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Optimal Structures and Maintenance Policies for PABX Power Systems

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A power system for a private automatic branch exchange (PABX) consists of n independent and identical rectifiers hooked up in parallel. The lifetime of each component is an exponential random variable with a known mean (*MTBF*). The system fails when the number of operating rectifiers is less than $m(m \leqq n)$.

Rectifiers are distinguished by two main characteristics: the current they carry and their *MTBF*. The minimal number of units, m, is determined by technological considerations: for each type of rectifier, m is calculated so that the system can carry a given maximal current. This maximal current is specified for each PABX.

The system is maintained by a technician who may visit the exchange regularly every v units of time or when it fails. Several maintenance policies are considered. For each type of exchange and for each form of maintenance policy, we determine the optimal set of parameters, $\{m, n, MTBF, \text{ and } v\}$, to minimize the operating cost for a required level of reliability. These results are then used as guidelines for standardizing PABX power systems.

A POWER SYSTEM for a private automatic branch exchange (PABX) consists of n independent and identical components (rectifiers) hooked up in parallel, as shown in Figure 1. The system converts AC into DC. Each of the n components may fail randomly. The system is operative so long as at least m units are operating; i.e., it fails when the number of operating components is less than $m(m \leqq n)$. Rectifiers are distinguished by two main characteristics: the current they carry and their mean time between failures (*MTBF*). The minimal number of units, m, is determined so that the system can carry a given maximal current, $I_{\max}$.

It is customary to add r "redundancy" rectifiers so that the total number of units in the system is $n=m+r$. The system is maintained by a technician who visits the exchange from time to time according to a specified maintenance policy. At each visit the technician replaces all non-operating components.

We study various types of PABX's characterized by their maximal

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currents. (The common types are $I_{\max}$=1, 3, 6, 10, 16, 24, 36, 48, 72, and 96 amperes.) The power system for an exchange may consist of different sets of identical rectifiers. For example, a 6-ampere exchange may have rectifiers of 1 ampere each, or 3, 6, or even 10 amperes.

Our objective is to find the optimal power system configuration for each PABX under various maintenance policies. For each of the maintenance policies considered, we find the optimal set of m, n and $MTBF$, and intervals between technician's visits, which minimize the average total cost under a reliability constraint. The constraint was specified by field engineers, who require a small probability of system failure because the time-dependent cost of such a failure is difficult or impossible to estimate.

We first describe the mathematical model and derive the probability distribution function of the system's lifetime. Next we describe various

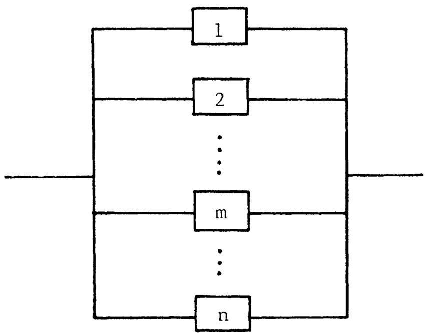

Figure 1. n independent and identical components (rectifiers), hooked up in parallel, are shown for a power system for a private automatic branch exchange (PABX).

maintenance policies, define and calculate the overall cost function for each policy, and analyze it. Finally, we give numerical results and discuss their implications.

1. DISTRIBUTION OF THE SYSTEM'S LIFETIME

Let n be the number of independent identical components in parallel. The lifetime of each component is exponentially distributed with mean $1/\mu = MTBF$. The system is in operating condition so long as the number of operating units is not less than $m (m \leqq n)$.

Let T_n^m be the time until system failure. T_n^m is the time until the $(n-m+1)$st unit fails. That is,

$$T_n^m = \sum_{i=1}^{n-m+1} X_i \qquad (m \leqq n), \tag{1}$$

where X_i $(i=1, \cdots, n-m+1;\ X_0=0)$ is the time between the $(i-1)$st and the ith failures of individual components. Since the lifetime of each component is exponentially distributed, X_i follows an exponential distribution with parameter $(n-i+1)\mu$ (e.g., see [1, p. 59]).

The following results will be needed (e.g., see [1, pp. 60–63]):

$$P\{T_n^m \leqq v\} \equiv F_{T_n^m}(v) = \sum_{k=0}^{m-1} \binom{n}{k} (\exp(-\mu v))^k (1-\exp(-\mu v))^{n-k}, \quad (2)$$

and

$$E\{T_n^m\} = (1/\mu) \sum_{k=m}^{n} 1/k. \quad (3)$$

2. MAINTENANCE POLICIES

Several maintenance policies are considered here. The policies differ according to the technician's visits. We make the following general assumptions: (1) The components are completely effective until they fail, after which they are completely ineffective; (2) When a replacement occurs, either a completely new component is inserted, or the old component is repaired so that the p.d.f. of its lifetime is that of a new component.

(a) *Periodic maintenance.* Following a fixed schedule, the technician visits the system every v units of time. He replaces all units that have failed during the time interval v. Note that the system might be in a non-operating condition for some time before the periodic maintenance takes place, and the technician would have to replace more than $n-m+1$ components. The system is therefore so designed that the probability of such an event will not exceed a specified level α.

(b) *Emergency maintenance.* The system is visited only when the number of operating units falls short of m. That is, when the $(n-m+1)$st failure occurs, the technician makes an emergency visit and replaces all $n-m+1=r+1$ non-operating units. (We assume that the time it takes to replace the units is negligible.) Instants of replacement constitute renewal points, where times between successive renewals are i.i.d. random variables distributed like T_n^m.

(c) *Periodic-emergency maintenance.* This policy is a combination of the first two. Basically, there is a planned schedule of visits every v units of time. However, if the system fails within this interval, the technician makes an immediate emergency visit and replaces all $n-m+1$ failed components. The next visit takes place v units of time after the emergency visit, unless the system again fails before that time.

(d) *Periodic maintenance and emergency replacements.* This is a modification of policy (c). Scheduled visits take place v units of time apart. In addition, each system-failure requires an emergency visit. The technician follows his preplanned schedule at times kv ($k=0, 1, 2, \cdots$) even if the emergency replacement takes place just before a periodic visit at time kv. On the surface, this policy seems to be inferior to the preceding one. However, since the technician usually maintains several exchanges, it might in certain cases be more efficient for him to stick to his planned schedule augmented by emergency visits.

3. THE COST FUNCTIONS

Let Y be the time between two successive visits, and let Z be the number of components that failed within the time interval $(0, Y]$. For each of the policies (a), (b), and (c), we would like to find the best set $\{m, n, MTBF, \text{and } v\}$ that minimizes the rate of cost

$$D_n^m = c_1 n + [c_2 E\{Z\} + K]/E\{Y\} \tag{4}$$

where

c_1 = investment cost of a component per unit time (for a system designed for a given technological lifetime);
c_2 = unit cost for replacement of failed component;
K = fixed cost for a technician's visit.

For policy (d) we wish to minimize the cost function

$$D_n^m = c_1 n + \Big[c_2 \begin{Bmatrix}\text{expected number of units}\\ \text{replaced within } (0, v]\end{Bmatrix} + \begin{Bmatrix}\text{expected number of}\\ \text{visits within } (0, v]\end{Bmatrix} K\Big]/v. \tag{5}$$

We now analyze in detail each of the four maintenance policies.

Periodic Maintenance

Here we readily have $E\{Y\} = v$ and $E\{Z\} = n(1-\exp(-\mu v))$. As mentioned above, for any given m, the optimal n, n^* is determined such that

$$P(T_{n^*}^m \leqq v) \leqq \alpha. \tag{6}$$

Substituting the above values in (4) yields $D_{n^*}^m = c_1 n^* + [c_2 n^* (1-\exp(-\mu v)) + K]/v$.

For each of the various PABX's detailed numerical calculations were performed (see [4]). The optimal values of m, n^*, $MTBF$, and v are given in Section 4.

Emergency Maintenance

In this case $E\{Y\} = E\{T_n^m\}$, while $E\{Z\} = n-m+1$. Thus, from (3),

$$D_n^m = c_1 n + [c_2(n-m+1) + K]\mu / \textstyle\sum_{i=m}^n (1/i). \tag{7}$$

Using the fact that, for given m and μ, D_n^m is a discrete convex function of n, we determine the optimal value of n that minimizes D_n^m to be either 1 or such that $D_n^m \leqq D_{n+1}^m$ and $D_n^m < D_{n-1}^m$.

Periodic-Emergency Maintenance

Visits take place either v units of time after the previous visit or when system-failure occurs—i.e., after T_n^m units of time. Specifically, $Y=\min\{T_n^m, v\}$. Hence, we have [3, p. 38] $E\{Y\}=v-\int_0^v F_{T_n^m}(t)dt$. Using (2) we derive

$$E\{Y\}=v-\sum_{k=0}^{m-1}\binom{n}{k}\int_0^v[\exp(-\mu kt)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\exp(-\mu it)]dt$$

$$=v-\sum_{k=0}^{m-1}\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}(-1)^i[1-\exp(-\mu(k+i)v)]/\mu(k+i). \tag{8}$$

The expected number of failures within (0, Y] is given by

$$\begin{aligned} E\{Z\} &= \sum_{j=0}^{n-m}j\binom{n}{j}(1-\exp(-\mu v))^j(\exp(-\mu v))^{n-j} \\ &\quad +(n-m+1)F_{T_n^m}(v) \\ &=\sum_{k=m}^{n}(n-k)\binom{n}{k}(\exp(-\mu v))^k(1-\exp(-\mu v))^{n-k} \\ &\quad +(n-m+1)F_{T_n^m}(v) \qquad (9) \\ &=n[1-F_{T_n^m}(v)]-\sum_{k=m}^{n}k\binom{n}{k}(\exp(-\mu v))^k(1-\exp(-\mu v))^{n-k} \\ &\quad +(n-m+1)F_{T_n^m}(v) \\ &=n-(m-1)F_{T_n^m}(v)-n\exp(-\mu v)[1-F_{T_{n-1}^m}(v)]. \end{aligned}$$

Note that by careful interpretation of (9) one may obtain the corresponding results for emergency maintenance and periodic maintenance as follows. For the former, $v=\infty$. Letting $F(\infty)=1$ in (9) yields $E\{Z\}=n-m+1$. For periodic maintenance, $F(v)$ represents the probability that the technician will make a visit before time v; obviously, this probability equals zero. Putting $F(v)=0$ in (9) yields $E\{Z\}=n(1-\exp(-\mu v))$, as was obtained directly.

To find the optimal set of $\{m, n, MTBF$ and $v\}$, we use (4) with $E\{Y\}$ and $E\{Z\}$ given by (8) and (9), respectively. Two procedures for deriving n may be employed. One way is to look for n^*, which, for every m, determines $D_{n^*}^m=\min_n\{D_n^m\}$. The second way (adopted here) is to find n^* such that $P\{T_{n^*}^m\leq v\}\leq\alpha$, as was done for the case of periodic maintenance.

Thus, our objective in this case is to minimize the cost function

$$D_{n^*}^m=c_1n^*+[c_2E\{Z\}+K]/E\{Y\}. \tag{10}$$

Periodic Maintenance and Emergency Replacement

Let $N(v)$ be the number of emergency visits within $(0, v]$, and let $M(v)=EN(v)$. $M(v)$ satisfies the renewal equation $M(v)=F_{T_n^m}(v)+\int_0^v M(v-t)dF_{T_n^m}(t)$. The Laplace transform of $M(v)$ is given by (see [3, p. 236]) $\tilde{M}(s)=\tilde{T}_n^m(s)/[1-\tilde{T}_n^m(s)]$, where $\tilde{T}_n^m(s)=E\{\exp(-sT_n^m)\}$ is the Laplace transform of T_n^m. Since T_n^m is the sum of exponentially distributed independent random variables (see (1)), its Laplace transform is the product of the transforms of the X_i's, namely, $\tilde{T}_n^m(s)=\Pi_{k=m}^n k\mu/(k\mu+s)$. Hence,

$$\tilde{M}(s)=\Pi_{k=m}^n(k\mu)/[\Pi_{k=m}^n(k\mu+s)-\Pi_{k=m}^n(k\mu)]. \tag{11}$$

To find $M(v)$, one has to invert $\tilde{M}(s)$ as given by (11). A general inversion seems to be difficult, if not impossible. However, for $m=n$, $m=n-1$, and $m=n-2$, we obtain explicit formulas. These formulas are applicable since, as will be evident from the numerical results, in most cases it is optimal to have $n=m+1$ components in the system.

For $m=n$, $\tilde{M}(s)=n\mu/s$. Thus, $M(v)=n\mu v$, as was expected because of the Markovian properties of the process.

For $m=n-1$, $\tilde{M}(s)=n\mu(n-1)\mu/s[s+(2n-1)\mu]=n(n-1)\mu(2n-1)^{-1}[1/s-1/(s+(2n-1)\mu)]$. Thus, $M(v)=n(n-1)\mu(2n-1)^{-1}v-n(n-1)(2n-1)^{-2}[1-\exp(-(2n-1)\mu v)]$.

For $m=n-2$,

$$\begin{aligned}\tilde{M}(s)&=n(n-1)(n-2)\mu^3/s[s^2+3(n-1)\mu s+(3n^2-6n+2)\mu^2]\\&=n(n-1)(n-2)\mu^3(a^2+b^2)^{-1}[1/s-(s-2a)/((s-a)^2+b^2)]\end{aligned}$$

where $a = 3\mu(1-n)/2$, $2b = \mu(3n^2-6n-1)^{1/2}$ and $a^2+b^2=\mu^2(3n^2-6n+2)$. From [2, p. 229] it follows that $dM(v)/dv=n(n-1)(n-2)\mu(3n^2-6n+2)^{-1}[1-\exp(av)(\cos(bv)-(a/b)\sin(bv))]$, which implies that

$$\begin{aligned}M(v)=n(n-1)(n-2)\mu(3n^2-6n+2)^{-1}\{&v-\exp(av)(a^2+b^2)^{-1}\\&[a\cos(bv)+b\sin(bv)\\&-(a/b)(a\sin(bv)-b\cos(bv))]-2a/(a^2+b^2)\}.\end{aligned}$$

The objective here (see (5)) is to find the optimal set that minimizes

$$D_{n^*}^m=c_1n^*+[c_2\left\{\begin{matrix}\text{expected number of units}\\ \text{replaced within } (0, v]\end{matrix}\right\}+\{M(v)+1\}K]/v, \tag{12}$$

where $\{M(v)+1\}$ = expected number of visits within $(0, v]$. Note that here too n^* is determined by (6).

The next step is to calculate the expected number of units replaced within $(0, v]$. Let $\{(T_n^m)_i\}$ be a sequence of i.i.d. random variables having the common distribution function $F_{T_n^m}(\cdot)$. Let $S_q=\sum_{i=1}^q(T_n^m)_i$ be the time until the qth system-failure within an interval of length v, and denote by $F_{S_q}(t)$ and $f_{S_q}(t)$ the distribution and density functions of S_q.

For $k \leqq n-m=r$, let $b(k;\, n,\, t) \equiv \binom{n}{k}(1-\exp(-\mu t))^k(\exp(-\mu t))^{n-k}$ be the probability of k component failures within $(0, t]$. Write $B(r;\, n,\, t)=\sum_{k=0}^{r} b(k;\, n,\, t)$. Clearly, $B(r;\, n,\, t)=1-F_{T_n^m}(t)=1-F_{S_1}(t)$ is the probability of no system failure within $(0, t]$.

Let R be the total number of units replaced within $(0, v]$. We wish to find $E\{R\}$. If no system failures occur within $(0, v]$, then the probability of k component failures ($k \leqq r$) is $b(k;\, n,\, v)$. If exactly $q \geqq 1$ system failures occur within $(0, v]$, then there are q emergency visits in addition to the planned one, and the total number of units replaced is $(r+1)q+k$ for some $k \leqq r$. The probability of such an event is

$$\int_{t=0}^{v} f_{S_q}(t)\, b(k;\, n,\, v-t)\, dt \equiv P(q,\, k), \tag{13}$$

and hence the probability of $q \geqq 1$ emergency visits within $(0, v]$ is given by

$$\begin{aligned} P\{N(v)=q\} &= \sum_{k=0}^{r} P(q,\, k) = \sum_{k=0}^{r} \int_{t=0}^{v} f_{S_q}(t)\, b(k;\, n,\, v-t)\, dt \\ &= \int_{t=0}^{v} f_{S_q}(t)[1-F_{S_1}(v-t)]\, dt = F_{S_q}(v)-F_{S_{q+1}}(v). \end{aligned} \tag{14}$$

It follows from (13) that $E\{R\}$ is

$$E\{R\} = \sum_{k=0}^{r} k\, b(k;\, n,\, v) + \sum_{q=1}^{\infty} \sum_{k=0}^{r} [(r+1)q+k] P(q,\, k).$$

Since $\sum_{k=0}^{r} k\, b(k;\, n,\, v) = n(1-\exp(-\mu v)) \sum_{k=0}^{r-1} b(k;\, n-1,\, v) = n(1-\exp(-\mu v))[1-F_{T_{n-1}^m}(v)]$ and $M(v) = \sum_{q=1}^{\infty} q P\{N(v)=q\}$, we get, using (14)

$$\begin{aligned} E\{R\} &= n(1-\exp(-\mu v))[1-F_{T_{n-1}^m}(v)] \\ &\quad + (r+1)M(v) + \sum_{q=1}^{\infty} \sum_{k=0}^{r} k P(q,\, k). \end{aligned} \tag{15}$$

The third term of the right-hand side of (15) is difficult to evaluate, but an approximation may be given, using (14):

$$\begin{aligned} \sum_{q=1}^{\infty} \sum_{k=0}^{r} k \int_{t=0}^{v} f_{S_q}(t)\, b(k;\, n,\, v-t)\, dt &\leqq \sum_{q=1}^{\infty} r \sum_{k=0}^{r} \int_{t=0}^{v} f_{S_q}(t)\, b(k;\, n,\, v-t)\, dt \\ &= r \sum_{q=1}^{\infty} [F_{S_q}(v)-F_{S_{q+1}}(v)] = rF_{T_n^m}(v). \end{aligned}$$

That is, $E\{R\} \leqq n(1-\exp(-\mu v))[1-F_{T_{n-1}^m}(v)]+(r+1)M(v)+rF_{T_n^m}(v)$. Note that, when $n>m+2$ and an explicit formula for $M(v)$ is difficult to obtain, one may use the approximation $M(v) \approx F_{T_n^m}(v)$ for small values of α.

To conclude, for any given pair of m and $\mu^{-1}=MTBF$, we wish to find the optimal values of n and v that minimize

$$D_n^m = c_1 n + [c_2 E\{R\} + [M(v)+1]K]/v,$$

where n and v are related by the constraint $P(T_n^m \leqq v) \leqq \alpha$.

4. NUMERICAL RESULTS

We examined 10 types of PABX's, distinguished from each other by the maximal current they can carry. We therefore specify each exchange by its maximal current, $I_{\max}$, which takes the values 1, 3, 6, 10, 16, 24, 36, 48, 72 and 96 amperes. Rectifiers are specified by their current and *MTBF*. The set of currents considered for rectifiers is the same as the set of values for $I_{\max}$; the values of *MTBF* are 5,000, 7,200, 10,000, 15,000, 20,000 and 25,000 hours. The minimum number of rectifiers with which a system can operate is m. For an exchange of 16 amperes, for example, we may put $m=16$ if we use rectifiers of 1A each, or $m=6$ if we use rectifiers of 3A each, or $m=3$, 2, 1, 1, 1 if we use rectifiers of 6A, 10A, 16A, 24A or 36A, respectively. For each of the above possibilities, we made calculations using all values of *MTBF*. For example, for a 16A PABX with 6A rectifiers, we put $m=3$ and calculated the total number of components, n, for different values of v, where v may take values of 1, 2, 3, 4, 5 or 6 months (no higher values of v were allowed because of various technological considerations). The values of m, n, *MTBF* and v were substituted in the various cost functions, and the optimal values were obtained.

TABLE I
OPTIMAL VALUES FOR PERIODIC MAINTENANCE

m	Current of each component	*MTBF*	n^*	v	$F_{T_n^m}(v)$
1	$I_{\max}$	20,000	2	6	0.038

Periodic maintenance. For $\alpha=0.05$ (see (6)) and for each exchange (characterized by $I_{\max}$), it was found that it is optimal to have $m=1$, $n^*=2$, $MTBF$=20,000h and $v=6$ months. These results are summarized in Table I. If we did not have the restriction that $v \leqq 6$ months, we would not have obtained the uniformity expressed in Table I.

Emergency maintenance. The optimal values that minimize (7) are presented in Table II. We conclude: (i) for all exchanges we have $m=n=1$; (ii) for exchanges with $I_{\max}=1$, 3, 6 or 10 amperes, the optimal value of *MTBF* is 25,000h, while for the others it is 20,000h.

Periodic-emergency maintenance. For $\alpha=0.05$ and for each exchange, the values that minimize (10) were found to be identical with the values given by Table I. That is, $m=1$, $n^*=2$, $MTBF$=20,000 and $v=6$.

Periodic maintenance and emergency replacement. In this case, too, the optimal values that minimize (12) are the same as those for the periodic maintenance and for the periodic-emergency maintenance. However, the values of the cost function for $m=1$, $n^*=2$, $v=6$ and $MTBF=25{,}000$ hours are very close to the optimal values obtained when $MTBF=20{,}000$ hours.

Standardization

From the numerical results given in [4], several "standard" schemes of sets of rectifiers to be used may be constructed. The actual scheme depends on various considerations, among which the cost factor is only one. A possible standard scheme for each of the maintenance policies considered, under the assumption of uniform distribution of types of exchanges, is given in Table III. In Table III A represents the current of

TABLE II
OPTIMAL VALUES OF PARAMETERS FOR EMERGENCY MAINTENANCE

I_{max}	m	Current of each unit	$MTBF$	n
1	1	1	25,000	1
3	1	3	25,000	1
6	1	6	25,000	1
10	1	10	25,000	1
16	1	16	20,000	1
24	1	24	20,000	1
36	1	36	20,000	1
48	1	48	20,000	1
72	1	72	20,000	1
96	1	96	20,000	1

the chosen rectifier, and $v=6$ for all PABX's. The set of exchanges is partitioned into three groups, within which the same type of rectifier should be used. For the group of 16, 24, or 36 amperes rectifiers with 36 amperes and $MTBF$ = 20,000 hours are favored, where $m=1$ and $n=2$. Similar results are obtained for the group of 48, 72, and 96 amperes, where the rectifiers being used are of 96 amperes. For the group of 1, 3, 6 and 10 amperes, different standard schemes are suggested depending on the specific maintenance policy.

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TABLE III
STANDARDIZATION

I_{max} \ Maintenance policy	(a) m	(a) A	(a) n	(a) $MTBF$	(b) m	(b) A	(b) n	(b) $MTBF$	(c) m	(c) A	(c) n	(c) $MTBF$	(d) m	(d) A	(d) n	(d) $MTBF$
1									1		2		1		2	
3	1	10	2	20,000	1	10	1	25,000	1	3	2	20,000	1	3	2	20,000
6									2		4		2		4	
10									4		7		4		7	
16																
24	1	36	2	20,000	1	36	1	20,000	1	36	2	20,000	1	36	2	20,000
36																
48																
72	1	96	2	20,000	1	96	1	20,000	1	96	2	20,000	1	96	2	20,000
96																

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