On Optimal Dimensioning of a Certain Local Network

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ABSTRACT

A group of m sources (telephone exchanges) is offering traffic to a group of n local exchanges. From each source S_k ($k=1,2,\ldots,m$) there are N_{ki} trunks leading directly to local exchange E_i ($i=1,2,\ldots,n$). A call originating at S_k is transmitted first to local exchange E_i with probability q_{ki} ($\Sigma_i q_{ki}=1$). If the call's destination is E_j ($j\neq i$) rather than E_i, the call is transferred from E_i to E_j.

For such a network, the optimal economic dimensioning of trunks (i.e., optimal allocation of the N_{ki}'s) is determined. It is shown that, for each source exchange, the optimal dimensioning is to direct all trunks to a single local exchange (which may differ for distinct sources). This single local exchange is determined as a function of the cost of trunks (i.e., distances) between the exchanges and the load offered by the sources to the various local exchanges.

1. INTRODUCTION

In this study we consider a telephone network comprised of two groups of exchanges. One group is a collection of local exchanges characterized by their common first digit. The other group is a collection of 'extraneous' exchanges offering traffic to the first group. The exchanges of the second group will be called source exchanges.

Each of the source exchanges is connected to the various local exchanges by trunks which carry the offered load. If the amount of traffic offered by a source exchange to the entire group of local exchanges can be measured or forecasted, then standard teletraffic methods may be employed to determine the total number of trunks required between this source exchange and the group of local exchanges. This number is determined so as to assure a pre-specified grade of service. If, in addition, the partitioning of the offered load among the various local exchanges is also known, then one would wish to find the best partitioning of the set of trunks that will optimize some measure of effectiveness (e.g., total investment in trunks or total flow in the system).

The special property of the network considered here is that the group of local exchanges is of the step-by-step type - and, because of various technological reasons, the flow of calls in the network is typically not a direct one. A call originating at a certain source exchange, say S_k, is routed randomly to one of the local exchanges. If, for example, the call arrives at local exchange E_i, but its destination is local exchange E_j, where $j\neq i$, then it is retransferred from E_i to E_j. The chance of a call originating at source exchange S_k arriving first at local exchange E_i is given by q_{ki} ($q_{ki} \geq 0$, $\Sigma_i q_{ki} =1$). Typically, this probability is proportional to the number of trunks leading directly from S_k to E_i. That is, if the total number of trunks outgoing from S_k is determined to be $N_{k\cdot}$, and if N_{ki} of these are connected directly to local exchange E_i, then the probability that a call originating at S_k will arrive first at E_i is given by $N_{ki}/N_{k\cdot}$.

A further complexity of the situation is the fact that, besides receiving calls originating at the various source exchanges, each local exchange offers traffic to its sister exchanges in the group. This traffic is routed directly according to its destination - but it is added to that part of the traffic which is generated by the source exchanges, and is rerouted among the local exchanges due to the random allocation of flow.

The objective of this study is to find the optimal dimensioning of trunks in the telephone network described above - i.e., the optimal partitioning and the optimal routing of trunks so as to achieve minimum investment costs while maintaining a required grade of service. It will be shown that the optimal economic allocation of trunks is to direct all trunks from a certain source exchange to a single local exchange. This local exchange may be different for distinct source exchanges; it is determined as a function of the cost of trunks (i.e., distances) between the exchanges and the load offered by the sources to the various local exchanges. This last result is an extension of an earlier study by the author [3] in which the measure of effectiveness to be minimized was taken to be the total flow in the network. It will be shown that the present framework reduces to the previous one if all costs of trunks (distances) between the exchanges are assumed to be alike.

In fact, the results obtained in [3] are qualitatively the same - namely, to direct all trunks from a given source exchange to a single local exchange. However, this local exchange is the one to which the offered load is maximal, and it may not coincide with the local exchange determined by the minimum-cost objective function.

The question of which objective function is most appropriate is usually left to be answered by the management. If one designs his network to minimize investment costs, he many not get the best possible grade of service. On the other hand, if he wants the best possible grade of service (i.e., with the minimum unnecessary flow in the system), which will be effective for a long period of time, he may find himself investing a little bit more.

2. THE MODEL

A group of m source exchanges offers traffic to a group of n local exchanges. Let A_{ki} (k=1,2,...,m; i=1,2,...,n) be the offered traffic from source exchange S_k to local exchange E_i. A_{ki} is given in Erlangs, and is assumed to be constant over the period under consideration. The total amount of traffic offered by source exchange S_k to all local exchanges is $A_{k.}=\Sigma_{i=1}^{n} A_{ki}$.

To assure a required grade of service, the total number of trunks leading from S_k to <u>all</u> local exchanges, $N_{k.}$, is determined by standard teletraffic methods [2]. The problem we are concerned with is the optimal allocation of the $N_{k.}$ trunks among the n local exchanges; that is, for each source exchange S_k (k=1,2,...,m), we wish to find the optimal partitioning of trunks $\{N_{ki}, i=1,2,...,n\}$ such that $\Sigma_{i=1}^{n} N_{ki} = N_{k.}$, where N_{ki} is the number of trunks leading <u>directly</u> from S_k to local exchange E_i (i=1,2,...,n).

An optimal partitioning is one whose total investment cost in trunks is minimal (the cost of switching equipment is assumed to be almost the same for every partitioning of $N_{k.}$). It follows immediately that one should take into consideration the distances between the various exchanges. These distances may be expressed in terms of the cost of trunks between the various exchanges. Thus, we let C_{ki} (k=1,2,...,m; i=1,2,...,n) be the cost of a single trunk from S_k to E_i (and, if one wishes, he may add to it the cost of the switching equipment).

There is another source of flow in the system. Every local exchange is offering traffic to each of its sister local exchanges. Denote by Z_{ij} (i,j=1,2,...,n) the local traffic offered by E_i to E_j. This traffic flows directly from E_i to E_j with no detours. Moreover, the Z_{ij}'s are independent of the A_{ki}'s. As a consequence, it follows that the Z_{ij}'s do not affect the optimal allocation of the N_{ki}'s.

Let d_{ij} (i,j=1,2,...,n) be the cost of a single trunk connecting local exchange E_i to local exchange E_j. (In many cases, the d_{ij}'s and the C_{ki}'s are linear functions of the distances between E_i and E_j, and S_k and E_i, respectively. However, this assumption is not required for the analysis of the system).

Now, consider a call originating at S_k. For ease of presentation and analysis, we assume that the grading is such that the traffic offered by S_k is distributed <u>evenly</u> among its $N_{k.}$ outgoing trunks. However, the final results remain the same if this assumption is relaxed and we assume that the traffic offered by S_k is distributed among the local exchanges according to some arbitrary distribution $\{q_{ki}, i=1,2,...,n\}$. Thus, assume that the probability of our call being directed to E_i is given by $q_{ki} = N_{ki}/N_{k.}$ (k=1,2,...,m;i=1,2,...,n).

This probability is <u>independent</u> of the destination of the call. If it turns out that the call arrives at E_i but its destination is E_j ($j \neq i$), then it is retransferred from E_i to E_j.

We assume that the system is designed for small losses - i.e., the $N_{k.}$'s are determined such that, for each source exchange S_k, the actual carried traffic from S_k to E_i is given by

$$Y_{ki} = A_{k.} q_{ki} \qquad (k=1,2,...,m;\ i=1,2,...,n). \tag{1}$$

In other words, from the total amount of $A_{k.}$ Erlangs offered by S_k to all local exchanges, a proportion q_{ki} flows directly to E_i. However, only $A_{ki}q_{ki}$ Erlangs of this amount have E_i as their destination. The rest of the traffic, amounting to $(A_{k.} - A_{ki})q_{ki}$ Erlangs, has to be retransferred to the various destinations. The amount of traffic to be transferred from E_i to E_j ($j \neq i$) is $A_{kj}q_{ki}$ - where, clearly, $\Sigma_{j \neq i} A_{kj}q_{ki} = (A_{k.}-A_{ki})q_{ki}$. Since E_i recieves calls from all source exchanges, the total amount of traffic which has to be retransferred from E_i to E_j is given by

$$W_{ij} = \Sigma_{k=1}^{m} A_{kj}q_{ki} \qquad (i,j = 1,2,...,n;\ i \neq j). \tag{2}$$

This retransferred traffic is added to Z_{ij}, the load offered by E_i to E_j, such that the total flow from E_i to E_j is

$$X_{ij} = Z_{ij} + W_{ij} \qquad (i,j=1,2,...,n;\ i \neq j). \tag{3}$$

Thus, every allocation of the N_{ki}'s determines the set of flows $\{X_{ij}\}$(through the q_{ki}'s) - and, for any such set, the number of trunks required to carry the flow in each route is determined by standard teletraffic tables that guarantee a high grade of service. The network is illustrated in Figure 1.

3. OPTIMAL DIMENSIONING

A quick study of standard teletraffic tables - or, equivalently, an analysis of Erlang's loss formula such as in [1] - reveals that if the offered traffic is large enough, the number of trunks required to carry the flow at a given grade of service is very closely proportional to the offered load. Let this proportion be denoted by α. To carry one Erlang of offered or transferred traffic, $\alpha > 1$ trunks are needed between the corresponding exchanges so that the network's performance will meet the specified grade of service. In other words, the number of trunks required between S_k and E_i is αY_{ki}, and the number of trunks to be allocated to carry the traffic between E_i and E_j is αX_{ij}.

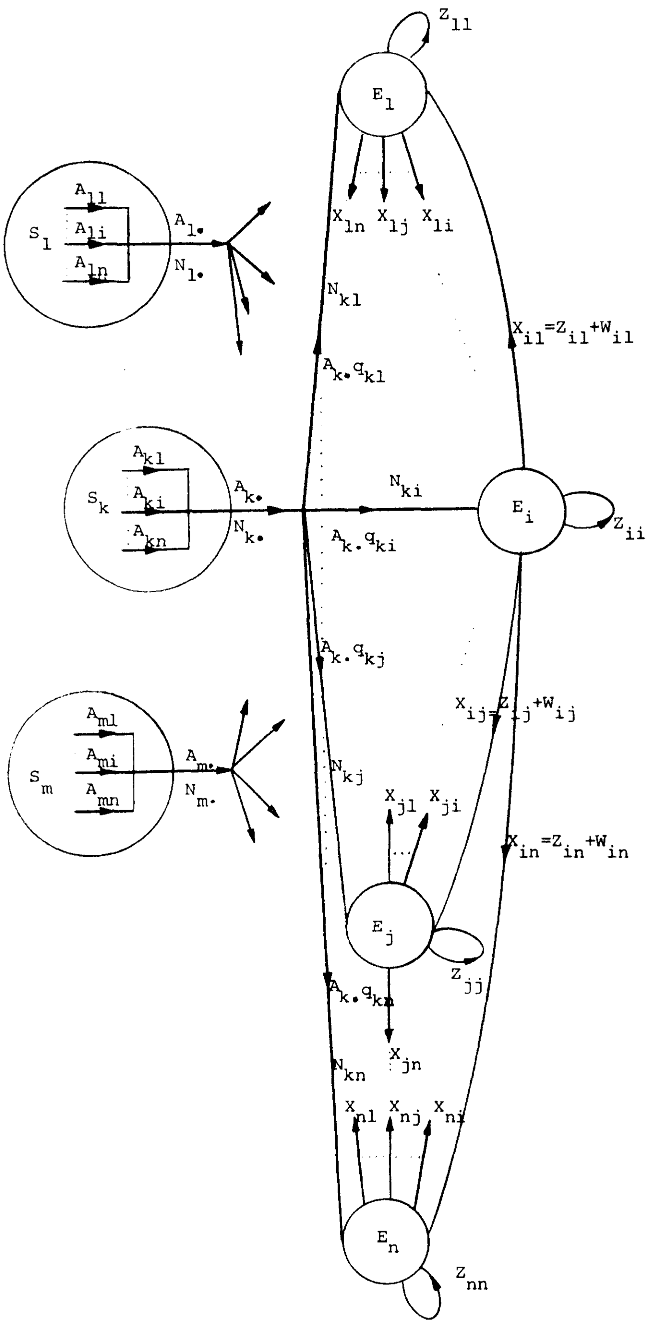

Figure 1: A Scheme of the Network

Our problem, therefore, is to find an allocation of trunks that will minimize the total investment cost in the network,

$$(4)\quad C = \Sigma_{k=1}^{m}\Sigma_{i=1}^{n} C_{ki}(\alpha y_{ki}) + \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} d_{ij}(\alpha X_{ij}).$$

Substituting the values of y_{ki} and X_{ij} from equations (1), (2) and (3) we get

$$(5)\quad C = \alpha\{\sum_{k=1}^{m}\sum_{i=1}^{n} C_{ki} A_{k\cdot}q_{ki} + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} \times (\sum_{k=1}^{m} A_{kj}q_{ki} + Z_{ij})\}.$$

Since $\Sigma_i\Sigma_j d_{ij}Z_{ij}$ is a constant term independent of the allocation of the N_{ki}'s, and we are using the relations $A_{k\cdot} = \Sigma_{j=1}^{n} A_{kj}$ and $q_{ki} = N_{ki}/N_{k\cdot}$, our problem is finally stated as: find non-negative integer-valued variables N_{ki}'s $(k=1,2,\dots,m;\ i=1,2,\dots,n)$, so as to minimize

$$(6)\quad \sum_{k=1}^{m} \frac{1}{N_{k\cdot}} \left[\sum_{i=1}^{n}\sum_{j=1}^{n} A_{kj}(C_{ki}+d_{ij})N_{ki}\right]$$

subject to

$$(7)\quad \sum_{i=1}^{n} N_{ki} = N_{k\cdot} \qquad (k=1,2,\dots,m).$$

It follows immediately that (6) and (7) can be separated into m independent problems. Each such problem corresponds to a single source exchange, and is an n-variable Integer Linear Programming problem with a single constraint. For each source exchange S_k $(k=1,2,\dots,m)$, the problem is

$$(8)\quad \text{minimize } \{\sum_{i=1}^{n}[\sum_{j=1}^{n} A_{kj}(C_{ki}+d_{ij})]N_{ki}\}$$

subject to

$$(9)\quad \sum_{i=1}^{n} N_{ki} = N_{k\cdot}.$$

The solution of (8) and (9) follows readily: find the local exchange $E_{i(k)}$ for which $\sum_{j=1}^{n} A_{kj}(C_{k,i(k)} + d_{i(k),j})$

$$= \min_{1\le i\le n}\{\sum_{j=1}^{n} A_{kj}(C_{ki}+d_{ij})\},$$ and direct <u>all</u> $N_{k\cdot}$ outgoing trunks from S_k to $E_{i(k)}$. This result has the qualitative advantage that it is independent of the actual value of $N_{k\cdot}$.

Before proceeding to a special case where $C_{ki}=1$ for all k and i, and $d_{ij} = 1$ for all $i \neq j$, $d_{ii}=0$, we want to give an interpretation of our result. The term

$\sum_{j=1}^{n} A_{kj}(C_{ki}+d_{ij}) = A_{k\cdot}C_{ki} + \sum_{j=1}^{n} A_{kj}d_{ij}$ in equation (8) has the dimension of Erlangs x cost. If al of the flow from S_k is directed to E_i, then $\alpha A_{k\cdot}C_{ki}$ is the investment cost of the $N_{k\cdot}$ trunks leading from S_k to E_i. In such a case, A_{kj} is the amount of traffic to be transferred from E_i to E_j. Thus, the investment cost for trunks between E_i and E_j is $\alpha A_{kj}d_{ij}$. Summation over all j gives the total investment in trunks for the transferred traffic. The total investment cost is therefore $(A_{k\cdot}C_{ki} + \sum_{j=1}^{n} A_{kj}d_{ij})$. Equation (5) may be interpreted in the same way. It should be noted that because of the proportionality role of α, the z_{ij}'s do not affect the (optimal) solution.

If the costs C_{ki} and d_{ij} are proportional to the corresponding distances between the exchanges, then the optimal partition is the one which minimizes the summation of products of Erlangs x distances. That is, the optimal solution is the one for which the overall distance of total traveled traffic is minimal. This last objective function may serve as a measure of effectiveness in other network studies.

4. A SPECIAL CASE

Suppose that $C_{ki} = 1$ for all k and i, $d_{ij} = 1$ for $i \neq j$, and $d_{ii} = 0$ for all i. That is, we assume that all distances between the various exchanges are alike, and we are interested only in minimizing the flow of traffic in the network. In such a case, each of the problems (8) becomes

$$\text{(10) minimize } \left\{ \frac{1}{N_{k\cdot}} \left[A_{k\cdot} N_{k\cdot} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} A_{kj} N_{ki} \right] \right\}$$

subject to (9).

Since $\sum_{k=1}^{m} A_{k\cdot}$ is a given constant, the problem is equivalent to

$$\text{(11) minimize } \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{kj} N_{ki} - \sum_{i=1}^{n} A_{ki} N_{ki} \right\}.$$

Since $\sum_{i=1}^{n} \sum_{j=1}^{n} A_{kj} N_{ki} = A_{k\cdot} N_{k\cdot}$, the problem is equivalent to

$$\text{(12) maximize } \left\{ \sum_{i=1}^{n} A_{ki} N_{ki} \right\}$$

subject to (9).

This case was considered in [3].

The optimal solution of (12) under (9) is, for each source exchange S_k, to find the local exchange $E_{i(k)}$ for which $A_{k,i(k)} = \max_i \{A_{ki}\}$, and to direct <u>all</u> $N_{k\cdot}$ trunks to that local exchange. In other words, all the traffic is directed to the local exchange to which the offered load is <u>maximal</u>.

As was pointed out in the Introduction, one should decide whether he wishes to design his network according to objective (8), which emphasizes the cost function, or according to equation (12), which emphasizes the traffic point of view.

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