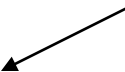
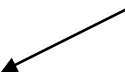




# An Introduction to wavelets.

# Overview of presentation

- What need do wavelets satisfy?  Understanding
- The STFT (Short Time Fourier Transform)
- The CWT (Continuous Wavelet Transform)  Practical
- The Discrete Versions of the Wavelet Transform
- Some Applications
- Very Brief Ridgelets and Curvelets

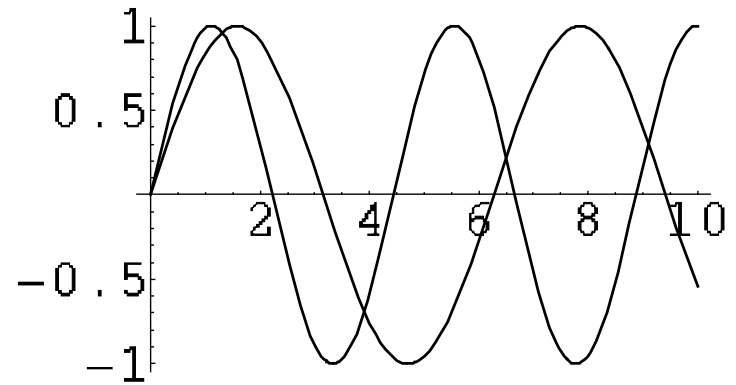
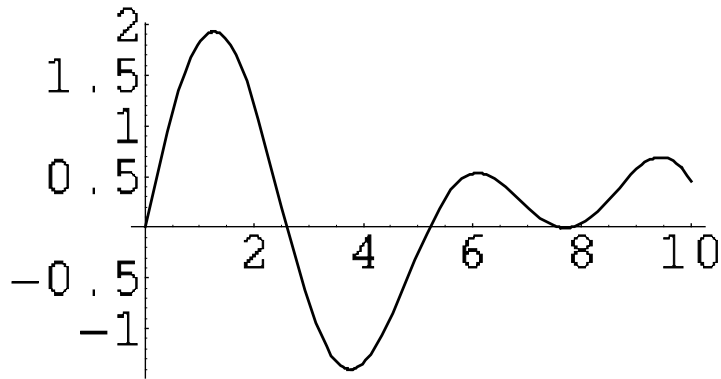
# What's wrong with Fourier?

Well what is it used for? To:

- Solve differential and difference equations.
- Accurately store and transmit data.
- Obtain a different perspective for a signal  
High and low frequencies

# What's wrong with Fourier?

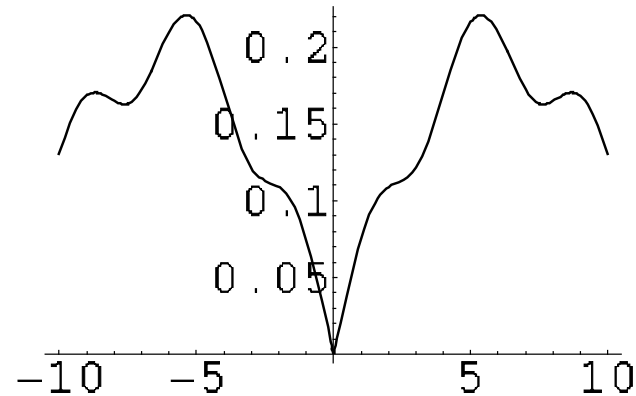
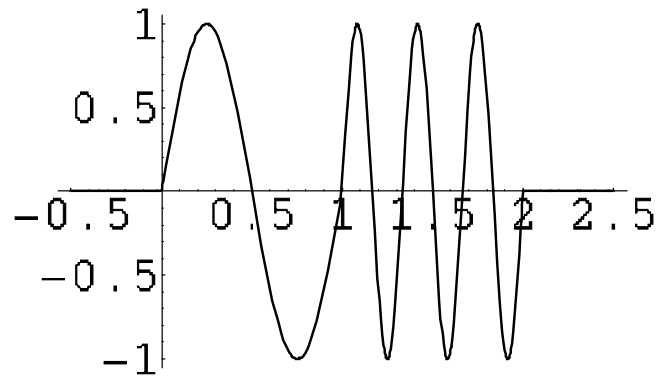
The Fourier Transform (Series) method is used to decompose a signal into its global frequency components. EXAMPLE 1:



Very helpful for this signal!

# What's wrong with Fourier?

The Fourier Transform (Series) method is used to decompose a signal into its global frequency components. EXAMPLE 2:



Not quite as helpful here.



# What's wrong with Fourier?.....

## **ANSWER:**

- 1.) The Fourier Transform is unable to pick out local frequency content.
- 2.) It has a “hard time” representing functions that are oscillatory or discontinuous (Gibbs phenomena)

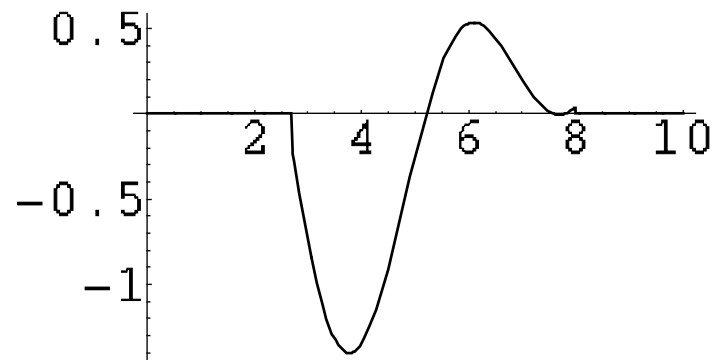
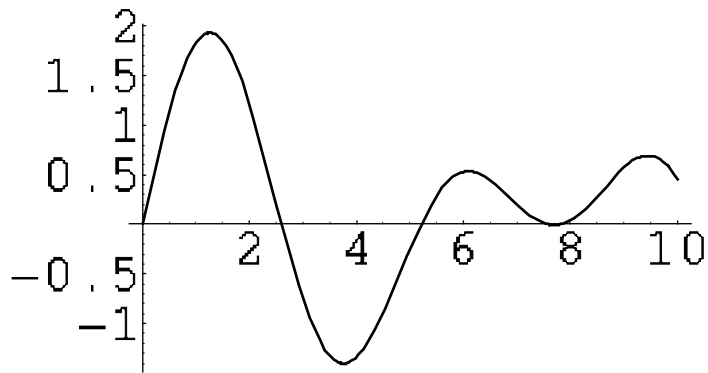


# What's wrong with Fourier?.....

So how can we get at the local frequency information? Most obvious solution: use a window!

# The WFT (Windowed Fourier Transform)

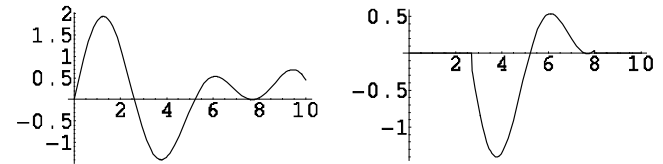
STEP 1: Window the function using  $g(t - t_0)$



Example of what happens when you window a function, with a given window function  $g(t-5)$ .

# The WFT (Windowed Fourier Transform)

STEP 2: Take the Fourier Transform of the resulting signal



$$STFT_g(\tau, k) = \int_{-\infty}^{\infty} f(t) g^*(t - \tau) e^{-2\pi i k t} dt$$

Complex conjugate

A function of two variables.

# The resolution of the WFT

FACT: There is a minimum time and frequency resolution associated with each window function  $g$ .

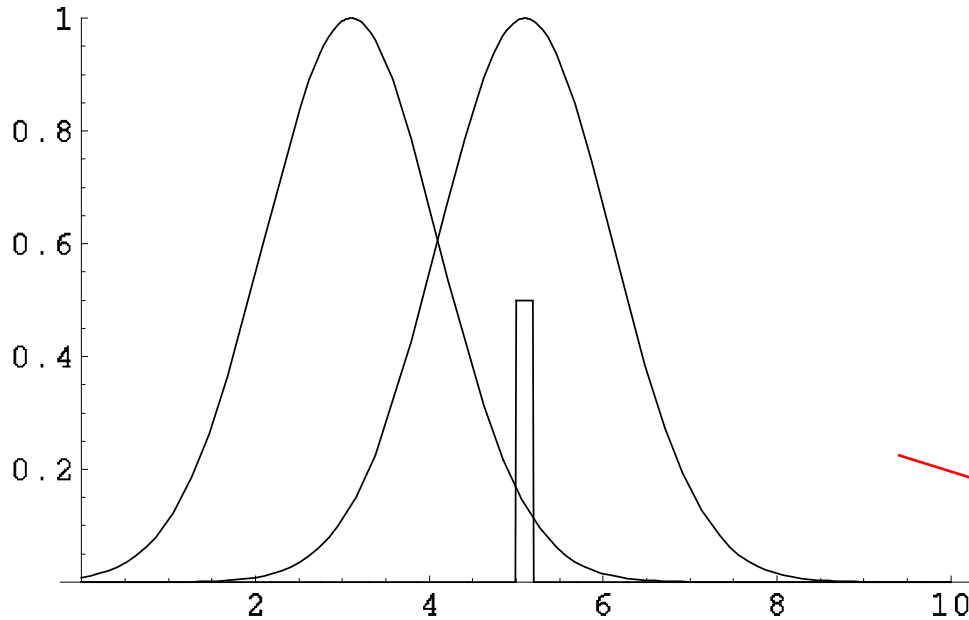
$$STFT_g(\tau, k) = \int_{-\infty}^{\infty} f(t)g^*(t - \tau)e^{-2\pi ikt} dt$$



A function of two variables.

# The resolution of the WFT

Consider the time resolution:



The window centered at 3 and the window centered at 5 both observe the impulse.

$$STFT_g(\tau, k) = \int_{-\infty}^{\infty} f(t)g^*(t - \tau)e^{-2\pi ikt} dt$$

$$\Delta t^2 \approx \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}.$$

# The resolution of the WFT

Consider the frequency resolution: What frequencies are present in the STFT of a pure sinusoid at time 0?

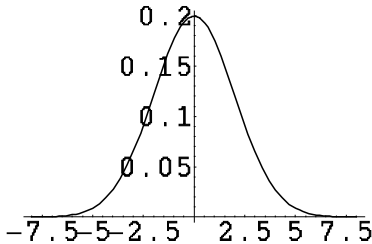
$$\begin{aligned}STFT_g(k, 0) &= \int_{-\infty}^{\infty} \sin(2\pi k_0 t) g^*(t - 0) e^{-2\pi i k t} dt \\ &= \frac{i}{4} \int_{-\infty}^{\infty} [\delta(x - k_0) - \delta(x + k_0)] G^*(k - x) dx\end{aligned}$$

By convolution theorem

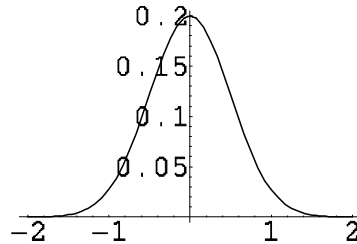
$$\Delta k^2 \approx \frac{\int_{-\infty}^{\infty} k^2 |G(k)|^2 dk}{\int_{-\infty}^{\infty} |G(k)|^2 dk}$$

# The resolution of the WFT

Delta t ~ width of g, while Delta k ~ width of G. But these widths are inversely proportional: We get an **uncertainty principle** relating the resolutions.



$$\Delta t^2 \approx \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}$$



$$\Delta k^2 \approx \frac{\int_{-\infty}^{\infty} k^2 |G(k)|^2 dk}{\int_{-\infty}^{\infty} |G(k)|^2 dk}$$



$$\Delta t \Delta k \geq \frac{1}{4\pi}$$

# Summary of the WFT

- 1.) To get local frequency information, a window function,  $g(t)$ , is first chosen.
- 2.) The WFT is taken as follows:

$$STFT_g(\tau, k) = \int_{-\infty}^{\infty} f(t)g^*(t - \tau)e^{-2\pi ikt} dt$$

- 3.) The WFT is a function of two variables, time and frequency. It allows one to observe the time-frequency distribution of the energy of a signal.
- 4.) With each window function,  $g$ , is associated both a time and a frequency minimum resolution. Once  $g$  is chosen, these resolutions are fixed and are the same at all frequencies and times. These resolutions are related by an uncertainty relation.

# Summary of the WFT

Typical Example: the WFT of a signal may tell us that at time  $\tau = 5 \pm 0.5$ , there are two dominant frequency components: one at  $k = 10 \pm 0.9$ , and one at  $10^{10} \pm 0.9$ .

Clearly, the frequency resolution is more significant for the lower frequency.

To treat each frequency “fairly”, we would prefer to have:

$$\frac{\Delta k}{|k|} = c$$

# Almost there....

So what can we do to get  $\frac{\Delta k}{|k|} = c$  ? Well...

$$c = \frac{\Delta k}{|k|} \approx \frac{\Delta G}{|k|} \approx \frac{1}{|k| \Delta g}$$

$$\Delta g = \frac{1}{c|k|} \longrightarrow$$

The width of the window should be allowed to vary inversely with the magnitude of the frequency.

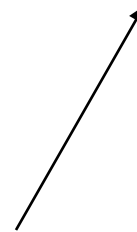
Finally, we get to our first Wavelet transform:

# Our first wavelet transform:

$$CWT(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} f(t) g^*\left(\frac{t - \tau}{\alpha}\right) e^{-2\pi i k_0 \frac{t - \tau}{\alpha}} dt$$



This is just here for weighting.



Note that as alpha increases: the frequency decreases, and the window function expands.

# Our first wavelet transform:

$$CWT(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} f(t) g^*\left(\frac{t - \tau}{\alpha}\right) e^{-2\pi i k_0 \frac{t - \tau}{\alpha}} dt$$

Time shift now also in exponential. Can write:

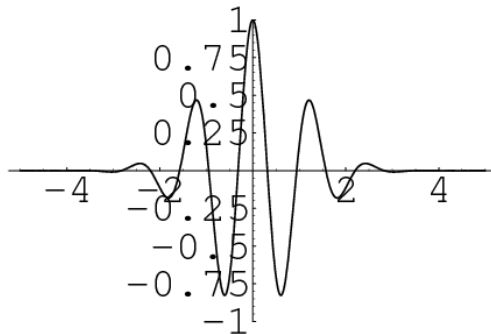
$$CWT_h(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} f(t) h^*\left(\frac{t - \tau}{\alpha}\right) dt$$

# Our first wavelet transform:

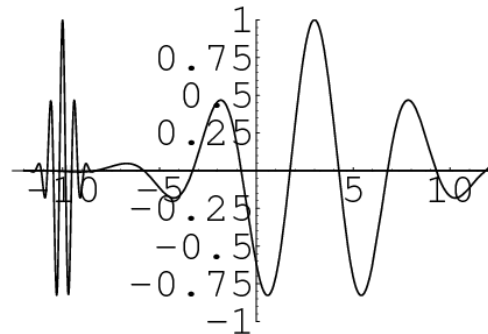
This is an inner product:

$$CWT(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} f(t) g^*\left(\frac{t - \tau}{\alpha}\right) e^{-2\pi i k_0 \frac{t - \tau}{\alpha}} dt$$

The basis is called a “wavelet” basis and consists of all the translated and dilated versions of the “mother” wavelet.



The mother Morlet wavelet.

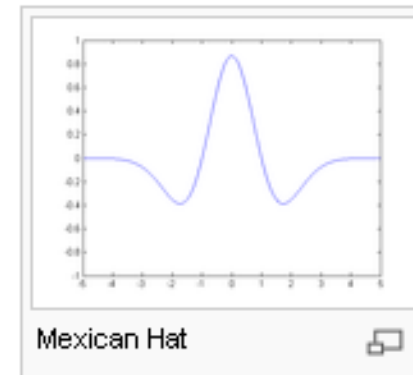
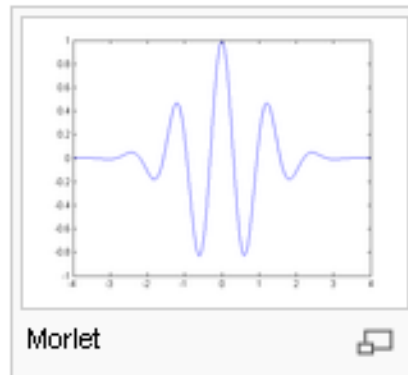
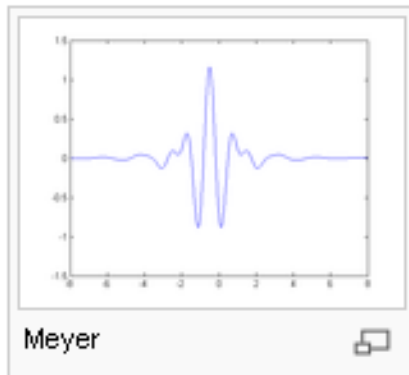


Two other Morlet wavelets

# The General CWT:

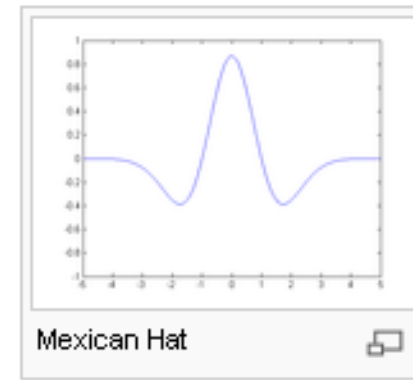
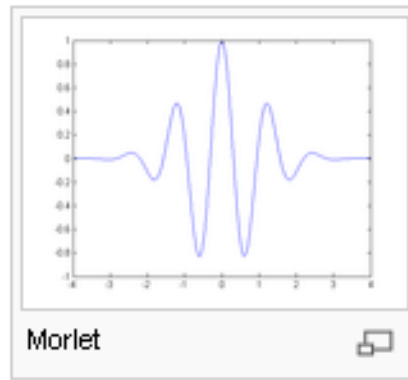
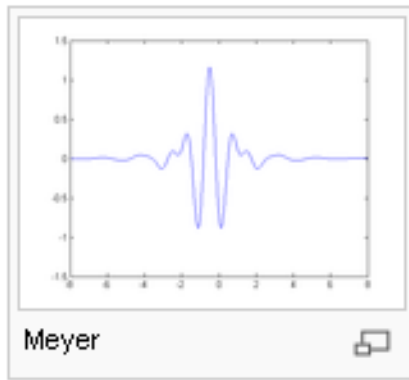
$$CWT_h(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} f(t) h^*\left(\frac{t - \tau}{\alpha}\right) dt$$

Can use any mother wavelet,  $h(t)$ , you want. Some typical examples:



Pic from wikipedia.org

# The General CWT:

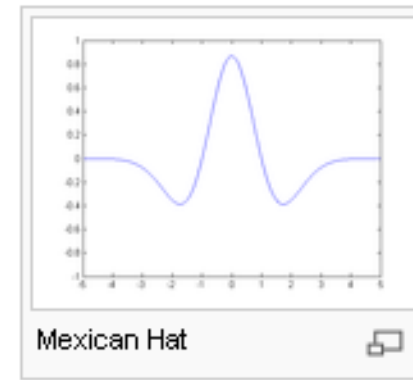
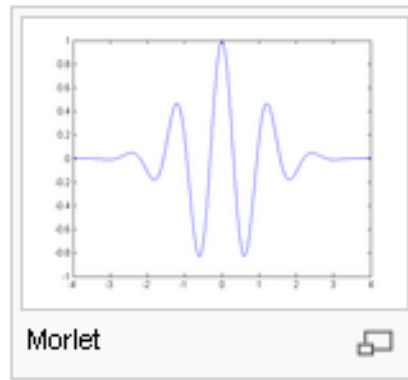
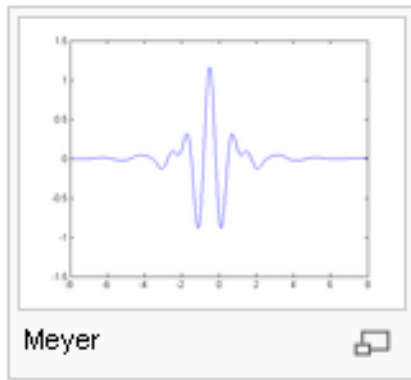


WHICH ONE?!

Pic from wikipedia.org

The task of determining which wavelet basis to use for a given application or signal is tough. One should consult with the Adapted Wavelet literature on this.

# The General CWT:



Can invert the CWT under certain conditions.

Pics from wikipedia.org

$$f(t) = \int_{-\infty}^{\infty} \int_0^{\infty} CWT_h(\tau, \alpha) h_{\tau, \alpha}(t) \frac{d\alpha d\tau}{\alpha^2}$$

$$h_{\tau, \alpha} \equiv h\left(\frac{t-\tau}{\alpha}\right)$$



# Moving on: Discrete Versions

## ■ Less Redundant Bases.

Would like to sample at discrete times and scales, so as to reduce redundancy, but also still want to be able to obtain a reconstruction. This can be done, and the theory of wavelet frames provides the theoretical background linking the restrictions on the mother wavelets allowed given the amount of redundancy used. A “frame” is a basis set such as this that allows for approximate reconstruction.

# Moving on: Discrete Versions

## ■ Less Redundant Bases.

Wavelet Frames: Typically, for a frame, one samples the signal at equally spaced scales in the Log Scale axis. That is, we sample at octaves.

$$h_{j,k}(t) = a_0^{-j/2} h(a_0^{-j} t - kT) \quad j, k \in \mathbb{Z}$$

$$c_{j,k} = \int x(t) h_{j,k}^*(t) dt$$

$$AE_x < \sum_{j,k} |c_{j,k}|^2 < BE_x$$

$$x(t) \approx \frac{2}{A+B} \sum_j \sum_k c_{j,k} h_{j,k}(t)$$

If  $A = B$ , get a “tight frame” and basis acts like an orthonormal basis.





# Moving on: Discrete Versions

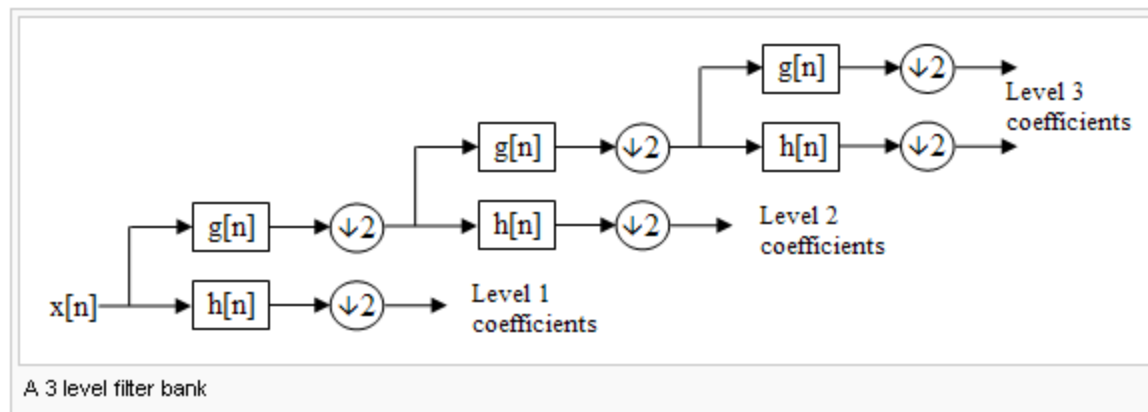
## ■ Discrete Wavelet Transform

When your signal is in vector form (or pixel form), the discrete wavelet transform may be applied. The idea of scale becomes slightly more difficult to define here.

# Moving on: Discrete Versions

## ■ Discrete Wavelet Transform

Effectively, the DWT is nothing but a system of filters. There are two filters involved, one is the “wavelet filter”, and the other is the “scaling filter”. The wavelet filter, is a high pass filter, while the scaling filter is a low pass filter.



Pic from wikipedia.org

Scaling Filter ~ Averaging Filter

Discrete Versions

Wavelet Filter ~ Details Filter

# Moving on: Discrete Versions

## ■ Discrete Wavelet Transform

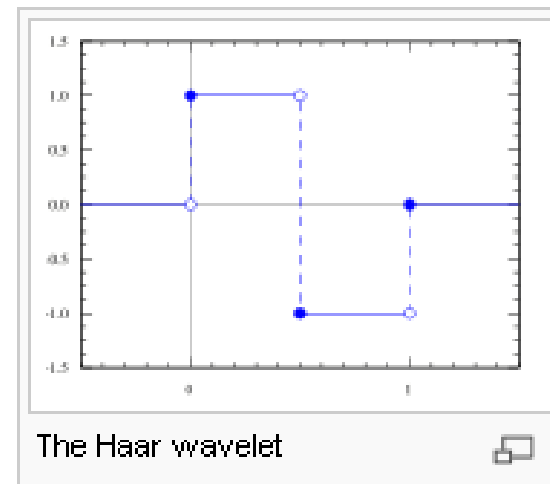
Example calculation: the Haar Wavelet.

$$\begin{aligned} [9 \ 7 \ 3 \ 5] & \quad (9+7)/2 = 8 \\ & \quad (9-7)/2 = 1 \\ & \quad (3+5)/2 = 4 \\ & \quad (3-5)/2 = -1 \end{aligned}$$

$$\begin{aligned} [8 \ 4 \ 1 \ -1] & \quad (8+4)/2 = 6 \\ & \quad (8-4)/2 = 2 \end{aligned}$$

$$[6 \ 2 \ 1 \ -1]$$

Observe: 1.) how the “scale” is changed 2.) the high pass is the QMF of the low pass (quadrature

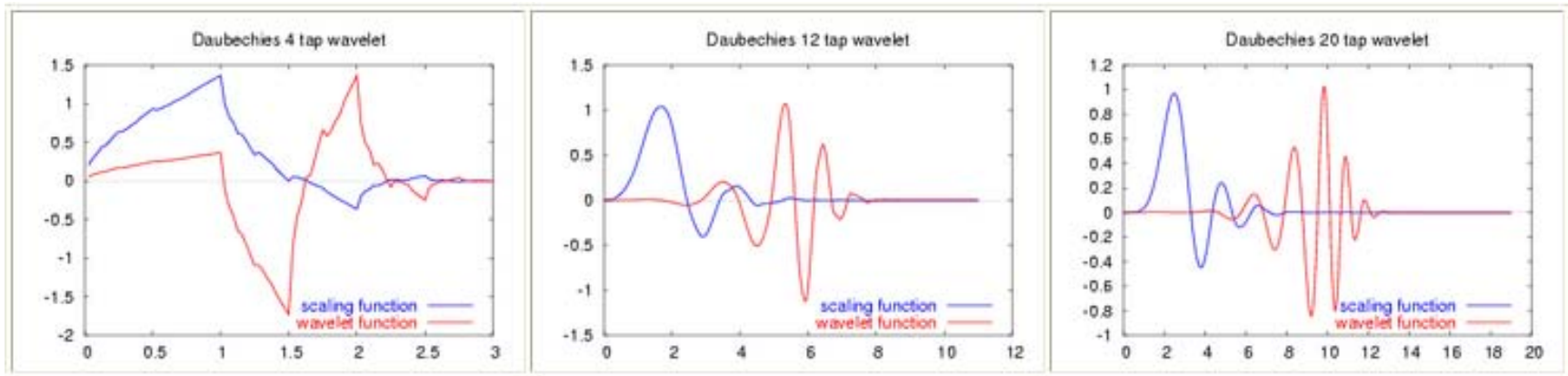


Pic from wikipedia.org

# Moving on: Discrete Versions

## ■ Discrete Wavelet Transform

For the DWT, it is difficult to construct orthogonal, continuous filters. Luckily other people have already done it for us. Ingrid Daubechies developed many such sets.

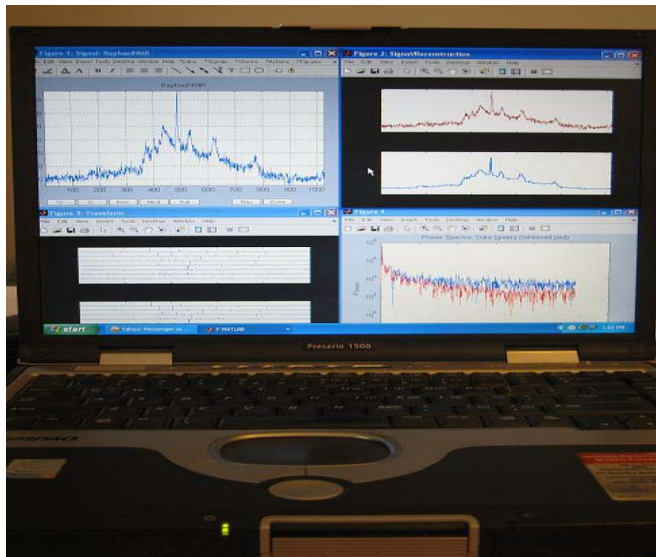


Pic from wikipedia.org

# Some Applications

“Most of the basic wavelet theory has been done...  
The future of wavelets lies in the as-yet uncharted  
territory of *applications*.” – Amara Graps, 1995

## WAVELAB



Available at:

<http://www.stat.stanford.edu/~wavelab/>

Created by David Donoho et. al.  
(including Ofer Levi)

# Some Applications

Thresholding for file size reduction.

(Pioneered by David Donoho)

$$[9 \ 7 \ 3 \ 5] \quad (9+7)/2 = 8$$

$$(9-7)/2 = 1$$

$$(3+5)/2 = 4$$

$$(3-5)/2 = -1$$

$$[8 \ 4 \ 1 \ -1] \quad (8+4)/2 = 6$$

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$$[6 \ 2 \ 1 \ -1]$$



↑ From [www.eetimes.com](http://www.eetimes.com)

# Some Applications

Thresholding for noise reduction.

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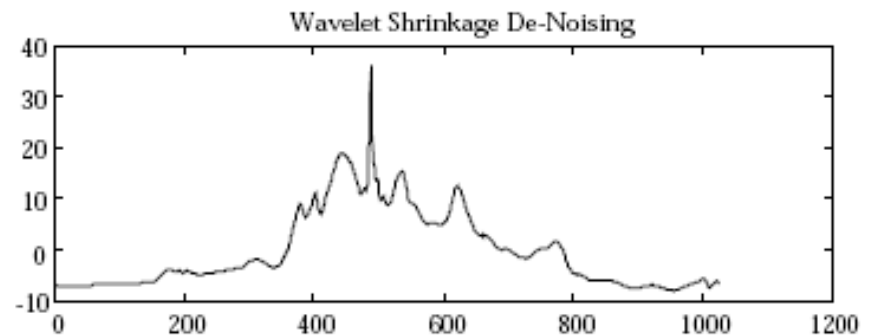
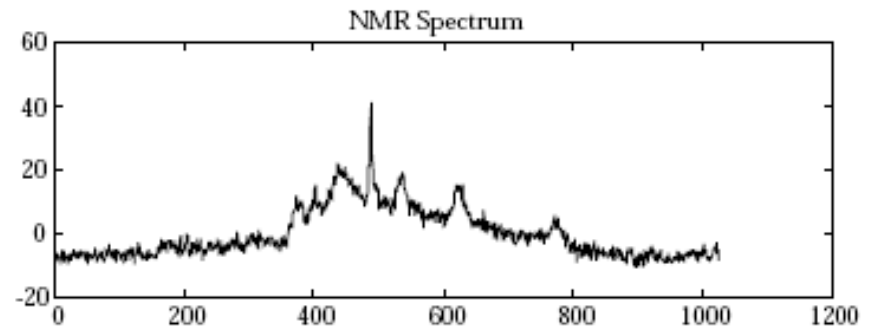
$$(3+5)/2 = 4$$

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$$[8 \ 4 \ 1 \ -1] \quad (8+4)/2 = 6$$

$$(8-4)/2 = 2$$

$$[6 \ 2 \ 1 \ -1]$$



From D. Donoho via Amara Graps  
Wavelet Introduction

# Some Applications

Image Enhancement.

$$[9 \ 7 \ 3 \ 5] \quad (9+7)/2 = 8$$

$$(9-7)/2 = 1$$

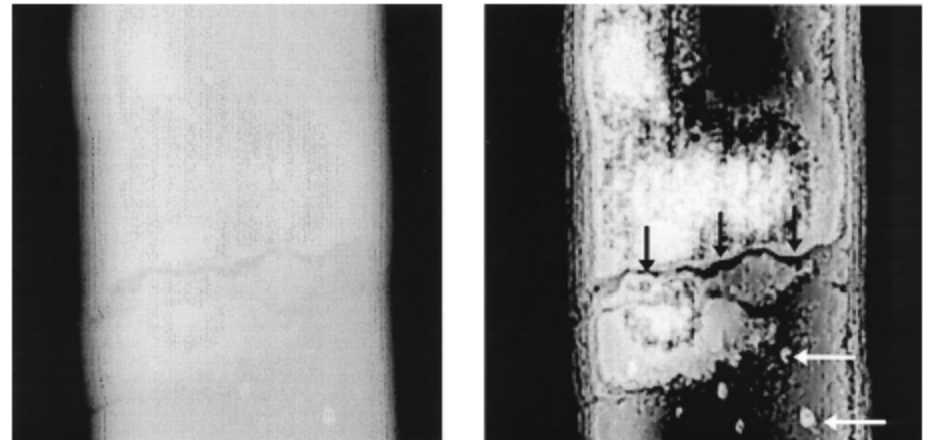
$$(3+5)/2 = 4$$

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$$[8 \ 4 \ 1 \ -1] \quad (8+4)/2 = 6$$

$$(8-4)/2 = 2$$

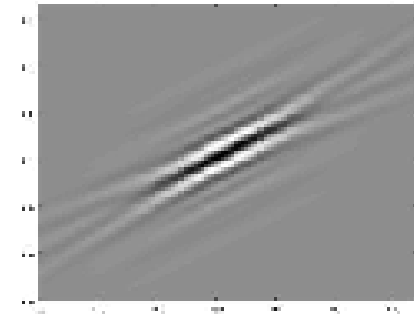
$$[6 \ 2 \ 1 \ -1]$$



From: "Wavelet-based image enhancement in x-ray imaging and tomography", by Bronnikov and Duifhuis

# Ridgelets and Curvelets

- A tight frame has been constructed.
- They provide efficient representation of images with “edge discontinuities”
- New enough so that as of 2002, the theory still in development.



$$\|f - \tilde{f}_m^F\|_2^2 \asymp m^{-1/2}, \quad m \rightarrow \infty$$

$$\|f - \tilde{f}_m^W\|_2^2 \asymp m^{-1}, \quad m \rightarrow \infty$$

$$\|f - \tilde{f}_m^C\|_2^2 \leq C \cdot m^{-2} (\log m)^3, \quad m \rightarrow \infty.$$



© Bruce Teleky