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SEQUENCING AN *N*-STAGE PROCESS WITH FEEDBACK

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N tasks must be successfully performed for a job to be completed. The tasks may be attempted in any order, where each attempt of task i requires an expected cost c_i and is successful with probability p_i. Whenever an attempt fails, the job is fed back to the initial stage and the entire sequence starts again. We show that the cost of completing a job is minimized if the tasks are sequenced via increasing values of $c_i/(1 - p_i)$. We further show that the same result holds when the feedback can be either to stage i itself or to the starting task.

1. INTRODUCTION

N tasks must be individually and successfully performed for a job to be completed. The tasks may be performed in any order (such as in a Flexible Manufacturing System), where task i requires random processing time X_i (equivalently, incurs expected cost c_i) when attempted. The job undergoes inspection after each attempt, passing the inspection at stage i with probability p_i. If the job passes the inspection the next task is attempted. With probability $(1 - p_i)$ the job fails the inspection at stage i, whereupon all the previous tasks must be attempted in the same order once more; that is, upon failure of any inspection, the job is fed back to the initial task and the entire sequence of operations starts again. The job is completed only when it first passes inspection after the last task, or equivalently, when all N tasks have been attempted successfully in one run.

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We derive the Laplace–Stieltjes transform of the total time required for successful completion of a job, and show that the optimal sequencing of the tasks to minimize the expected cost to completion is via increasing values of $c_i/(1 - p_i)$. We further show that the same result holds when the feedback can be either to state i itself or to the initial stage.

2. OPTIMAL SEQUENCING

Sequence the tasks in an increasing order of their indices. Let X_i $(1 \leqq i \leqq N)$ denote the random processing time requirement of task i at each attempt, and let p_i denote the probability of a successful attempt of task i whereupon the job moves to stage $i + 1$.

Define the sequence of random variables $\{Y_j\}_1^N$ and $\{Z_j\}_1^N$ as follows:

$$Y_1 = X_1, \qquad Z_1 = \sum_{m=1}^{N_1} X_{1m} \qquad Y_j = Z_{j-1} + X_j, \qquad Z_j = \sum_{m=1}^{N_j} Y_{jm}, \qquad j \geqq 2, \tag{1}$$

where $\{X_{1m}\}_{m=1}^{\infty}$ is a sequence of i.i.d. random variables all distributed as X_1; $Y_j(Z_j)$ is the first (successful) passage time through stages 1 to j; N_j is a geometric random variable with probability of success p_j; and $\{Y_{jm}\}_{m=1}^{\infty}$ is a sequence of i.i.d. random variables all distributed like Y_j. Z_N is then the total time required for a successful completion of the job.

Let $\tilde{X}(s) = E[e^{-sX}]$ denote the Laplace–Stieltjes transform of a nonnegative random variable X. Then

$$\tilde{Z}_1(S) = \frac{p_1\tilde{X}_1(s)}{1 - (1 - p_1)\tilde{X}_1(s)}, \tag{2}$$

and, for $j \geqq 2$

$$\tilde{Z}_j(s) = \frac{p_j\tilde{Y}_j(s)}{1 - (1 - p_j)\tilde{Y}_j(s)} = \frac{p_j\tilde{Z}_{j-1}(s)\tilde{X}_j(s)}{1 - (1 - p_j)\tilde{Z}_{j-1}(s)\tilde{X}_j(s)}, \tag{3}$$

as Z_{j-1} and X_j are independent.

Taking expectations leads to the set of difference equations

$$EZ_1 = EX_1/p_1, \qquad EZ_j = (EZ_{j-1} + EX_j)/p_j, \qquad j \geqq 2, \tag{4}$$

the solution of which is

$$EZ_j = \sum_{i=1}^{j} EX_i \Big/ \left(\prod_{n=i}^{j} p_n\right), \qquad j \geqq 1. \tag{5}$$

The expected cost for the successful completion of a job is then

$$C = \sum_{i=1}^{N} c_i \Big/ \left(\prod_{n=i}^{N} p_n\right). \tag{6}$$

A direct probabilistic argument that leads to Eq. (6) was suggested by the referee. Considering the ordering $1,2,\ldots,N$, the number of times task i will be performed is a geometric random variable with mean $(p_i p_{i+1} \cdots p_N)^{-1}$ since each attempt at stage i will be the last one if all of the remaining attempts (on tasks i, $i+1,\ldots,N$) are all successful. Hence, the expected cost under this sequencing is the expression given by Eq. (6).

Let Π denote the set of all $N!$ permutations of the index set $\{1,2,\ldots,N\}$. A policy $\pi \in \Pi$ is a permutation of the set $\{1,2,\ldots,N\}$ such that $\pi(i) = j$ means that task j is the ith one to be performed.

Our goal is to find a permutation $\pi \in \Pi$ such that

$$C(\pi) \equiv \sum_{i=1}^{N} c_{\pi(i)} \bigg/ \left(\prod_{n=i}^{N} p_{\pi(n)} \right) \tag{7}$$

is minimized.

THEOREM: *$C(\pi)$ is minimized if the tasks are sequenced in an increasing order of $c_i/(1-p_i)$.*

PROOF: Consider the permutation $\pi_0 = (1,2,\ldots,N)$, and the permutation $\pi_1 = (1,2,\ldots,j-1, j+1, j,\ldots,N)$ obtained from π_0 by interchanging the jth and $(j+1)$st terms. Then, by decomposing $C(\pi_0)$ and $C(\pi_1)$ into terms up to and beyond j, it follows that $C(\pi_0) \leqq C(\pi_1)$ if and only if

$$c_j/(p_j p_{j+1} P) + c_{j+1}/(p_{j+1} P) \leqq c_{j+1}(p_j p_{j+1} P) + c_j/(p_j P), \tag{8}$$

where $P = \prod_{n=j+2}^{N} p_n$.

That is, $C(\pi_0) \leqq C(\pi_1)$ if and only if $c_j/(1-p_j) \leqq c_{j+1}/(1-p_{j+1})$. By successive pairwise interchanges the proof is complete. ■

3. A TWO-WAY FEEDBACK

The results above can be extended to the case where feedback occurs either to stage i itself or all the way back to the initial stage. Specifically, suppose that each attempt of task i culminates in one of three possibilities: (i) with probability f_i, it is successful and the job is routed to the next stage; (ii) with probability q_i, it is fed back to stage i itself, whereupon it is immediately attempted once more; or (iii) with probability $(1-p_i) = 1-(f_i+q_i)$, it is returned to the initial stage to start the sequence of operations all over again.

This case can be readily reduced to the one-way feedback scheme. Note that the expected cost to move away from job i is equal to $c_i/(1-q_i)$, and the (conditional) probability that this move is onward rather than to the beginning is $f_i/(1-q_i)$. Now, the optimal sequence is, again, to order the tasks by increasing values of $[c_i/(1-q_i)]/[1-f_i/(1-q_i)] = c_i/(1-p_i)$.