method of Descent

In this section, we use Kirchoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimensions. This technique is known as the **method of descent**. This technique can be used in general to find the solution of the wave equation in even dimensions, using the solution of the wave equation in odd dimensions.

Solution for $\mathbf{n} = 2$.

Suppose u is a solution of the initial value problem for the wave equation in two dimensions,

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^2, t \geq 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

We will find a solution in the 2-D case, by using the solution to the 3-D problem. Let $u(x_1, x_2, t)$ be the solution to the 2-D problem. Define

$$\widetilde{u}(x_1, x_2, x_3, t) \equiv u(x_1, x_2, t).$$

Therefore,

$$\begin{aligned} \widetilde{u}(x_1, x_2, x_3, 0) &\equiv u(x_1, x_2, 0) = \phi(x_1, x_2) \\ \widetilde{u}_t(x_1, x_2, x_3, 0) &\equiv u(x_1, x_2, 0) = \psi(x_1, x_2). \end{aligned}$$

Clearly, $\widetilde{u}(x_1, x_2, x_3, t)$ is a solution of the 3D wave equation with initial data $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$,

$$\begin{cases} \widetilde{u}_{tt} - \widetilde{u}_{x_1x_1} - \widetilde{u}_{x_2x_2} - \widetilde{u}_{x_3x_3} = 0 \\ \widetilde{u}(x_1, x_2, x_3, 0) = \widetilde{\phi}(x_1, x_2, x_3) = \phi(x_1, x_2) \\ \widetilde{u}_t(x_1, x_2, x_3, 0) = \widetilde{\psi}(x_1, x_2, x_3) = \psi(x_1, x_2). \end{cases}$$

Now we can solve the 3D wave equation using Kirchoff's formula. In particular, our solution is given by

$$\widetilde{u}(x_1, x_2, 0, t) = \fint_{\partial\overline{B}(\overline{x},t)} [\widetilde{\phi}(y) + \nabla\widetilde{\phi}(y) \cdot (y - x) + t\widetilde{\psi}(y)]\, dS(y)$$

where $\overline{B}(\overline{x}, t)$ is the ball of radius t in $\mathbb{R}^3$ about the point $\overline{x} = (x_1, x_2, 0)$. Now we note that

$$\begin{aligned}
\fint_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y) &= \frac{1}{4\pi t^2} \int_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y) \\
&= \frac{1}{2\pi t^2} \int_{B(x,t)} \phi(y)(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy
\end{aligned}$$

where $B(x, t)$ is the ball in $\mathbb{R}^2$ of radius t about the point $x = (x_1, x_2)$ and $\gamma(y) = (t^2 - |y - x|^2)^{1/2}$. Therefore,

$$\nabla\gamma(y) = -\frac{y - x}{(t^2 - |y - x|^2)^{1/2}}$$

which implies

$$(1 + |\nabla\gamma(y)|^2)^{1/2} = \left(\frac{t^2}{t^2 - |y - x|^2}\right)^{1/2}.$$

Therefore,

$$\fint_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\phi(y)}{(t^2 - |y - x|^2)^{1/2}}\, dy.$$

Similarly,

$$\fint_{\partial\overline{B}(\overline{x},t)} t\widetilde{\psi}(y)\, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t^2\psi(y)}{(t^2 - |y - x|^2)^{1/2}}\, dy$$

and

$$\fint_{\partial\overline{B}(\overline{x},t)} \nabla\widetilde{\phi}(y) \cdot (y - x)\, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\nabla\phi(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}}\, dy.$$

Therefore, the solution of the initial-value problem for the wave equation in $\mathbb{R}^2$ (with $c = 1$) is given by

$$\boxed{u(x, t) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\phi(y) + t^2\psi(y) + t\nabla\phi(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}}\, dy.} \tag{7.4}$$

Again, by making a change of variables, we see that the solution of the wave equation in two dimensions is given by

$$u(x, t) = \frac{1}{2\pi c^2 t^2} \int_{B(x,ct)} \frac{ct\phi(y) + ct^2\psi(y) + ct\nabla\phi(y) \cdot (y - x)}{(c^2t^2 - |y - x|^2)^{1/2}}\, dy.$$

7.3 Huygen's Principle

Note that for the initial-value problem for the wave equation in three dimensions, the value of the solution at any point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$ depends only on the values of the initial data on the *surface of the ball* of radius ct about the point $x \in \mathbb{R}^3$; that is, on $\partial B(x,ct)$. That is to say, disturbances all travel at exactly speed c. This is known as **Huygens's principle**. In contrast, in two dimensions, the value of the solution u at the point (x,t) depends on the initial data *within* the ball of radius ct about the point $x \in \mathbb{R}^2$. Signals don't all travel at speed c. In fact, as we will see, for $n \geq 3$ and odd, Huygens's principle holds. That is, all signals travel at exactly speed c. In even dimensions, however, that is not the case.

7.4 Wave Equation in $\mathbb{R}^n$, $n > 3$

Ref: Evans, Sec. 2.4.1

Note: In this section, we assume $c = 1$. *For* $c \neq 1$, *we can make a change of variables to find the solution.*

Odd dimensions.

For the case of odd dimensions, we use the method of spherical means as we did for the case of $n = 3$. Let $n = 2k+1$. Let $x \in \mathbb{R}^n$. Define

$$\begin{aligned} v(x;r,t) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{u}(x;r,t)) \\ g(x;r) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\phi}(x;r)) \\ h(x;r) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\psi}(x;r)). \end{aligned}$$

Notice that for $k = 1$, these definitions reduce to those functions introduced in the case $n = 3$.

First, we will show that $v(x;r,t)$ solves the wave equation on the half-line with Dirichlet boundary conditions.

Lemma 4. *For each integer* $k \geq 1$, *for each* $x \in \mathbb{R}^n$, *the function* $v(x;r,t)$ *defined above solves*

$$\begin{cases} v_{tt} - v_{rr} = 0 & r > 0 \\ v(x;r,0) = g(x;r) \\ v_t(x;r,0) = h(x;r) \\ v(x;0,t) = 0. \end{cases}$$

The proof relies on the following lemma.

Lemma 5. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be* C^{k+1}. *Then for* $k = 1, 2, \ldots$

1.

$$\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right)$$

One of the "holy Trinity" of partial differential equations is the second–order wave equation, the canonical example of a hyperbolic PDE. In n dimensions the equation takes the form

$$\boxed{u_{tt} = \Delta u,} \tag{1}$$

where Δ is the Laplacian operator, $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. A wave speed c can be included by a factor c^2 on the right-hand side. Since (1) is of second order in t, a well-posed initial-value problem for this equation would normally involve two initial conditions such as $u(x,0)$ and $u_t(x,0)$.

The wave equation describes linear, nondispersive wave propagation. For example, Figure 1 presents a pair of images that show the outward spread of a circular pulse in 2D. At $t = 0$ we begin with a cone of radius 0.1 with $u_t(0) = 0$. At $t = 2$, the cone has spread to a concentric ring of outer radius exactly 2.1.

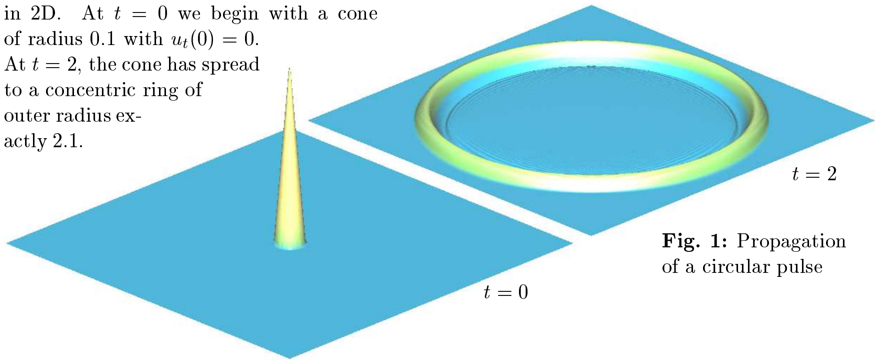

Fig. 1: Propagation of a circular pulse

The wave equation arises in numerous applications. The classical 1D example is the vibration of an ideal string ($\to$ *ref*), and in 2D this becomes the vibration of an ideal membrane or drum ($\to$ *ref*). In 3D, the most famous example is the propagation of sound waves in a gas or liquid. Indeed, equation (1) is often called the *acoustic wave equation* to distinguish it from the more complicated *elastic wave equation* ($\to$ *ref*), where the presence of stiffness as well as compressibility leads to the appearance of two distinct kinds of waves.

Being hyperbolic, the wave equation has finite speed of propagation for all information—namely 1, for the equation as written in (1). A curious property known as *Huygens' principle* is as follows. In dimensions $n = 3, 5, 7, 9, \ldots$, all information propagates under (1) at speed exactly 1, never slower. Thus, the light from a bulb flashed at $t = 0$ passes the observer at a later time as a pure delta function. In dimensions $n = 1, 2, 4, 6, 8, \ldots$, on the other hand, a finite fraction of the energy may travel more slowly than at speed 1, so the observer sees a delta function flash followed by a decaying tail. To illustrate this phenomenon, Figure 2 shows the result at time $t = 1$ of the initial condition $u_t(x,0) = \max\{0, 1 - 10|x|\}$ in dimensions $1, 2, 3, 4, 5, 6$, where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

In an unbounded domain, the wave equation is readily investigated by Fourier analysis. Separation of variables leads to the observation that for any n-vector k, known as the *wave number*, there are

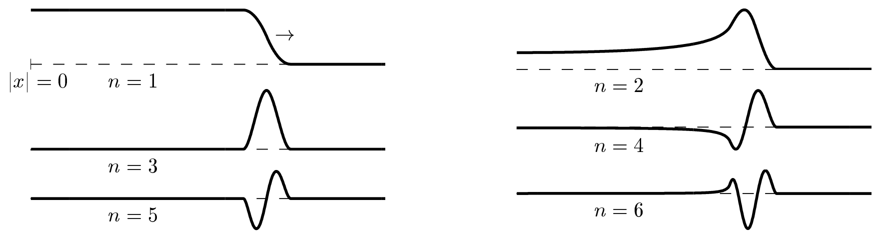

Fig. 2: Huygens' principle: zero tails in odd dimensions $n \geq 3$

plane wave solutions of (1) of the form

$$u(x,t) = e^{\mathrm{i}(\omega t + k\cdot x)}, \tag{2}$$

where $k \cdot x = k_1 x_1 + \cdots + k_n x_n$, so long as $\omega = \pm |k|$. This condition relating the frequency to the wave number is the *dispersion relation* for (1). By a Fourier integral, general solutions to (1) can be obtained by the superposition of plane waves (2), and under suitable technical assumptions, all solutions can be written this way.

In a bounded domain Ω, separation of variables in (1) leads to oscillatory solutions of the form $e^{\mathrm{i}\omega_j t}\phi_j(x)$, where the functions $\phi_j(x)$ are eigenfunctions of the Laplacian operator for Ω ($\to$ *ref*). The allowed frequencies ω_j now belong to a discrete set, and general solutions can be obtained via superpositions as series rather than integrals. If Ω is a rectangle, a disk, or a ball, the eigenfunctions are trigonometric functions, Bessel functions, or spherical harmonics, respectively.

Another technique in the study of the wave equation is *Hadamard's method of descent*. The idea here is that any solution in dimension n can be thought of as a solution in dimension $n+1$ that happens to be invariant with respect to one coordinate. In particular, solutions in even dimensions can be obtained from solutions in the odd dimension one higher, which are relatively elementary superpositions of expanding spheres thanks to Huygens' principle.

In applications of the wave equation, boundaries and variable coefficients are important, including discontinuities in the sound speed. Among the phenomena that arise are reflection, refraction, and diffraction. Just as the field of fluid mechanics can be described without too much exaggeration as the study of the Navier–Stokes equations ($\to$ *ref*), so the field of acoustics is more or less the study of the wave equation. There are enough subtleties here to fill books, and careers—even if we confine our attention to the fascinating subfield of the physics of musical instruments.

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2.

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{d^j\phi}{dr^j}(r)$$

where each β_j^k is independent of ϕ.

3.

$$\beta_0^k = 1 \cdot 3 \cdot 5 \cdots (2k-1).$$

Proof. Use induction. □

Proof of Lemma 4.

$$\begin{aligned}
v_{rr} &= \partial_r^2\left[\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}(x;r,t)\right)\right] \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\overline{u}_r(x;r,t)\right) \qquad \text{by Lemma 5} \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}\frac{d}{dr}\right)\left(r^{2k}\overline{u}_r(x;r,t)\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}[2kr^{2k-1}\overline{u}_r + r^{2k}\overline{u}_{rr}]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left[\frac{2k}{r}\overline{u}_r + \overline{u}_{rr}\right]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left[\frac{n-1}{r}\overline{u}_r + \overline{u}_{rr}\right]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}_{tt}\right) \\
&= \partial_t^2\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}_{tt}\right) \\
&= v_{tt}
\end{aligned}$$

Clearly, $v(x;r,0) = g(x;r)$, $v_t(x;r,0) = h(x;r)$ and $v(x;0,t) = 0$. Therefore, the lemma is proved. □

Now $v(x;r,t)$ is a solution of the one-dimensional wave equation on the half-line with Dirichlet boundary condition implies for $0 \le r \le t$, the solution is given by

$$v(x;r,t) = \frac{1}{2}[g(x;r+t) - g(x;t-r)] + \frac{1}{2}\int_{t-r}^{t+r} h(x;y)\,dy.$$

Recall:

$$u(x,t) = \lim_{r\to 0} \overline{u}(x;r,t).$$

Now

$$
\begin{aligned}
v(x;r,t) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}(x;r,t)\right) \\
&= \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}\overline{u}(x;r,t) \\
&= \beta_0^k r\overline{u}(x;r,t) + \beta_1^k r^2\overline{u}_r(x;r,t) + \ldots + \beta_{k-1}^k r^k\frac{\partial^{k-1}}{\partial r^{k-1}}\overline{u}(x;r,t).
\end{aligned}
$$

Therefore,

$$
\beta_0^k r\overline{u}(x;r,t) = v(x;r,t) - \beta_1^k r^2\overline{u}_r(x;r,t) - \ldots - \beta_{k-1}^k r^k\frac{\partial^{k-1}}{\partial r^{k-1}}\overline{u}(x;r,t),
$$

which implies

$$
\overline{u}(x;r,t) = \frac{v(x;r,t)}{\beta_0^k r} - \frac{\beta_1^k}{\beta_0^k}r\overline{u}_r(x;r,t) - \ldots - \frac{\beta_{k-1}^k}{\beta_0^k}r^{k-1}\frac{\partial^{k-1}}{\partial r^{k-1}}\overline{u}(x;r,t).
$$

Therefore,

$$
\begin{aligned}
u(x,t) &= \lim_{r\to 0}\left[\frac{v(x;r,t)}{\beta_0^k r} - \frac{\beta_1^k}{\beta_0^k}r\overline{u}_r(x;r,t) - \ldots - \frac{\beta_{k-1}^k}{\beta_0^k}r^{k-1}\frac{\partial^{k-1}}{\partial r^{k-1}}\overline{u}(x;r,t)\right] \\
&= \lim_{r\to 0}\frac{v(x;r,t)}{\beta_0^k r} \\
&= \lim_{r\to 0}\frac{1}{\beta_0^k}\left[\frac{g(x;t+r)-g(x;t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r}h(x;y)\,dy\right] \\
&= \frac{1}{\beta_0^k}\left[\partial_t g(x;t) + h(x;t)\right]
\end{aligned}
$$

where $\beta_0^k = 1\cdot 3\cdot 5\cdots(2k-1)$. Recall

$$
g(x;r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}\overline{\phi}(x;r)).
$$

Now $n = 2k+1$ implies $k = (n-1)/2$, and, therefore,

$$
g(x;t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)}\phi(y)\,dS(y)\right).
$$

And,

$$
h(x;r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}\overline{\psi}(x;r)).
$$

Therefore,

$$
h(x;t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)}\psi(y)\,dS(y)\right).
$$

Therefore,

$$u(x,t) = \frac{1}{\gamma_n}[\partial_t g(x;t) + h(x;t)]$$

implies

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} &\left(t^{n-2} \fint_{\partial B(x,t)} \phi(y)\,dS(y)\right) \\ + \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} &\left(t^{n-2} \fint_{\partial B(x,t)} \psi(y)\,dS(y)\right) \end{aligned}}$$

where $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$.

Even dimensions.

As in the case of $n = 2$ dimensions, we use the method of descent. In particular, suppose $u(x_1, \ldots, x_n, t)$ is a solution of the wave equation in $\mathbb{R}^n$ with initial data $u(x_1, \ldots, x_n, 0) = \phi(x_1, \ldots, x_n)$ and $u_t(x_1, \ldots, x_n, 0) = \psi(x_1, \ldots, x_n)$. Then define

$$\begin{aligned} \widetilde{u}(x_1, \ldots, x_{n+1}, t) &\equiv u(x_1, \ldots, x_n, t) \\ \widetilde{\phi}(x_1, \ldots, x_{n+1}) &\equiv \phi(x_1, \ldots, x_n) \\ \widetilde{\psi}(x_1, \ldots, x_{n+1}) &\equiv \psi(x_1, \ldots, x_n). \end{aligned}$$

Therefore, $\widetilde{u}$ is a solution of the wave equation in $\mathbb{R}^{n+1}$, where now $n+1$ is odd. Therefore, from the formula above for the case when the dimension is odd, our solution at the point $(\overline{x}, t) = (x_1, \ldots, x_n, 0, t)$ is given by

$$\begin{aligned} \widetilde{u}(\overline{x},t) = \frac{1}{\gamma_{n+1}}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} &\left(t^{n-1} \fint_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y)\right) \\ + \frac{1}{\gamma_{n+1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} &\left(t^{n-1} \oint_{\partial \overline{B}(\overline{x},t)} \widetilde{\psi}(y)\,dS(y)\right) \end{aligned}$$

where $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdots (n-1)$, and where $\overline{B}(\overline{x}, t)$ is the ball in $\mathbb{R}^{n+1}$ of radius t about the point $\overline{x} = (x_1, \ldots, x_n, 0)$.

Now,

$$\fint_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y) = \frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y).$$

But, notice $\partial \overline{B}(\overline{x}, t) \cap \{y_{n+1} \geq 0\}$ is the graph of the function $\gamma(y) \equiv (t^2 - |y-x|^2)^{1/2}$. And, similarly, $\partial \overline{B}(\overline{x}, t) \cap \{y_{n+1} \leq 0\}$ is the graph of $-\gamma$. Therefore,

$$\frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y) = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \phi(y)(1 + |\nabla\gamma(y)|^2)^{1/2}\,dy$$

Now

$$(1 + |\nabla\gamma(y)|^2)^{1/2} = t(t^2 - |y-x|^2)^{-1/2}.$$

Therefore,

$$
\begin{aligned}
⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y) &= \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{t\phi(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy \\
&= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)\alpha(n)t^n} \int_{B(x,t)} \frac{\phi(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy \\
&= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)} ⨍_{B(x,t)} \frac{\phi(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy.
\end{aligned}
$$

Therefore, our solution formula is given by

$$
\begin{aligned}
u(x,t) &= \frac{1}{\gamma_{n+1}} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y)\right) \\
&\quad + \frac{1}{\gamma_{n+1}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\psi}(y)\, dS(y)\right) \\
&= \frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \left[\left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n ⨍_{\partial B(x,t)} \frac{\phi(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy\right) \right. \\
&\quad \left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n ⨍_{\partial B(x,t)} \frac{\psi(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy\right) \right].
\end{aligned}
$$

Now $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdots (n-1)$ and

$$
\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)},
$$

where $\Gamma(n)$ is the gamma function,

$$
\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx.
$$

Therefore,

$$
\begin{aligned}
\frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (n-1)} \cdot \frac{2\frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}}{(n+1)\frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+3}{2})}} \\
&= \frac{1}{1 \cdot 3 \cdot 5 \cdots (n+1)} \cdot \frac{1}{\pi^{1/2}} \cdot \frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+2}{2})}.
\end{aligned}
$$

Using properties of the gamma function, namely that

$$
\Gamma(m+1) = m\Gamma(m)
$$

and

$$
\Gamma(1/2) = \pi^{1/2},
$$

we can conclude that

$$\Gamma\left(\frac{n+3}{2}\right) = \left(\frac{n+1}{2}\right) \cdot \left(\frac{n-1}{2}\right) \cdots \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)$$

and

$$\Gamma\left(\frac{n+2}{2}\right) = \left(\frac{n}{2}\right) \cdot \left(\frac{n-2}{2}\right) \cdots \left(\frac{2}{2}\right).$$

And, therefore,

$$\frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} = \frac{1}{2 \cdot 4 \cdots (n-2) \cdot n}$$

Therefore, the solution of the wave equation in even dimensions is given by

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{\gamma_n}\Bigg[&\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n \fint_{B(x,t)} \frac{\phi(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy\right) \\ &+ \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n \fint_{B(x,t)} \frac{\psi(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy\right)\Bigg] \end{aligned}}$$

where $\gamma_n \equiv 2 \cdot 4 \cdots (n-2) \cdot n$.

7.5 Wave Equation in $\mathbb{R}^n$ with a source.

In this section, we consider the inhomogeneous wave equation in $\mathbb{R}^n$. First, recall Duhamel's Principle. If $S(t)$ is the solution operator for the first-order initial-value problem

$$\begin{cases} U_t + AU = 0 \\ U(0) = \Phi, \end{cases}$$

then the solution of the inhomogeneous problem

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi \end{cases}$$

"should" be given by

$$U(t) = S(t)\Phi + \int_0^t S(t-s)F(s)\,ds.$$

Now consider the initial-value problem for the wave equation in $\mathbb{R}^n$,

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{7.5}$$