the maximum of $u_i$ variables with the common distribution

$$\Pr\{Y \leqq y\} = y^{1/t}$$

for $y \in [0, 1]$. Homogeneity of the expression (1) completes the proof as before. For irrational values of the $w_i$, the proof can be extended by continuity. Thus, in any case, our scheme gives the value (1) for $v_R$. Since expectations are additive, it follows that this scheme coincides with $\Phi$ for all games.

We point out, finally, that the form of the scheme guarantees that, for monotonic games, the modified value $\Phi$ will be an imputation.

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**References**

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# A STOCHASTIC BOTTLENECK ASSIGNMENT PROBLEM*,†

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Consider an $n$-job stochastic bottleneck assignment problem in which the production rates are independent random variables rather than constants. Assume that the production rate, $R_{ij}$, ($R_{ij} \geqq 0$; $i, j = 1, 2, \cdots, n$) of man $i$ when assigned to job $j$ is: 1) exponentially distributed with mean $1/\lambda_{ij}$ or 2) Weibull distributed with scale parameter $\lambda_{ij}$ and shape parameter $\beta$. It is shown that in either case, the assignment that maximizes the expected rate of the entire line is found by solving the deterministic assignment problem with 'cost' matrix $[\lambda_{ij}]$.

The classical 'Bottleneck Assignment Problem' [2] is described [1] as follows: $n$ men are available to be assigned to $n$ operations or jobs comprising a production line. Associated with each man $i$–job $j$ combination is a known and constant production rate $R_{ij}$ ($R_{ij} \geqq 0$; $i, j = 1, 2, \cdots, n$). The production rate of the line for any given assignment will be equal to the slowest rate in the line, i.e., the bottleneck. The problem then is to maximize the rate of the line over all $n!$ possible assignments.

Mathematically, given the set $\{R_{ij} \geqq 0; i, j = 1, 2, \cdots, n\}$, find a permutation $\pi^*$ on the integers $1, 2, \cdots, n$ so as to achieve

$$\text{(1)} \qquad \text{Max}_\pi \{\text{Min}_i R_{i,\pi(i)}\}.$$

Now, suppose that for each man-job combination, the corresponding $R_{ij}$ is not

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a constant but a continuous nonnegative random variable with distribution function $F_{ij}(\cdot)$. For any permutation $\pi$ the rate of production, $R$, of the entire line is also a random variable defined by: $R = \mathrm{Min}_i R_{i,\pi(i)}$. Our objective then is to find an assignment that will maximize the expected rate of production of the line, i.e., we seek a permutation $\pi^*$ so as to achieve

$$\mathrm{Max}_\pi E\{R\}. \tag{2}$$

Assuming that the $R_{ij}$'s are independent random variables with finite means, the distribution function of $R$, $F_R(\cdot)$, is found to be:

$$F_R(r) = 1 - \prod_{i=1}^n [1 - F_{i,\pi(i)}(r)], \tag{3}$$

and the expected rate of production is given by:

$$E\{R\} = \int_0^\infty \{\prod_{i=1}^n [1 - F_{i,\pi(i)}(r)]\}\, dr. \tag{4}$$

We consider now two cases:

*CASE 1:* Let $R_{ij}$ be exponentially distributed with a known parameter $\lambda_{ij} > 0$, i.e., with p.d.f.:

$$\begin{aligned} f_{ij}(r) &= \lambda_{ij} \exp\{-\lambda_{ij} r\}, & r > 0, \\ &= 0, & \text{otherwise,} \end{aligned} \tag{5}$$

for $i, j = 1, 2, \cdots, n$.

Consider also the following 'Assignment Problem': Given $\lambda_{ij} > 0$, $i, j = 1, 2, \cdots, n$, find a permutation $\pi^*$ so as to achieve

$$\mathrm{Min}_\pi \sum_{i=1}^n \lambda_{i,\pi(i)}\,. \tag{6}$$

We show that in the exponential case, a solution of (6) is a solution of (2). To see this we write:

$$\begin{aligned} E\{R\} &= \int_0^\infty \exp\{-(\textstyle\sum_{i=1}^n \lambda_{i,\pi(i)}) r\}\, dr \\ &= 1 / \textstyle\sum_{i=1}^n \lambda_{i,\pi(i)}\,. \end{aligned} \tag{7}$$

It is clear that $\mathrm{Max}_\pi E\{R\}$ is achieved when $\mathrm{Min}_\pi \sum_{i=1}^n \lambda_{i,\pi(i)}$ is achieved, thus proving the assertion.

*CASE 2:* Let $R_{ij}$ be distributed according to the Weibull probability law with scale parameter $\lambda_{ij}$ and shape parameter $\beta$ (equal for all man-job combinations). The p.d.f. of $R_{ij}$ is:

$$\begin{aligned} f_{ij}(r) &= \lambda_{ij}\beta r^{\beta-1} \exp\{-\lambda_{ij} r^\beta\}, & r > 0, \\ &= 0, & \text{otherwise,} \end{aligned} \tag{8}$$

$\lambda_{ij}, \beta > 0$, and the distribution function:

$$F_{ij}(r) = 1 - \exp\{-\lambda_{ij} r^\beta\}. \tag{9}$$

For a given permutation $\pi$, the expected rate of production equals:

$$E\{R\} = \int_0^\infty \exp\{-(\textstyle\sum_{i=1}^n \lambda_{i,\pi(i)})r^\beta\}\, dr = (1/\textstyle\sum_{i=1}^n \lambda_{i,\pi(i)})^{1/\beta}\cdot\Gamma(1/\beta + 1), \tag{10}$$

where $\Gamma(\cdot)$ denotes the gamma function. Since $\beta$ is fixed, $E\{R\}$ is again maximized when $\mathrm{Min}_\pi \sum_{i=1}^n \lambda_{i,\pi(i)}$ is achieved.

Note that the exponential case is seen to be a special case of the Weibull with $\beta = 1$ for all $i$ and $j$.

To summarize, it is shown that, in both cases, the assignment that maximizes the expected rate of production of the entire line in a randomized bottleneck assignment problem, is found by solving a 'deterministic' assignment problem with 'cost' matrix $[\lambda_{ij}]$.

It should also be pointed out that even if a positive lower bound, $G$,—'location parameter'—is imposed on each $R_{ij}$, (i.e., $R_{ij} \geqq G > 0$, all $i, j$)—the results still hold.


**References**

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## A Clarification in LIFO vs. FIFO*

In his investigation of inventory depletion policies, Bomberger [1] examines conditions on the field life function, $L(S)$, for which either LIFO (last in, first out) or FIFO (first in, first out) is an optimal issue policy. His Theorem I states "If $L^{-1}(S)$ is concave increasing, then LIFO is optimal for $n \geqq 2$." In this theorem $L^{-1}(S)$ is not explicitly defined, and although the author refers to this function as the *inverse* of $L(S)$, the reader should be cautioned that he intends $L^{-1}(S)$ to be the reciprocal of $L(S)$ and not the usual inverse function whose composition with $L(S)$ yields the identity function.

That this may be a point of confusion is evidenced, for example, by the fact that Hillier and Lieberman [2] cite the above result in their recent textbook (page 370), rephrasing it in the form, "Finally, Bomberger has shown that if the inverse of the field life function, *i.e.*, $L^{-1}(S)$, is concave increasing, then LIFO is optimal for $n \geqq 2$." Again, a question arises as to the intended meaning of the phrase "inverse of the field life function." The reason why this clarification is needed is that the writer has shown [3] that if, for example, $L(S)$ is convex with $L'(S) \geqq 1$ for all $S \geqq 0$, in which case the usual inverse function $L^{-1}(S)$ is indeed concave increasing, then not only iFO nouLI bmal,t optis e t thopposing policy FIFO is

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