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# ON THE HOTEL OVERBOOKING PROBLEM—AN INVENTORY SYSTEM WITH STOCHASTIC CANCELLATIONS*

VARDA LIBERMAN† AND URI YECHIALI†

$M$ hotel rooms are available at a date $n$ periods from now. Reservations are made by customers for that date, which is at the peak of the high season. Typically, for such a time period, a policy of overbooking is exercised by the hotel management. Customers, however, may cancel their previously confirmed reservations at any time prior to their arrival, with no penalty. On the other hand, new requests for rooms for that particular date are generated anew. At the end of each period the hotel management reviews both the "inventory" level of remaining uncanceled (previously confirmed) reservations and the total number of not-yet-confirmed new requests. At that time a decision is made regarding the inventory level of confirmed reservations with which to start the next period. A decision is one of three actions: (i) to keep the inventory at its present level (i.e., declining all new requests); (ii) to increase the level of overbooking by confirming some of the new requests and, if necessary, by trying to obtain some additional reservations (at some extra cost); (iii) to decrease the level of inventory by canceling some of the previously confirmed reservations (incurring a penalty for each such cancellation). Each occupied room at the target day carries a given profit, while each unhonored reservation at that time incurs a penalty. The problem is to find the optimal over-booking strategy that will maximize net profit.

For both criteria, maximization of the expected total net profit, and maximization of the expected discounted net profit, it is shown that the optimal strategy is a 3-region policy as follows: For each period there exist upper and lower bounds and an intermediate point such that, (a) if the overbooking level at the end of a period is greater than the upper bound, it should be decreased to that bound; (b) if the inventory level is below the lower bound, two cases may occur: (i) if the discrepancy is greater than the number of new requests, all new requests should be confirmed and additional reservations should be acquired such that the inventory level will be equal to the lower bound; and (ii) if the discrepancy is smaller than the number of new requests, some of the new requests are confirmed but the inventory level may not exceed the intermediate point; (c) if the inventory level is between the two bounds there are two possibilities: (i) if it is above the intermediate point none of the new requests are confirmed, but (ii) if it is below that point, some of the new requests should be confirmed provided that the new inventory level will not exceed the intermediate point.

## Introduction

The hotel overbooking problem arises as a consequence of the option given to prospective guests to cancel their reservations—with no penalty—at any time prior to their arrival date. As a means of (partially) overcoming this problem, hotel managements practice overbooking, expecting that, due to (probabilistic) cancellations, the number of actual "show ups" at a given date will be as close as possible to the hotel's nominal capacity. The problem then is to find the optimal overbooking policy such that some measure of effectiveness, say expected profit, will be maximized.

The day-to-day handling of overbookings is a complicated and tedious task. In this study we focus on overbooking problems arising at a specific period of time where demand for rooms is high and the number of room-requests exceeds the hotel's nominal capacity. We restrict ourselves to a single day (e.g., New Year's Eve), or a given period of time which is "sold" as a single entity. We consider a hotel (or part of it) in which all rooms are assumed to be identical with regard to their contribution to the hotel's income. Several months before the target day the hotel offers an option to "buy" some portion of its rooms to various agents. The remaining capacity is "sold"

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† Tel Aviv University.

by the hotel's management itself. The agents will keep their options until some predetermined time before the target date, and then release the unsold rooms. (No direct monetary costs are involved in taking or releasing the options. It is just that both sides are interested in these arrangements for various reasons which will not be discussed here.) The time interval from the moment the agents release their unsold rooms until the target day is divided into $n$ periods and at the end of each period a decision is made by the hotel management regarding the level of confirmed reservations. Two competing processes are in effect: (i) customers with confirmations cancel their reservations in some random manner, and (ii) new requests for rooms for the target day are randomly generated. The hotel management wishes to keep the number of overbookings at a controlled level: not too high and not too low. In case it is too low, some or all of the new requests are confirmed. If it is still too low, an effort is usually made to "sell" more rooms to the above mentioned agencies or to some other organizations. In such situations an attractive deal is offered, usually at a lower rate per room, or a higher commission is given to the agents for their extra effort in selling rooms. We term this discount in price or the higher commission given as the *cost of acquiring* a room. If only a few prospective guests cancel their reservations and the number of confirmed reservations is considered too high, all new requests will be rejected (at no direct cost to the hotel), and an additional effort will be made to cancel some of the already-confirmed reservations. The hotel management will contact the various agents and try to persuade them to transfer some of their clients to other hotels. Usually, the clients are compensated by having a better room or better facilities and the hotel management bears the extra costs. We term such an arrangement as the *penalty* of cancellation of a previously confirmed rereservation. In case a confirmed reservation cannot be honored at the target day, a high penalty is incurred. In addition to legal actions that may be taken by customers or the authorities, the hotel management would try its best to secure an accommodation for the guest with the unhonored reservation. Usually the customer will be accommodated at a better place.

Few authors have addressed the hotel overbooking problem in the literature. Rothstein [4] formulates the problem as a Markovian Sequential Decision Process where at each stage the states of the process are the number of recorded reservations, and the transition probabilities are determined from the customers' demand, cancellations and "no-show" probabilities. He demonstrates how an optimal booking policy, which maximizes the expected revenue over an $n$-stages problem, may be *numerically* calculated, using Howard's procedure, but the *structure* of the optimal booking policy is not studied. Rothstein also compares the hotel overbooking problem with the airline overbooking problem and indicates the common aspects as well as the differences between these two problems. Ladany, in two closely related papers [2], [3] presents a model for finding the maximum allowable number of bookings for single and double-bed rooms at each time period prior to the target day. No costs or penalties for management's actions regarding acquiring additional bookings or cancellation of confirmed reservations are imposed. A numerical example for hotel capacity of 6 double-bed rooms and 2 single-bed rooms is given. As in [4], no attempt is made to study the structure of the optimal policy.

Shlifer and Vardi [6] present a model for determining an overbooking policy for an airline. They study three cases of airline overbooking: (a) single-leg flight carrying a single type of passenger (this case resembles the model proposed by Rothstein and Stone [5]), (b) single-leg flight carrying two types of passengers, and (c) two-leg flight. They assume that all types of passengers generate the same profit for a given flight-leg and that the number of show-ups is normally distributed with mean and variance proportional to the number of reservations. Letting the booking policy be of the form that puts an upper bound, $N^*(t)$, on the number of reservations on hand at time $t$ before take-off, they calculate the optimal values of $N^*(t)$ for various criteria. Their decision rules, however, do not make use of the (probabilistic) properties of future demand.

In this paper we develop an $n$-period control model of the overbooking problem where the objective is to find the strategy that maximizes net profit. We consider two criteria: maximization of the expected total net profit, and maximization of the expected discounted net profit. For both criteria it is shown that the optimal strategy is a 3-region policy as follows: For each period $i$ there exists a lower bound, $U_i^*$, an upper bound, $V_i^*$, and an intermediate level, $Z_i^*$, such that if $X_i$ is the inventory level just prior to the beginning of the period (after cancellations by customers have taken place during the preceding period), and $Y$ is the number of new requests (not yet confirmed) then:

(i) if $X_i + Y < U_i^*$, all new requests should be confirmed and additional ones acquired such that the level of inventory at the beginning of the period will be equal to $U_i^*$.

(ii) if $U_i^* < X_i + Y \leqslant Z_i^*$, all new requests should be confirmed.

(iii) if $Z_i^* < X_i + Y$ but $X_i \leqslant V_i^*$, $\max(0, Z_i^* - X_i)$ new requests are confirmed.

(iv) if $V_i^* < X_i$, $X_i - V_i^*$ reservations should be canceled.

It is interesting to note that a somewhat similar form of an optimal strategy has been derived by Fukuda [1] for an inventory model where disposal of surplus items is possible and any excess demand is to be backlogged. For his problem, Fukuda shows that the optimal policy for the single period model is a two-region policy such that if the inventory level, $X$, is greater than some specific number, $V$, then $X - V$ items should be disposed; but if $X \leqslant V$, nothing should be done. For the $n$-period model he demonstrates that a simple three-region policy is an optimal one for the minimization of the expected discounted cost.

The presentation in this paper develops as follows: we first describe the mathematical model. Then we consider the expected total net profit criterion and with the aid of two lemmas develop the theory assuming that the new requests at each period are ignored (Theorem 1). In Theorem 2 we drop this restriction and derive the general results. The discounted case is considered in Theorem 3.

## The Model

Let $M$ be the number of rooms available at the *target day*, $T$ days from now. Suppose that the hotel management makes decisions regarding the overbooking inventory level of confirmed reservations at specified times

$$T = t_n > t_{n-1} > \cdots > t_1 > t_0 = 0,$$

where $t_0 = 0$ is the target day.

A decision at time $t_i$ ($i = n, n-1, \ldots, 2, 1$) is one of three possible actions:

(i) to increase the level of overbooking;

(ii) to cancel confirmed reservations subject to a *cancellation penalty*;

(iii) to keep the inventory at its present level.

Action (i) may be achieved in two steps. At first, new requests that have been accumulated since the last decision point are confirmed. Then, if necessary, some additional reservations are acquired. The second step, however, is associated with an *acquiring cost*. If either action (ii) or (iii) is taken, all new requests are declined and consequently lost.

*Customer Cancellation.* A customer may cancel his reservation with no penalty at any time prior to the arrival date.

*Notation.*

$a_0$—The profit from an occupied room (at time $t_0 = 0$).

$c_i$—The cost of acquiring a reservation at time $t_i$ ($i \geqslant 1$).

$b_i$—The penalty for cancellation of a confirmed reservation at time $t_i$. ($b_0$ is the penalty of not honoring a confirmed reservation at the target day.)

$Z_i$—The inventory level of confirmed reservations at time $t_i - 0$, immediately after one of the three aforementioned decisions has been made.

$X_i$—The inventory level at instant $t_i + 0$, i.e., at the end of time interval $(t_{i+1}, t_i)$ and just before taking an action at time $t_i$.

$Y_i$—The number of new requests for rooms accumulated during the time interval $(t_i, t_{i-1})$. $Y_i$ is random at time $t_i$ but is known at time $t_{i-1}$.

$Q_i(\cdot)$—The distribution function of $Y_i$.

$P_i$—A random variable such that $X_i = P_{i+1} \cdot Z_{i+1}$.

$F_i(\cdot)$—The distribution function of $P_i$, possesses a finite mean, $E(P_i)$, and is defined and positive over (0, 1).

$G_i(Z)$—The expected total net profit realized over the time interval $(t_i, 0]$ starting with inventory level $Z$ and following an optimal strategy.

Note that $Z_i$, $X_i$ and $Y_i$ are assumed to be continuous variables. This assumption is made for ease of representation. Results similar to the ones developed in this paper can also be derived for descrete variables.

*The Problem.* Find an optimal booking policy that will maximize the expected net profit (or the discounted net profit) realized over the period $[T, 0]$.

## Expected Total Net Profit

In this section we derive the optimal $n$-period strategy for the expected total net profit criterion. We formulate the decision problem as an $n$-stage dynamic programming problem. In Lemmas 1, 2 and in Theorem 1 the optimal strategy is derived under the restrictive assumption that the random new requests arriving during each time period are ignored. In Lemma 3 and Theorem 2 we drop this restriction and prove the optimality of the 3-region policy stated above.

LEMMA 1. *Let $Z$ be the level of inventory at time $t_1$, and let $G_1(Z)$ be the expected net profit. Then $G_1(Z)$ is maximized at the inventory level $Z_1^*$ where*

$$[a_0/(a_0 + b_0)]E(P_1) = \int_{M/Z_1^*}^{1} p \, dF_1(p) \tag{1}$$

*and the maximal expected net profit, $G_1(Z_1^*)$, is given by*

$$G_1(Z_1^*) = (a_0 + b_0)M[1 - F_1(M/Z_1^*)]. \tag{2}$$

PROOF. The net profit realized at time $t_0 = 0$ is $a_0 P_1 Z$ if $P_1 Z \leqslant M$, and is $a_0 M - b_0(P_1 Z - M)$ if $P_1 Z \geqslant M$.

Since, for $a_0 \geqslant 0$, $Z_1^*$ should be at least $M$, the expected net profit is given by:

$$G_1(Z) = \int_0^{M/Z} a_0 p Z \, dF_1(p) + \int_{M/Z}^{1} [a_0 M - b_0(pZ - M)] dF_1(p). \tag{3}$$

$G_1(Z)$ is differentiable and concave. Hence, if it possesses a maximum, it is a unique one. Differentiating $G_1(\cdot)$ results in

$$G_1'(Z) = -(a_0 + b_0)\int_{M/Z}^{1} p\, dF_1(p) + a_0 E(P_1). \tag{4}$$

Letting $G_1'(Z_1^*) = 0$ yields (1), while (2) is obtained by substituting (1) into (3).

Our next lemma will specify the single period (= last period) optimal strategy if no new requests are accepted after $t_1$, at which time one of the three possible actions specified above has to be taken. The costs involved are $c_1 > 0$ and $b_1 > 0$ for a unit increase or a unit decrease in the overbooking level, respectively. We further assume that $a_0 E(P_1) > c_1$ and $b_0 E(P_1) > b_1$. These assumptions are natural ones, since $a_0 E(P_1)$ is the expected profit realized at time $t_0$ from a single standing reservation at time $t_1$. If this quantity is smaller than $c_1$, then there is no point in increasing the inventory level. Similarly, the cancellation cost $b_1$ should not exceed the expected loss, $b_0 E(P_1)$, due to an overbooking at time $t_0$.

LEMMA 2. *Suppose at time $t_1$ there are $X_1$ uncanceled reservations left, and assume that no new requests are accepted during the time interval $(t_1, 0]$. Then there exist numbers $U_1^*$ and $V_1^*$, independent of $X_1$, such that $U_1^* < Z_1^* < V_1^*$, and the optimal policy is as follows*:

(i) *if $X_1 < U_1^*$, the inventory level should be increased to $U_1^*$.*
(ii) *if $U_1^* \leqslant X_1 \leqslant V_1^*$, nothing should be done.*
(iii) *if $X_1 > V_1^*$, the inventory level should be reduced to $V_1^*$.*

PROOF. We distinguish between the two possibilities: $X_1 \leqslant Z_1^*$ and $X_1 \geqslant Z_1^*$. Suppose $X_1 \leqslant Z_1^*$. Then it does not pay to cancel reservations since $G_1(Z)$ is increasing for $Z < Z_1^*$. Thus, suppose that the number of reservations is increased by $S$ such that $X_1 + S = U$. The expected total net profit will be $G_1(X_1 + S) - c_1 S$ and we wish to find $S \geqslant 0$ so as to maximize $\{G_1(X_1 + S) - c_1 S\}$. Since $X_1$ is a given number, the above is equivalent to finding $U$ so as to

$$\operatorname*{Maximize}_{U \geqslant X_1} \{G_1(U) - c_1 U\}.$$

The function $H(U) = G_1(U) - c_1 U$ is concave with a unique maximum. Denote this maximum by $U_1^*$, where $U_1^*$ satisfies

$$G_1'(U_1^*) = c_1 > 0. \tag{5}$$

Moreover, from Lemma 1, $G_1'(Z_1^*) = 0$, and $G_1'(\cdot)$ is a monotone decreasing function ($G_1$ is concave); hence, $U_1^* < Z_1^*$. It follows that if $X_1 < U_1^*$, the inventory level should be increased up to $U_1^*$; if $U_1^* \leqslant X_1 \leqslant Z_1^*$, no action should be taken. This completes part (i) and the left hand side of part (ii) of the theorem.

If $X_1 \geqslant Z_1^*$, then, similarly it does not pay to increase the level of reservations, since $G_1(Z)$ is decreasing for $Z > Z_1^*$ and it costs $c_1$ to acquire a new reservation. Thus, suppose that we decrease the inventory level by $S$ such that $X_1 - S = V$. In this case our objective is to $\mathrm{MAX}_{V \leqslant X_1}\{G(V) + b_1 V\}$. $\tilde{H}(V) = G_1(V) + b_1 V$ is a concave function with a unique maximum, $V_1^*$, such that

$$G_1'(V_1^*) = -b_1 \tag{6}$$

and $V_1^* > Z_1^*$. It follows that if $X_1 > V_1^*$, the inventory level should be reduced to $V_1^*$, and if $Z_1^* \leqslant X_1 \leqslant V_1^*$, nothing should be done. This completes the proof.

It is of interest to point out the intuitive meaning of results (5) and (6). $c_1$ is the marginal cost of increasing the inventory level. Thus, since $G$ is concave with a monotone decreasing derivative, it pays to increase the number of reservations only as long as the marginal cost of doing it is not greater than the marginal increase in profit. A similar argument explains result (6).

COROLLARY 1.
(i) $\int_{M/U_1^*}^{1} p\, dF_1(p) = [a_0 E(P_1) - c_1]/(a_0 + b_0)$.
(ii) $\int_{M/V_1^*}^{1} p\, dF_1(p) = [a_0 E(P_1) + b_1]/(a_0 + b_0)$.

PROOF. Applying equation (4) with $U_1^*$ and $V_1^*$, and using results (5) and (6), yields (i) and (ii), respectively.

REMARK 1. Lemma 2 holds for any concave function $G$. If each of the functions $H(U)$ and $\tilde{H}(V)$ possesses a maximum, then exactly the same results apply. If $H(U)$ $[\tilde{H}(V)]$ does not have a maximum, then, since $H(U) \to -\infty$ $[\tilde{H}(V) \to +\infty]$ as $U \to \infty$ $[V \to \infty]$, it is a monotone decreasing [increasing] function and hence $U_1^* = 0$ $[V_1^* = \infty]$.

REMARK 2. Although it is somewhat simpler to assume linear cost functions ($cU$ and $bV$) for acquisition or cancellation of reservations, respectively, Lemma 2 and the following theorems remain true for appropriate (but otherwise arbitrary) convex functions, $c(U)$ and $b(V)$.

In Theorem 1 we ignore the flow, $Y_i$, of new requests during each time interval $(t_i, t_{i-1})$. In Theorem 2, however, we relax this assumption and show that the structure of the optimal policy derived in Theorem 1 holds true for the general case.

THEOREM 1. *Assume that no requests arrive during the time intervals* $(t_i, t_{i-1})$ *for* $i = n, n-1, \ldots, 2, 1$. *Then* (a) *for any* $i$, *there exist numbers* $U_i^*$, $Z_i^*$, *and* $V_i^*$ $(U_i^* < Z_i^* < V_i^*)$ *which determine a 3-region optimal policy with the following actions at time* $t_i$:
*if* $X_i < U_i^*$, *the inventory level should be increased to* $U_i^*$.
*if* $U_i^* \leqslant X_i \leqslant V_i^*$, *nothing should be done.*
*if* $X_i > V_i^*$, *the inventory level should be decreased to* $V_i^*$.
(b) *the optimal values* $U_i^*$, $Z_i^*$ *and* $V_i^*$ *satisfy*

$$G_i'(U_i^*) = c_i, \quad G_i'(Z_i^*) = 0 \quad \text{and} \quad G_i'(V_i^*) = -b_i.$$

PROOF. The proof will be carried out by induction. For $i = 1$, claims (a) and (b) were derived explicitly in Lemma 2. Assume now that the optimal policy at time $t_i$ $(i \geqslant 1)$ is as stated in (a) with $U_i^*$ and $V_i^*$ satisfying (b). We wish to show that claims (a) and (b) also hold for time $t_{i+1}$. Define $G_{i+1}(Z)$ for $i = 1, 2, \ldots, n-1$ recursively as a consequence of the following facts: if $Z$ is the inventory level at time $t_{i+1}$, then the number of reservations at the end of the interval $(t_{i+1}, t_i)$ is $P_{i+1}Z$. By the induction assumption, if $P_{i+1}Z < U_i^*$, we increase the number of reservations up to $U_i^*$ (at a unit cost of $c_i$ per reservation). If $U_i^* \leqslant P_{i+1}Z \leqslant V_i^*$, nothing is done; and if $P_{i+1}Z > V_i^*$, $P_{i+1}Z - V_i^*$ reservations are canceled at a penalty of $b_i$ per reservation canceled. Thus:

$$\begin{aligned} G_{i+1}(Z) = &\int_0^{\min(U_i^*/Z,\, 1)} [G_i(U_i^*) - c_i(U_i^* - pZ)]\, dF_{i+1}(p) \\ &+ \int_{\min(U_i^*/Z,\, 1)}^{\min(V_i^*/Z,\, 1)} G_i(pZ)\, dF_{i+1}(p) \\ &+ \int_{\min(V_i^*/Z,\, 1)}^{1} [G_i(V_i^*) - b_i(pZ - V_i^*)]\, dF_{i+1}(p). \end{aligned}$$

We now show that $G_{i+1}(Z)$ is concave and satisfies claim (b). At the same time part (a) of the theorem will be proven.

$G_1(Z)$ was shown to be concave, satisfying (b). Assume that $G_i(Z)$ is concave and satisfies claim (b). Then, differentiating $G_{i+1}$ with respect to $Z$ yields

$$G'_{i+1}(Z) = c_i \int_0^{U_i^*/Z} p\, dF_{i+1}(p) + \int_{U_i^*/Z}^{V_i^*/Z} pG_i'(pZ)\, dF_{i+1}(p)$$
$$- b_i \int_{V_i^*/Z}^{1} p\, dF_{i+1}(p).$$

Differentiating again, and using the induction assumption that $G_i'(U_i^*) = c_i$ and $G_i'(V_i^*) = -b_i$, one gets

$$G''_{i+1}(Z) = \int_{U_i^*/Z}^{V_i^*/Z} p^2 G_i''(pZ)\, dF_{i+1}(p).$$

By the induction assumption, $G_i'' < 0$, and hence, $G''_{i+1}(Z) < 0$. That is, $G_{i+1}$ is concave. Moreover, $G_{i+1}$ possesses a maximum since, for $Z = U_i^*$, $G'_{i+1}(U_i^*) = c_i E(P_{i+1}) > 0$ and, for $Z \to \infty$, $G'_{i+1}(Z) < 0$. Hence, there exists $Z_{i+1}^*$ that satisfies $G'_{i+1}(Z_{i+1}^*) = 0$.

Suppose $X_{i+1}$ is the inventory level just prior to time $t_{i+1}$. As in Lemma 2, if $X_{i+1} < Z_{i+1}^*$, it does not pay to cancel reservations. Suppose we increase the inventory level up to $U_{i+1}$. Then, we wish to find

$$\operatorname*{Max}_{U_{i+1} \geqslant X_{i+1}} \{ -c_{i+1} U_{i+1} + G_{i+1}(U_{i+1}) \}. \tag{7}$$

Considering Remark 1, there exists $U_{i+1}^*$ such that if $X_{i+1} < U_{i+1}^*$, the inventory level is increased to $U_{i+1}^*$. This shows the first part of claim (a).

Differentiating the term in brackets in (7) yields claim (b) for $U_{i+1}^*$. Since, for $c_{i+1} > 0$, $G'_{i+1}(U_{i+1}^*) > 0$, it follows that $U_{i+1}^* < Z_{i+1}^*$. A similar argument regarding $V_{i+1}^*$ completes the proof.

We now generalize Theorem 1 for the case where the $Y_{i+1}$ new requests for booking which flow into the system during time interval $(t_{i+1}, t_i)$ are taken into consideration. A decision at time $t_i$ is one of the following three actions: (i) confirming some or all of the $Y_{i+1}$ new requests (where no costs are involved); (ii) increasing the inventory level by more than $Y_{i+1}$ (at a cost of $c_i$ for each reservation in excess of $Y_{i+1}$); (iii) declining all $Y_{i+1}$ new requests and canceling some of the already confirmed reservations at a penalty of $b_i$ per cancellation.

LEMMA 3. *For fixed $y$, define*

$$G_1(Z, y) = \int_0^{(M-y)/Z} a_0(pZ + y)\, dF_1(p) + \int_{(M-y)/Z}^{M/Z} a_0 M\, dF_1(p)$$
$$+ \int_{M/Z}^{1} [a_0 M - b_0(pZ - M)]\, dF_1(p),$$

*and let*

$$E_{Y_1} G_1(Z, Y_1) = \int_0^1 G_1(Z, y)\, dQ_1(y)$$

*be the expected value of $G_1(Z, Y_1)$ with respect to $Y_1$. Then, $E_{Y_1} G_1(Z, Y_1)$ is a concave function of $Z$.*

PROOF. For fixed $y$, differentiating twice with respect to $Z$, yields

$$G_1''(Z, y) = [-a_0(M-y)^2/Z^3] f_1((M-y)/Z) - [b_0 M^2/Z^3] f_1(M/Z) < 0$$

where $f_1(\cdot)$ is the probability density function of $P_1$. Since $Q_1(\cdot)$ is a distribution function, it readily follows that $\int_0^1 G_1(Z, y)\, dQ_1(y)$ is concave. Q.E.D.

Note that $E_{Y_1}G_1(Z, Y_1) = G_1(Z)$, the expected net profit, when the new requests are taken into account.

THEOREM 2. *Allowing the possibility of flows of requests, $Y_i$'s, during the time intervals $(t_i, t_{i-1})$, $i = n, n-1, n-2, \ldots, 2, 1$, there exist numbers $U_i^*$, $Z_i^*$, $V_i^*$ satisfying $U_i^* < Z_i^* < V_i^*$ and $G_i'(U_i^*) = c_i$, $G_i'(Z_i^*) = 0$, $G_i^*(V_i^*) = -b_i$, such that the optimal strategy is a 3-region policy as follows*:

(i) *if $X_i + Y_{i+1} \leqslant U_i^*$, all new requests should be confirmed and additional $U_i^* - X_i - Y_{i+1}$ reservations acquired.*

(ii) *if $U_i^* < X_i + Y_{i+1} \leqslant Z_i^*$, all $Y_{i+1}$ new requests should be confirmed.*

(iii) *if $Z_i < X_i + Y_{i+1}$ but $X_i \leqslant V_i^*$,* $\max(0, Z_i^* - X_i)$ *new requests are confirmed.*

(iv) *if $V_i^* < X_i$, $X_i - V_i^*$ reservations should be canceled.*

PROOF. Consider period 1. Let $Z_1$ be the inventory level after an action has been taken at time $t_1$, and suppose that $Y_1$ new requests have arrived during $(t_1, t_0)$. If at time $t_0 = 0$ the number of uncanceled bookings, $P_1Z_1$, is less than $M$, then $\min(M - P_1Z_1, Y_1)$ reservations out of the new $Y_1$ should be confirmed since the total number of confirmed reservations at that time may not exceed $M$. If, on the other hand, $P_1Z_1 > M$, none of the $Y_1$ new requests will be confirmed. Hence, the net profit realized at time $t_0 = 0$ is $a_0(P_1Z_1 + Y_1)$ if $P_1Z_1 + Y_1 \leqslant M$; it is $a_0M$ if $M - Y_1 < P_1Z_1 \leqslant M$, and is equal to $a_0M - b_0(P_1Z_1 - M)$ if $P_1Z_1 > M$. Thus, for given $Y_1 = y$, the net profit is $G_1(Z_1, y)$ defined in Lemma 3 and the expected net profit is $G_1(Z_1) = E_{Y_1}G_1(Z_1, Y_1)$.

Now, from Lemma 3, $G_1(Z)$ is concave. Hence, by Remark 1, and following the reasoning of Lemma 2, there exist numbers $U_1^*$, $Z_1^*$ and $V_1^*$ (depending on $Q_1(\cdot)$) comprising a 3-region optimal strategy where $G_1'(U_1^*) = c_1$, $G_1'(V_1^*) = -b_1$, and $G_1'(Z_1^*) = 0$. Also, by differentiation, it readily follows that $Z_1^*$ satisfies

$$a_0E(P_1)/(a_0 + b_0)$$
$$= \int_{M/Z_1^*}^{1} p\, dF_1(p) + (a_0/(a_0 + b_0)) \int_{y=0}^{\infty} \left[ \int_{(M-y)/Z_1^*}^{M/Z_1^*} p\, dF_1(p) \right] dQ_1(y).$$

Let $G_i(Z)$ be the expected net profit over $(t_i, 0]$ if the inventory level at time $t_i = 0$ is $Z$ and optimal overbooking strategy is followed thereafter. Assume that $G_i(Z)$ is concave with a maximum at $Z_i^*$ and the optimal strategy at time $t_i$ is a 3-region policy with boundaries $U_i^*$ and $V_i^*$ ($U_i^* < Z_i^* < V_i^*$) such that $G_i(U_i^*) = c_i$, and $G_i'(V_i^*) = -b_i$. We now show that $G_{i+1}(\cdot)$ also possesses the properties of $G_i(\cdot)$. Let $Z$ be the number of confirmed reservations at time $t_{i+1} - 0$. At time $t_i + 0$ the system may be found in various states:

(a) if $P_{i+1}Z < U_i^*$ then there are three possibilities:

(1) $Y_{i+1} + P_{i+1}Z < U_i^*$. In such a case the inventory level should be increased to $U_i^*$.

(2) $U_i^* \leqslant Y_{i+1} + P_{i+1}Z \leqslant Z_i^*$. This implies that all $Y_{i+1}$ reservations should be accepted.

(3) $Z_i^* < Y_{i+1} + P_{i+1}Z$. This event dictates that $Z_i^* - P_{i+1}Z$ requests out of the $Y_{i+1}$ new ones should be accepted such that the inventory level will become equal to $Z_i^*$.

(b) if $U_i^* \leqslant P_{i+1}Z < Z_i^*$, two cases may occur:

(1) $Y_{i+1} + P_{i+1}Z < Z_i^*$. This implies that all $Y_{i+1}$ new requests are accepted.

(2) $Z_i^* \leqslant Y_{i+1} + P_{i+1}Z$. Only $Z_i^* - P_{i+1}Z$ requests out of the $Y_{i+1}$ are accepted.

(c) if $Z_i^* \leqslant P_{i+1}Z < V_i^*$ nothing should be done.

(d) if $V_i^* \leqslant P_{i+1}Z$, $P_{i+1}Z - V_i^*$ reservations should be canceled.

The explicit expression for $G_{i+1}$ should be written separately for each of the following three cases depending on the actual value, $y$, of $Y_{i+1}$: (i) $y < U_i^*$, (ii) $U_i^* \leqslant y \leqslant Z_i^*$, (iii) $Z_i^* < y$.

For the case where $y < U_i^*$ we have:

$$G_{i+1}(Z) = \int_{p=0}^{(U_i^*-y)/Z} [G_i(U_i^*) - c_i(U_i^* - y - pZ)]\, dF_{i+1}(p)$$
$$+ \int_{(U_i^*-y)/Z}^{\min((Z_i^*-y)/Z,\, U_i^*/Z)} G_i(y + pZ)\, dF_{i+1}(p)$$
$$+ \int_{\min((Z_i^*-y)/Z,\, U_i^*/Z)}^{U_i^*/Z} G_i(Z_i^*)\, dF_{i+1}(p)$$
$$+ \int_{U_i^*/Z}^{\max((Z_i^*-y)/Z,\, U_i^*/Z)} G_i(y + pZ)\, dF_{i+1}(p)$$
$$+ \int_{\max((Z_i^*-y)/Z,\, U_i^*/Z)}^{Z_i^*/Z} G_i(Z_i^*)\, dF_{i+1}(p)$$
$$+ \int_{Z_i^*/Z}^{V_i^*/Z} G_i(pZ)\, dF_{i+1}(p) + \int_{V_i^*/Z}^{1} [G_i(V_i^*) - b_i(pZ - V_i^*)]\, dF_{i+1}(p).$$

For both possibilities, $Z_i^* - y \leqslant U_i^*$, or $Z_i^* - y > U_i^*$, we derive:

$$G'_{i+1}(Z) = c_i \int_0^{(U_i^*-y)/Z} p\, dF_{i+1}(p) + \int_{(U_i^*-y)/Z}^{(Z_i^*-y)/Z} pG_i'(y + pZ)\, dF_{i+1}(p)$$
$$+ \int_{Z_i^*/Z}^{V_i^*/Z} pG_i'(pZ)\, dF_{i+1}(p) - b_i \int_{V_i^*/Z}^{1} p\, dF_{i+1}(p).$$

Taking the second derivative of $G_{i+1}(Z)$ while using $G_i'(Z_i^*) = 0$, $G_i'(U_i^*) = c_i$, and $G_i'(V_i^*) = -b_i$ we get

$$G''_{i+1}(Z) = \int_{(U_i^*-y)/Z}^{(Z_i^*-y)/Z} p^2 G_i''(y + pZ)\, dF_{i+1}(p) + \int_{Z_i^*/Z}^{V_i^*/Z} p^2 G_i''(pZ)\, dF_{i+1}(p).$$

Since $G_i''(\cdot) < 0$, it follows that $G''_{i+1}(Z) < 0$, i.e., $G_{i+1}$ is concave for $y < U_i^*$.

Following a similar analysis it may be shown that $G_{i+1}(\cdot)$ is concave when $U_i^* \leqslant y \leqslant Z_i^*$ or when $Z_i^* < y$. The detailed calculation, however, will be omitted.

Considering Remark 1, it follows that there exist numbers $U_{i+1}^*$, $Z_{i+1}^*$ and $V_{i+1}^*$ ($U_{i+1}^* < Z_{i+1}^* < V_{i+1}^*$) which determine a 3-region policy as stated in the theorem, and satisfy $G'_{i+1}(U_{i+1}^*) = c_{i+1}$, $G'_{i+1}(Z_{i+1}^*) = 0$, and $G'_{i+1}(V_{i+1}^*) = -b_{i+1}$. This completes the proof.

## Expected Discounted Total Profit

In this section we show that the three-region structure of the optimal policy prevails for the expected discounted total profit objective function as well. The results are summarized in Theorem 3 below. For ease of presentation (as we did in developing Theorem 1), we ignore the flow of new requests $Y_i$ during period $i$. However, as was shown in Theorem 2, the analysis may be carried out without this restriction.

THEOREM 3. *Let $\beta_i \in (0, 1)$ be the discount factor for period i and, for the time interval $(t_{i+1}, 0]$, let $G_{i+1}(Z)$ be the maximal expected discounted net profit. Then, there exist numbers $U_i^*(\beta_i)$ and $V_i^*(\beta_i)$, $i = 1, 2, \ldots, n$, which determine a 3-region optimal policy (in the sense of Theorems 1 and 2) satisfying $G_i'(U_i^*(\beta_i)) = c_i/\beta_i$ and $G_i'(V_i^*(\beta_i)) = -b_i/\beta_i$.*

PROOF. Let $X_1$ be the inventory level just prior to making a decision at time $t_1$, and let $G_1(Z)$ and $G_1(Z^*)$ be given by equations (3) and (2), respectively. Then, as in Lemma 2, if $X_1 \leqslant Z^*$ we wish to find $U_1^* = U_1^*(\beta_1)$ so as to

$$\underset{U \geqslant X_1}{\text{Maximize}} \{ -c_1 U + \beta_1 G_1(U) \} \tag{8}$$

and if $X_1 \geqslant Z^*$ we wish to find $V_1^* = V_1^*(\beta_1)$ so as to

$$\underset{V \leqslant X_1}{\text{Maximize}} \{ b_1 V + \beta_1 G_1(V) \} \tag{9}$$

Since $G_1(\cdot)$ is concave the 3-region optimality is readily established with $G_1'(U_1^*) = c_1/\beta_1$ and $G_1'(V_1^*) = -b_1/\beta_1$.

To develop the induction step, assume that $G_i(\cdot)$ possesses the required properties and write:

$$\begin{aligned} G_{i+1}(Z) = &\int_0^{U_i^*/Z} [-c_i(U_i^* - pZ) + \beta_i G_i(U_i^*)]\, dF_{i+1}(p) \\ &+ \int_{U_i^*/Z}^{V_i^*/Z} \beta_i G_i(pZ)\, dF_{i+1}(p) \\ &+ \int_{V_i^*/Z}^{1} [b_i(pZ - V_i^*) + \beta_i G_i(V_i^*)]\, dF_{i+1}(p) \end{aligned}$$

Differentiating twice with respect to $Z$ and substituting $G_i'(U_i^*) = c_i/\beta_i$ and $G_i'(V_i^*) = -b_i/\beta_i$, yields

$$G''_{i+1}(Z) = \int_{U_i^*/Z}^{V_i^*/Z} \beta_i p^2 G_i''(pZ)\, dF_{i+1}(p).$$

Since, by assumption, $G_i(\cdot)$ is concave, it follows that so is $G_{i+1}(\cdot)$. Now, if $X_{i+1}$ is the inventory level at time $t_{i+1}$ our problem is similar to (8) and (9) above, with the index $i+1$ replacing the index 1. It readily follows that $G_{i+1}'(U_{i+1}^*) = c_{i+1}/\beta_{i+1}$ and $G_{i+1}'(V_{i+1}^*) = -b_{i+1}/\beta_{i+1}$. This completes the proof.

## Discussion

We have studied a restrictive model of the hotel overbooking phenomena. Many extensions may now be carried out. Examples are: (1) where lengths of reservations are for periods of several days rather than for a single day; (2) when dealing with several types of rooms rather than with a single type; and (3) for finding the appropriate allocation of rooms among the various agents.

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