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DYNAMIC PRIORITY RULES FOR CYCLIC-TYPE QUEUES

SID BROWNE,* *Columbia University*
URI YECHIALI,** *Tel Aviv University*

Abstract

A cyclic service system is composed of K channels (queues) and a single cyclically roving server who typically takes a positive amount of time to switch between channels. Research has previously focused on evaluating and computing performance measures (notably, waiting times) of fixed template routing schemes under three main service disciplines, the *exhaustive, gated* and *limited service* regimes.

In this paper, probabilistic results are derived that allow control strategies and optimal policies to be considered for the first time. By concentrating on a new objective function, we are able to derive rules of index form amenable for direct implementation to dynamically control the system at suitably defined decision epochs. These rules utilize current system information, are of an adaptive nature, and are shown to emanate from a general physical principle.

CYCLIC QUEUES; POLLING SYSTEMS; STOCHASTIC SCHEDULING

1. Introduction

A polling system, or cyclic queue, is composed of K queues, (channels) labelled $i = 1, \cdots, K$. Customers arrive to channel i in a Poisson stream of intensity λ_i, independently of the other channels. There is a single server in the system who moves from channel to channel in a 'cyclic' fashion, i.e., the server stays at channel $i (i = 1, \cdots, K)$ for a length of time determined by the queue discipline and then moves to channel $i + 1$. Upon 'completion' of channel K, the server reverts to channel 1 and so on, hence the name 'cyclic'.

Each customer in channel i carries an independent random service requirement distributed as V_i having distribution function $G_i(\cdot)$, $i = 1, \cdots, K$. The queue discipline determines how many customers are to be served in each channel. The disciplines most often studied are the *exhaustive, gated* and *limited service* regimes. To illustrate these regimes, assume the server arrives to channel i to find n_i customers waiting. Under the *exhaustive* regime, the server must service channel i until it is empty before he is allowed to move on. This amount of time is distributed as the sum of n_i *ordinary busy periods* in an $M/G_i/1$ queue. Under the *gated* regime, the server gates off those customers already present upon his arrival to channel i, and serves only them before moving on to channel $i + 1$. As such, the total service

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* Postal address: Graduate School of Business, 402 Uris Hall, Columbia University, New York, NY 10027, USA.
** Postal address: Department of Statistics, Tel Aviv University, Tel Aviv, Israel.

time in channel i is distributed as the sum of n_i *ordinary service requirements*. Under the *limited service* regimes, the server must serve either 1, at most k_i (where k_i is some fixed integer), or deplete the queue at channel i by 1 (i.e., stay one busy period of $M/G_i/1$ type).

Typically, the server takes a non-negligible amount of time to switch between channels. These switching times are assumed random and complicate the analysis greatly.

Cooper (1970) was the first to explicitly name and study cyclic queues, and Eisenberg (1972) gave a thorough analysis of equilibrium results for both the exhaustive and gated regimes, later improved upon by Baker and Rubin (1987). The exhaustive discipline with zero switching times between channels has been analyzed for $K = 2$ as an 'alternating priority queue' (Avi-Itzhak et al. (1965) and Takács (1968)). For $K = 1$ and positive switching times, the system can be modelled as a single server queue with vacations (Levy and Yechiali (1975)). Konheim and Meister (1974) studied the exhaustive discipline in depth in relation to communications systems. Results on waiting times are notoriously difficult to obtain and (pseudo)- conservation laws are surveyed in Watson (1984). Boxma and Groenendijk (1987) present a probabilistic proof of a pseudo- conservation law allowing a mixture of different service strategies at the various queues, generalizing and unifying previous results. There is an enormous literature on the theory and applications of these models surveyed and presented in Takagi (1986) and (1987).

Previous research has centered on models where the server traverses the same fixed route from cycle to cycle. As such, the switching time from channel to channel need be functionally indexed only by the channel being exited.

In this paper we seek to determine a routing scheme, or policy, to optimize some measure of system performance. If we no longer hold the server to a fixed cyclic strategy, but allow him to service channel j after channel i, $j \neq i + 1$, we must consider a *general switching time* from channel i to channel j, say $r_{ij} \geqq 0\ \forall i,j$. This of course complicates the analysis of the system by increasing the number of switching variables from K to K^2 (or $K(K-1)$ if $r_{ii} = 0$, $\forall i$). We assume that r_{ij} may be decomposed into $r_{ij} = \theta_i + \tau_j$, where θ_i is the time to switch-out of channel i and τ_j is the time to switch-into channel j. Within this framework we will also allow the server to remain idle when no customers are present in the system.

In Section 2 we model the exhaustive regime with zero switching times as a Markov decision process over an unbounded horizon to minimize total weighted waiting time in the system. The difficulties associated with this criterion are illustrated and we are led to consider a 'limited horizon lookahead criterion', *minimizing cycle time* (to be defined later). To that end, we first introduce in Section 3 some probabilistic results on cycle times for the exhaustive and gated regimes. In Section 4 we define the criteria of minimizing cycle time, and use the results of Section 3 to obtain some surprising rules of a simple form for both the exhaustive and gated regimes with positive switching times. The results are then unified via a general principle. In Section 5 we discuss the implementation of these rules and some of their ramifications in applications.

2. Formulation as a Markov decision process

In this section we model the polling system under the exhaustive regime with zero switching times as a (semi) Markov decision process to minimize the expected weighted (by costs) waiting time accruing to the system per unit time. The control variable is simply the index of the next channel to be visited, as we allow the server to choose to serve any channel with a positive number of inhabitants.

The state of the system at any point in time, t, is $Q(t)=(Q_1(t),\cdots,Q_K(t))$, where $Q_i(t)$ is the number of customers in channel i at time t. *Decision epochs* occur when the server has emptied a channel and must now move to some other channel with a positive number of occupants (as idleness is not permitted). The state space is therefore N^K where $N=\{0,1,\cdots\}$, and the action space is $A=\{1,2,\cdots,K\}$ with generic element a.

By looking at the system only at these decision epochs, we may define an embedded Markov chain over the system. If $\{t_j\}_0^\infty(t_0\equiv 0)$ are the completion times of the channels, and a_j denotes the action taken at the jth transition (i.e., a_j denotes the channel served during (t_j,t_{j+1})) then

$$P(Q(t_j)\in S \mid Q(t_{j-1}),\, a_{j-1};\, Q(t_{j-2}),\, a_{j-2};\cdots;Q(0),\, a_0)$$
$$=P(Q(t_j)\in S\mid Q(t_{j-1}),\, a_{j-1}),$$

for $S\subset N^K$. For this model it is necessary and sufficient to assume $\sum_1^K\rho_i<1$ to ensure the t_j are well-defined stopping times.

Let $Q_i(t_r)=n_i$, $i=1,\cdots,K$. Note that $Q_{a_{r-1}}(t_r)=n_{a_{r-1}}=0$ by the exhaustive property. If X_{a_r} denotes the service sojourn in channel a_r, then it is known (Takács (1967), Cooper (1981), p. 231) that

$$P(X_{a_r}<t\mid Q(t_r),\, a_r)=\sum_{n=n_{a_r}}^{\infty}\frac{n_{a_r}}{n}\int_0^t\frac{(\lambda_{a_r}x)^{n-n_{a_r}}}{(n-n_{a_r})!}e^{-\lambda_{a_r}x}\,dG_{a_r}^{*n}(x)\equiv F_{x_{a_r}}(t) \tag{2.1}$$

where G^{*n} denotes the nth convolution of G. Alternatively, conditional upon $Q(t_r)$, $X_{a_r}=\sum_{i=1}^{n_{a_r}}B_{a_r i}$, where $B_{a_r i}$ is a random variable distributed as an ordinary $M/G_{a_r}/1$ busy period. As such, if for any random variable Y we let $\tilde{Y}(s)$ denote its Laplace–Stieltjes transform, we have (where all expectations are conditioned by $Q(t_r),\, a_r$),

$$\tilde{X}_{a_r}(s)=(\tilde{B}_{a_r}(s))^{n_{a_r}}=\tilde{V}_{a_r}(s+\lambda_{a_r}-\lambda_{a_r}\tilde{B}_{a_r}(s))^{n_{a_r}}, \tag{2.2}$$

leading to

$$E(X_{a_r})=n_{a_r}\frac{E(V_{a_r})}{1-\rho_{a_r}} \tag{2.3}$$

$$E(X_{a_r}^2)=\frac{n_{a_r}}{(1-\rho_{a_r})^2}\left(\frac{E(V_{a_r}^2)}{1-\rho_{a_r}^2}+(n_{a_r}-1)(E(V_{a_r}))^2\right). \tag{2.4}$$

Note too that

$$Q_i(t_{r+1}) = \begin{cases} n_i + m_i & i \neq a_r \\ 0 & i = a_r \end{cases} \tag{2.5}$$

where m_i denotes the number of arrivals to channel i during the service sojourn at a_r, i.e., $m_i \equiv N_i(X_{a_r})$, where $N_i(X_{a_r})$ is a Poisson random variable with intensity λ_i.

Letting $\mathbf{Q}(t_r) \equiv \mathbf{Q}$, and $\mathbf{Q}(t_{r+1})(\equiv \mathbf{Q}(t_r + X_{a_r})) \equiv \mathbf{Q}'$, the process $\mathbf{Q}(\cdot)$ is semi-Markov with kernel

$$P(\mathbf{Q}', t_{r+1} - t_r \leqq t \mid \mathbf{Q}, a_r) = \int_0^t \exp\left(\left(-\sum_{i \neq a_r} \lambda_i\right)x\right) \prod_{i \neq a_r} \frac{(\lambda_i x)^{m_i}}{m_i!} dF_{x_{a_r}}(x). \tag{2.6}$$

Let

$$P^{(a)}_{\mathbf{Q},\mathbf{Q}'} \equiv P(\mathbf{Q}', t_{r+1} - t_r \leqq \infty \mid \mathbf{Q}, a_r = a) \tag{2.7}$$

so that $\{P^{(a)}_{\mathbf{Q},\mathbf{Q}'}\}$ is the set of one-step transition probabilities for the system.

To complete the formulation as a (semi) Markov decision process, we need the costs associated with a transition. We assume *linear holding costs* in that the system is charged $\$c_i$ per unit time for each type i customer waiting in the system.

The costs incurred in channel i, $i \neq a$ is the cost of holding the original n_i customers through X_a, i.e. $c_i n_i X_a$, plus the cost of holding those arriving during X_a. Following Yechiali (1976), by Poisson arrivals, the *expected total wait of the new arrivals* during X_a can be written as the product

$$E(\text{wait of an } \textit{arbitrary} \text{ arrival}) \cdot E(\text{number of new arrivals}).$$

The expected number of arrivals is $\lambda_i E(X_a)$, and the expected waiting time of an arbitrary arrival during X_a is, by Poisson arrivals, the *random modification,* or forward recurrence time of X_a, $E(X_a^2)/2E(X_a)$. Therefore, the total expected costs incurred in the non-serviced channels is

$$\sum_{i \neq a} c_i \left(n_i E(X_a) + \frac{\lambda_i}{2} E(X_a^2) \right). \tag{2.8}$$

The total expected waiting time in Channel a is

$$E(\text{wait of original } n_a) + E(\text{wait of new arrivals during } X_a). \tag{2.9}$$

Now, once again by Poisson arrivals, the expected wait of new arrivals (under a first-come first-served discipline) during X_a is equivalent to

$$E(\text{number of new } a\text{-arrivals during } X_a) \cdot E(\text{wait for arbitrary new arrival}).$$

Treating the service time of the original n_a customers as a 'vacation' in an $M/G_a/1$ queue, we recall Equation (38) of Levy and Yechiali (1975):

$$E(W) = \frac{\lambda E(V^2)}{2(1-\rho)} + \frac{E(U^2)}{2E(U)}, \tag{2.10}$$

where U corresponds to the 'vacation period', and W is the average wait (exclusive of service) experienced by an arriving customer in an $M/G/1$ queue with arrival rate λ, service times distributed as V and a vacation period U. As such, we have (where V_{ak} is distributed as V_a)

$$
\begin{aligned}
U &= \sum_{k=1}^{n_a} V_{ak} \\
E(U) &= n_a E(V_a) \\
E(U^2) &= n_a E(V_a^2) + n_a(n_a-1)(E(V_a))^2.
\end{aligned}
\tag{2.11}
$$

From Equations (2.10), (2.11), we may write the expected wait of an arbitrary arrival to channel a during a service sojourn at channel a with n_a initial customers in queue, as

$$
\frac{1}{1-\rho_a}\frac{E(V_a^2)}{2E(V_a)} + \frac{n_a-1}{2}E(V_a). \tag{2.12}
$$

The expected waiting time for the original n_a customers during X_a is

$$
E\left(\sum_{j=1}^{(n_a-1)}\sum_{k=1}^{j} V_{ak}\right) = \frac{n_a(n_a-1)}{2}E(V_a). \tag{2.13}
$$

The expected number of arrivals to channel a during X_a is $n_a\rho_a/(1-\rho_a)$, therefore using (2.9), (2.12), (2.13), the total expected wait incurred at channel a during X_a is

$$
\begin{aligned}
&\frac{n_a(n_a-1)}{2}E(V_a) + \frac{n_a\rho_a E(V_a^2)}{2E(V_a)(1-\rho_a)^2} + \frac{n_a(n_a-1)\rho_a E(V_a)}{2(1-\rho_a)} \\
&= \frac{n_a(n_a-1)E(V_a)}{2(1-\rho_a)} + \frac{n_a\lambda_a E(V_a^2)}{2(1-\rho_a)^2}.
\end{aligned}
\tag{2.14}
$$

Using (2.8), (2.3), (2.4) and (2.14) we may now write the *total expected cost* incurred by a transition from $\boldsymbol{Q}$ during an exhaustive sojourn at channel a as

$$
\begin{aligned}
C(\boldsymbol{Q}, a) &= \sum_{i\neq a} c_i\left[n_i\frac{n_a E(V_a)}{1-\rho_a} + \frac{\lambda_i}{2}\frac{n_a}{(1-\rho_a)^2}\left(\frac{E(V_a^2)}{1-\rho_a} + (n_a-1)(E(V_a))^2\right)\right] \\
&\quad + c_a\left(\frac{n_a(n_a-1)E(V_a)}{2(1-\rho_a)} + \frac{n_a\lambda_a E(V_a^2)}{2(1-\rho_a)^2}\right).
\end{aligned}
\tag{2.15}
$$

We have now in (2.3), (2.7) and (2.15) a complete characterization of a semi-Markov decision process, with (Ross (1970), Theorem 7.6) optimality equation,

$$
h(\boldsymbol{Q}) = \min_a\left(C(\boldsymbol{Q}, a) + \sum_{Q'} P^{(a)}_{Q,Q'}h(\boldsymbol{Q}') - gE(X_a)\right). \tag{2.16}
$$

Where if a bounded function $h(\cdot)$, and a constant g exist to satisfy Equation (2.16), then a stationary policy, π^*, exists such that g is the minimal average cost from

$Q(\forall Q)$ under π^*, which prescribes (for each Q) the action that minimizes the right-hand side of (2.16).

A similar approach can be utilized to derive the optimality equations for the *gated* as well as all forms of the *limited service* disciplines. Unfortunately, Bellman's curse of dimensionality plagues us, as is obvious from the required search and summation over N^{K-1} in (2.16). This leads us to consider a truncation to a horizon where optimal policies can be explicitly calculated. To that end we will examine a set of '*limited lookahead policies*' (see e.g. Bertsekas (1987), p. 149), where we redirect our attention to the 'cyclic' nature of polling systems. We truncate the time horizon to a *single cycle* and attempt to minimize the cycle time. In the next section we derive probabilistic results for the *cycle times* of both the gated and exhaustive regimes with or without switch-in and switch-out times.

3. Cycle times

We analyze a single cycle where the server traverses the fixed template $(1, 2, \cdots, K)$, denoted π_0, where by a single cycle we mean the total time from the moment the server first *enters* channel 1 until he first *exits* channel K thereafter. Without loss of generality, we set 0 to be the epoch at which this cycle starts and denote the initial state of the system at the beginning of this cycle as $Q(0) = (n_1, n_2, \cdots, n_K)$. The reason for this approach will become clear once we study the dynamic evolution of the system from cycle to cycle. Let X_i denote the service sojourn at channel i within this cycle, set $S_i = \sum_{j=1}^{i} X_j$, $Z_i = E(S_i)$, and let $N_i(t)$ be the number of arrivals to channel i in $(0, t)$, where $N_i(t)$ is distributed as a Poisson random variable with intensity λ_i.

3.1. *Exhaustive regime, zero switching times.* Consider first the *exhaustive* discipline with zero switching times. If B_i denotes a random variable distributed as an $M/G_i/1$-type busy period, with Laplace–Stieltjes transform (LST) $\tilde{B}_i(s) \equiv \tilde{V}_i(s + \lambda_i \tilde{B}_i(s))$, then it is immediate that $\tilde{X}_1(s) = \tilde{B}_1(s)^{n_1}$, and $E(X_1) = n_1 EV_1/(1 - \rho_1)$.

With zero switching times, the state at the start of service in channel 2 is

$$Q(X_1) = (0, n_2 + N_2(X_1), n_3 + N_3(X_1), \cdots, n_K N_K(X_1)). \tag{3.1}$$

Let $\{B_{jk}\}_{k=1}^{\infty}$ denote a sequence of i.i.d. random variables, all distributed as B_j. Decompose X_2 as

$$X_2 = \sum_{j=1}^{n_2} B_{2j} + \sum_{k=1}^{N_2(X_1)} B_{2k},$$

the collection of busy periods carried by the original n_2 customers and the busy periods carried by the new arrivals to channel 2 during X_1.

In general, at the completion of the $(j-1)$th channel $(2 \leqq j \leqq K)$ the state of the system is

$$Q(S_{j-i}) = (N_1(S_{j-1} - S_1),\ N_2(S_{j-1} - S_2),\ \cdots,\ N_{j-2}(S_{j-1} - S_{j-2}),\ 0,$$
$$n_j + N_j(S_{j-1}),\ \cdots,\ n_K + N_K(S_{j-1})) \tag{3.2}$$

so that the service sojourn in channel j can be written as

$$X_j = \sum_{i=1}^{n_j} B_{ji} + \sum_{k=1}^{N_j(S_{j-1})} B_{jk}, \tag{3.3}$$

with LST

$$\tilde{X}_j(s) = \tilde{B}_j(s)^{n_j}\tilde{S}_{j-1}(\lambda_j[1 - \tilde{B}_j(s)]), \qquad j = 1, \cdots, K.$$

Equation (3.3) leads to

$$E(X_j) = \frac{n_j EV_j}{1 - \rho_j} + \frac{\rho_j}{1 - \rho_j} \sum_{i=1}^{j-1} E(X_i) \tag{3.4}$$

which, using the definition of $Z_j = E(S_j)$, can be written as a difference equation in Z_j, i.e.,

$$Z_j - \left(\frac{1}{1 - \rho_j}\right) Z_{j-1} = \frac{n_j EV_j}{1 - \rho_j}, \qquad j = 1, \cdots, K, \qquad Z_0 = 0. \tag{3.5}$$

The solution of (3.5) is

$$Z_j = \sum_{i=1}^{j} n_i EV_i \prod_{r=i}^{j} \left(\frac{1}{1 - \rho_r}\right), \qquad j = 1, \cdots, K. \tag{3.6}$$

Letting $T[\pi_0; e]$ denote the cycle time (conditional on $Q(0)$) under π_0 in an exhaustive regime, we have

$$E(T[\pi_0; e]) \equiv Z_K = \sum_{i=1}^{K} n_i EV_i \prod_{r=i}^{K} \left(\frac{1}{1 - \rho_r}\right). \tag{3.7}$$

Equation (3.7) exhibits quite clearly the evolution of the cycle. Specifically, consider the jth term in (3.7), $n_j EV_j / [\prod_{r=j}^{K} (1 - \rho_r)]$, in which the contribution of the n_j original customers in channel j to the cycle time is demonstrated. These n_j original customers require expected work $n_j EV_j$ which causes a delay busy period in channel j of expected duration $n_j EV_j / (1 - \rho_j)$. In turn, *this* delay busy period causes a delay busy period in channel $j + 1$ of expected duration $(n_j EV_j / (1 - \rho_j)) / (1 - \rho_{j+1})$, etc. $\cdots$. So that the jth term in (3.7) contains the total expected delay caused to the cycle by the n_j initial customers in channel j. The expected cycle time is simply the sum of the delays caused by all initial customers present at the start of the cycle.

3.2. *Gated regime, zero switching times.* The *gating* discipline, with zero switching times, interestingly, requires only a slight modification to the above analysis. Under the gating regime, if the server traverses policy π_0 from the initial state

$Q(0) = (n_1, n_2, \cdots, n_K)$, the service sojourn at channel j is now

$$X_j = \sum_{i=1}^{n_j} V_{ji} + \sum_{k=1}^{N_j(S_{j-1})} V_{jk} \tag{3.8}$$

with LST

$$\tilde{X}_j(s) = \tilde{V}_j(s)^{n_j} \tilde{S}_{j-1}(\lambda_j[1 - \tilde{V}_j(s)]),$$

as only customers arriving to channel j prior to time S_{j-1} are served in this cycle.

Taking expectations on (3.8) gives

$$E(X_j) = n_j EV_j + \rho_j Z_{j-1}, \qquad j = 1, \cdots, K \tag{3.9}$$

which can be written as a system of difference equations

$$\begin{gathered} Z_j - (1 + \rho_j) Z_{j-1} = n_j EV_j, \qquad j = 1, \cdots, K, \\ Z_0 = 0. \end{gathered} \tag{3.10}$$

The system (3.10) yields the solution

$$Z_j = \sum_{i=1}^{j} n_i EV_i \prod_{r=i+1}^{j} (1 + \rho_r), \qquad j = 1, \cdots, K. \tag{3.11}$$

Letting $T[\pi_0; g]$ denote the cycle time under gating starting from $Q(0)$ and utilizing π_0, we have

$$E(T[\pi_0; g]) = Z_K = \sum_{i=1}^{K} n_i EV_i \prod_{r=i+1}^{K} (1 + \rho_r). \tag{3.12}$$

The jth term in Equation (3.12), $n_j EV_j [\prod_{r=j+1}^{K} (1 + \rho_r)]$, exhibits the contribution of the original n_j customers found in channel j to our total cycle time. The n_j customers cause a delay of expected duration $n_j EV_j$ in channel j. During this delay in j, we expect $\lambda_{j+1}(n_j EV_j)$ arrivals to channel $j + 1$, who will all be served under the gating regime during this cycle. *Their* expected service requirement is $\rho_{j+1} n_j EV_j$. So the total delay in channels j and $j + 1$ is $n_j EV_j(1 + \rho_{j+1})$. But this allows an expected $\lambda_{j+2}[n_j EV_j(1 + \rho_{j+1})]$ customers to enter into channel $j + 2$, whose service during this cycle will take an expected time $\rho_{j+2} n_j EV_j(1 + \rho_{j+1})$, so the total delay in channels j, $j + 1$, $j + 2$ is now $n_j EV_j(1 + \rho_{j+1})(1 + \rho_{j+2})$, etc. Once again the cycle is simply the sum of the delays caused by the initial customers in queue.

3.3. *Exhaustive regime, swap-in and switch-out times.* Consider now the exhaustive discipline where the server must take a random time, τ_i, to 'switch *into*' channel i, no matter which channel he just completed. One can think of τ_i as a *set-up* time for channel i. We refer to it as the '*swap-in*' time. After swapping into channel i, the server stays for a service sojourn denoted X_i, after which he must take a random time θ_i to *switch out* of or exit channel i regardless of which channel he will switch into next. τ_i, θ_i are assumed to be independent of X_j, θ_j and τ_j for all j (θ_j is also assumed independent of τ_j, $\forall j$). X_j, however *is* obviously dependent on τ_j. Let Y_j denote the 'total server occupation time with channel j during our cycle', and now let $S_j \equiv \sum_{i=1}^{j} Y_i$, $Z_j = E(S_j)$, i.e., S_j is the exit time, or completion time of channel j in our cycle. As such, we may decompose Y_j as

$$Y_j = \tau_j + X_j + \theta_j = \tau_j + \sum_{i=1}^{n_j} B_{ji} + \sum_{k=1}^{N_j(S_{j-1})} B_{jk} + \sum_{r=1}^{N_j(\tau_j)} B_{jr} + \theta_j \tag{3.13}$$

with LST

$$\tilde{Y}_j(s) = \tilde{B}_j(s)^{n_j}\tilde{S}_{j-1}(\lambda_j[1-\tilde{B}_j(s)])\tilde{\tau}_j(s+\lambda_j[1-\tilde{B}_j(s)])\tilde{\theta}_j(s).$$

Taking expectations we obtain

$$E(Y_j) = \frac{n_j EV_j}{1-\rho_j} + \frac{\rho_j}{1-\rho_j}\sum_{i=1}^{j-1} E(Y_i) + \frac{E(\tau_j)}{1-\rho_j} + E(\theta_j). \tag{3.14}$$

That is, the expected total server occupation with channel j is composed of the delay busy period caused by the original n_j customers plus the delay busy period caused by the swap-in time τ_j, plus the delay busy period caused by all the arrivals to channel j during the cycle so far (S_{j-1}), and the switch-out time θ_j.

We rewrite (3.14) as

$$Z_j - \left(\frac{1}{1-\rho_j}\right)Z_{j-1} = \frac{n_j EV_j + E(\tau_j) + (1-\rho_j)E(\theta_j)}{1-\rho_j}, \qquad j = 1, \cdots, K \tag{3.15}$$
$$Z_0 = 0$$

with the (by now familiar) solution

$$Z_j = \sum_{i=1}^{j} \frac{n_i EV_i + E(\tau_i) + (1-\rho_i)E(\theta_i)}{\prod_{r=i}^{j}(1-\rho_r)}, \qquad j = 1, \cdots, K. \tag{3.16}$$

Letting $T[\pi_0; e, \theta, \tau]$ denote the cycle time under the exhaustive regime with swap-in and switch-out times utilizing π_0, $E(T[\pi_0; e, \theta, \tau]) = Z_K$ in Equation (3.16).

3.4. *Gated regime, swap-in and switch-out times.* We assume that customers at channel j are gated only after the server has switched into channel j, i.e., at time $S_{j-1} + \tau_j$. For this case, we mimic our results for the exhaustive discipline in that now we decompose Y_j as

$$Y_j = \tau_j + \sum_{i=1}^{n_j} V_{ji} + \sum_{k=1}^{N_j(S_{j-1})} V_{jk} + \sum_{r=1}^{N_j(\tau_j)} V_{jr} + \theta_j \tag{3.17}$$

with LST

$$\tilde{Y}_j(s) = \tilde{V}_j(s)^{n_j}\tilde{S}_{j-1}(\lambda_j[1-\tilde{V}_j(s)])\tilde{\tau}_j(s+\lambda_j[1-\tilde{V}_j(s)])\tilde{\theta}_j(s)$$

yielding

$$E(Y_j) = n_j EV_j + \rho_j \sum_{i=1}^{j-1} E(Y_i) + (1+\rho_j)E(\tau_j) + E(\theta_j), \tag{3.18}$$

leading to

$$Z_j - (1+\rho_j)Z_{j-1} = n_j EV_j + (1+\rho_j)E(\tau_j) + E(\theta_j), \qquad j = 1, \cdots, K$$
$$Z_0 = 0$$

so that

$$Z_j = \sum_{i=1}^{j} [n_i EV_i + (1+\rho_i)E(\tau_i) + E(\theta_i)] \prod_{r=i+1}^{j} (1+\rho_r). \tag{3.19}$$

Following our notation, $E(T[\pi_0; g, \theta, \tau]) = Z_K$ in Equation (3.19).

4. Minimizing cycle times

Many optimization criteria may be considered within this truncated horizon, i.e., looking just K steps ahead. A classical objective would be to minimize the sum of (weighted) waiting times obtained during a cycle. Unfortunately, this problem appears to be computationally 'hard'. Another (greedy) objective is to minimize the cycle time, where the cycle time is the total time from 0, the epoch when a cycle starts, with $Q(0) = (n_1, n_2, \cdots, n_K)$, until the server visits every channel exactly once. This allows the server to dispense with the work potentially generated (for the next K steps) by the current workload in the system as quickly as possible. To expand, we consider an arbitrary, or 'free' cycle where the server faced with $Q(0)$ may choose the *path* freely to minimize the expected time of traversing this path. A *path* or *tour* is simply a permutation of $\pi_0 = (1, 2, \cdots, K)$. We first consider the case where the server has only one option, to set the *path* at the beginning of a cycle for the next K steps, and shall later explore dynamic aspects.

4.1. *Exhaustive and gated regimes with zero switching times.* Consider first the exhaustive discipline without switching times. The server's objective, being to minimize the expected cycle time, cannot be cast in standard stochastic scheduling models as his time spent in each channel is dependent on his entry time into that channel due to accruing customers. We compare the expected cycle time under policy π_0, with the expected cycle time under a policy $\pi_1 = (1, 2, \cdots, j-1, j+1, j, j+2, \cdots, K)$, i.e., π_1 is the policy π_0 with the jth and $(j+1)$th channels interchanged.

In an obvious notation corresponding to Equation (3.7),

$$\begin{aligned} E(T[\pi_1; e]) = {} & \sum_{i=1}^{j-1} n_i EV_i \prod_{r=i}^{K} \left(\frac{1}{1-\rho_r}\right) \\ & + n_{j+1} EV_{j+1} \prod_{r=j}^{K} \left(\frac{1}{1-\rho_r}\right) + n_j EV_j (1-\rho_{j+1}) \prod_{r=j}^{K} \left(\frac{1}{1-\rho_r}\right) \\ & + \sum_{i=j+2}^{K} n_i EV_i \prod_{r=i}^{K} \left(\frac{1}{1-\rho_r}\right). \end{aligned} \tag{4.1}$$

Subtracting Equation (4.1) from (3.7) we find

$$E(T[\pi_0; e]) - E(T[\pi_1; e]) = \frac{\rho_{j+1} n_j EV_j - \rho_j n_{j+1} EV_{j+1}}{\prod_{r=j}^{K} (1 - \rho_r)}. \tag{4.2}$$

By the stability assumption, the denominator in Equation (4.2) is positive implying

$$E(T[\pi_0; e]) < E(T[\pi_1; e]) \text{ if and only if}$$
$$\rho_{j+1} n_j EV_j < \rho_j n_{j+1} EV_{j+1},$$

or equivalently, iff

$$\frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}}. \tag{4.3}$$

Observation (4.3) yields immediately the form of the optimal policy, derived upon iterating the pairwise interchange, which is to schedule the channels at the initialization of the cycle in an *increasing* order of the values of n_i/λ_i.

Similarly, applying the comparison of π_0 and π_1 to the *gating* discipline with zero switching times (Equation (3.12)), we may show

$$E(T[\pi_0; g]) - E(T[\pi_1; g]) = (n_j EV_j \rho_{j+1} - n_{j+1} EV_{j+1} \rho_j) \prod_{r=j+2}^{K} (1 + \rho_r). \tag{4.4}$$

This results, surprisingly, in the *identical* criterion as the exhaustive, i.e.

$$E(T[\pi_0; g]) < E(T[\pi_1; g]) \text{ iff}$$
$$\frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}}. \tag{4.5}$$

Relabel the channels so that lower indices correspond to lower values of n_j/λ_j, and call this resulting permutation π^*. We summarize the above as follows.

Theorem 1. Policy π^* (determined at the initialization of the cycle), by which the server faced with the initial state $Q(0) = (n_1, \cdots, n_K)$ serves the channels via the path set in increasing order of the values of n_i/λ_i *minimizes* the expected cycle time in *both* the *exhaustive* and *gated* systems when switching times are zero, i.e., $E(T[\cdot; e])$ and $E(T[\cdot; g])$ are both minimized under the permutation π^*.

4.2. *Discussion.* The policy described in Theorem 1 seems at first counterintuitive as it is independent of the (differing) service times, and is of identical form for both the exhaustive and gated regimes. Before we attempt to gain intuition into Theorem 1 a few remarks are in order.

1. We considered the case where the server chooses the full path at the start of a cycle (a K-step horizon). By the *principle of optimality,* however, if the server is able to limit his horizon further, i.e., a J-step horizon, $1 \leqq J \leqq K$, then at the

completion time of the Jth step, say at time t, it is optimal to utilize again the above strategy based now on $\mathbf{Q}(t)$ for the next J steps. In particular, if $J = 1$, π^* dictates to always serve next the *unserved* channel (in that cycle) with minimal value of $n_i(t)/\lambda_i$, where t is a channel completion epoch, and $n_i(t)$ is the number of occupants of channel i at t. As such, π^* is easily implementable as it makes use only of the on-line information $\mathbf{Q}(t)$, and $\boldsymbol{\lambda}$, at any point in time t, where $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_K)$. We explore the issue of dynamic implementation in Section 5.

2. We chose π^* under the assumption of a 'greedy' server, or system. If, however, economic conditions warrant a longer cycle, e.g., if a cycle is 'expensive' in some sense, and we wish to maximize its expected length, the policy π_* with the channels being sequenced via *decreasing* values of n_i/λ_i is optimal. For example, in the symmetric service case, such as the important *slotted-time* models (see Takagi (1986)), where $EV_i = EV \forall i$, maximizing the expected cycle length is equivalent to maximizing the expected number of customers served during the cycle. This is seen by noting that if $M(\pi, e)$ and $M(\pi, g)$ denote respectively the number of customers served during the cycle starting from $\mathbf{Q}$ under the exhaustive and gated regimes following policy π, then

$$E(M(\pi, \cdot)) = E(T[\pi, \cdot])/EV,$$

which is obviously maximized over π_*.

4.3. *Exhaustive and gated regimes with swap-in and switch-out times.* It is of interest to see if the above noted similarities between the *exhaustive* and *gated* regimes as well as the lack of dependence on the service times remain when switching times are permitted. Turning to the general cases of Sections 3.3 and 3.4, the analysis of Section 4.1 can be repeated for the *exhaustive* case (using Equation (3.16)) to show

$$\begin{aligned} &E(T[\pi_0; e, \theta, \tau]) - E(T[\pi_1; e, \theta, \tau]) \\ (4.6)\qquad &= \frac{\rho_{j+1}(n_j EV_j + E\tau_j + (1-\rho_j)E\theta_j) - \rho_j(n_{j+1}EV_{j+1} + E\tau_{j+1} + (1-\rho_{j+1})E\theta_{j+1})}{\prod_{r=j}^{K}(1-\rho_r)}. \end{aligned}$$

Similarly for the *gated* case, using Equation (3.19),

$$\begin{aligned} &E(T[\pi_0; g, \theta, \tau]) - E(T[\pi_1; g, \theta, \tau]) \\ (4.7)\qquad &= (\rho_{j+1}[n_j EV_j + (1+\rho_j)E\tau_j + E\theta_j] \\ &\quad - \rho_j[n_{j+1}EV_{j+1} + (1+\rho_{j+1})E\tau_{j+1} + E\theta_{j+1}]) \prod_{r=j+2}^{K}(1+\rho_r). \end{aligned}$$

Let $\pi^*(e, \theta, \tau)$ and $\pi^*(g, \theta, \tau)$ denote respectively the permutations obtained when the channels are relabelled in an *increasing* order of the respective values

$$(4.8)\qquad [n_i EV_i + E\tau_i + (1-\rho_i)E\theta_i]/\rho_i$$

and

$$[n_i EV_i + (1+\rho_i)E\tau_i + E\theta_i]/\rho_i. \tag{4.9}$$

Upon iterating the pairwise interchange, we can readily derive the following result from Equations (4.6) and (4.7).

Theorem 2. $E(T[\cdot; e, \theta, \tau])$ is minimized when calculated over the permutation $\pi^*(e, \theta, \tau)$, and $E(T[\cdot; g, \theta, \tau])$ is minimized when calculated over the permutation $\pi^*(g, \theta, \tau)$.

Remark. We may again define policies $\pi_*(\cdot, \theta, \tau)$ for both the exhaustive and gated regimes that maximize the expected cycle length by following $\pi^*(\cdot, \theta, \tau)$ in reverse order. For the symmetric service case ($EV_i = EV, \forall i$), this policy will maximize the *expected number of customers served* during the cycle. Letting $m(\pi, \cdot)$ denote the number of customers served in the cycle following policy π with swap-in and switch-out times included, we have

$$E(M(\pi, \cdot)) = \frac{E(T[\pi; \cdot, \theta, \tau]) - \sum_{i=1}^{K} E(\theta_i + \tau_i)}{EV},$$

which is maximized under $\pi_*(\cdot, \theta, \tau)$.

We collect our results in Table 1.

It is obvious from the forms of π^* in Table 1 that when any switching times are included service times *do* play a role in the determination of the optimal policy, and policies are obviously of different forms for the exhaustive and gated regimes.

TABLE 1

	Exhaustive (e)	Gated (g)
$\pi^*(\cdot)$ Zero-switch times ($\theta_i = 0,\ \tau_i = 0,\ \forall i$)	$\dfrac{n_i}{\lambda_i}$	$\dfrac{n_i}{\lambda_i}$
$\pi^*(\cdot, \theta)$ Switch-out alone ($\theta_i \geqq 0,\ \tau_i = 0,\ \forall i$)	$\dfrac{n_i EV_i + (1-\rho_i)E\theta_i}{\rho_i}$	$\dfrac{n_i EV_i + E\theta_i}{\rho_i}$
$\pi^*(\cdot, \tau)$ Swap-in alone ($\theta_i = 0,\ \tau_i \geqq 0$)	$\dfrac{n_i EV_i + E\tau_i}{\rho_i}$	$\dfrac{n_i EV_i + (1+\rho_i)E\tau_i}{\rho_i}$
$\pi^*(\cdot, \theta, \tau)$ in-out ($\theta_i \geqq 0,\ \tau_i \geqq 0$)	$\dfrac{n_i EV_i + E\tau_i + (1-\rho_i)E\theta_i}{\rho_i}$	$\dfrac{n_i EV_i + (1+\rho_i)E\tau_i + E\theta_i}{\rho_i}$

Expected Cycle Time

$$E(T[\pi; e, \theta, \tau]) = \sum_{i=1}^{K} \frac{n_{\pi(i)} EV_{\pi(i)} + E\tau_{\pi(i)} + (1-\rho_{\pi(i)})E\theta_{\pi(i)}}{\prod_{r=i}^{K} (1-\rho_{\pi(r)})}$$

$$E(T[\pi; g, \theta, \tau]) = \sum_{i=1}^{K} \left[(n_{\pi(i)} EV_{\pi(i)} + (1+\rho_{\pi(i)})E\tau_{\pi(i)} + E\theta_{\pi(i)}) \prod_{r=i+1}^{K} (1+\rho_{\pi(r)}) \right]$$

However, a unifying principle does exist by which all the results are clearly generated.

4.4. *Unification via a general principle.* Consider K tasks that must be sequentially performed in a non-preemptive manner by a single processor. All tasks are available at time 0 (as is the processor). Task i carries a random initial processing requirement of expected size a_i, called its *core,* but if processing is delayed until t, the *expected* requirement has grown to $a_i + \alpha_i t$ (i.e., α_i is the expected growth per unit time delay in performing task i) $i = 1, \cdots, K$. Browne and Yechiali (1987) showed that the dynamics of this process is such that if the policy $\pi = (\pi(1), \pi(2), \cdots, \pi(K))$ is followed, the total time to process all K tasks has expectation

$$\sum_{i=1}^{K} a_{\pi(i)} \prod_{r=i+1}^{K} (1 + \alpha_{\pi(r)}), \tag{4.10}$$

which is minimized when following the permutation based on *increasing* values of the critical quantity a_i/α_i, *the ratio of each task's core to its growth rate.* Call this permutation π^*.

Browne and Yechiali (1987) further showed that within this π^*

$$\frac{a_j}{\alpha_j} > \frac{\sum_{i=1}^{j-1} a_i \prod_{r=i+1}^{j-1} (1 + \alpha_r)}{\prod_{r=1}^{j-1} (1 + \alpha_r) - 1}, \qquad j = 1, \cdots, K. \tag{4.11}$$

Consider now the exhaustive regime with both swap-in and switch-out times at the initialization of our cycle from state $\boldsymbol{Q}(0)$. As the server is free to just choose the *path* for the next K steps but committed to serve all unserviced channels before returning to those serviced previously in the cycle, we may immediately apply the above results. The core at channel i is the expected total server occupation at i if the server chooses i at time 0. As there are n_i customers in queue there, this is simply the *delay busy period* caused by those n_i waiting customers plus the delay busy period caused by the swap-in time, plus the switch-out time, i.e.,

$$a_i \equiv \frac{n_i E V_i}{1 - \rho_i} + \frac{E\tau_i}{1 - \rho_i} + E\theta_i.$$

The *expected growth* in service requirement per unit time delay in channel i is $\alpha_i = \rho_i/(1 - \rho_i)$, as the expected flow of work to channel i per unit time is ρ_i, causing a delay busy period there of expected length $\rho_i/(1 - \rho_i)$. Upon application of the above principle, it is seen that scheduling via increasing values of a_i/α_i corresponds to $\pi^*(e, \theta, \tau)$. Note too, that in conjunction with Equation (4.10), $E(T[\pi; e, \theta, \tau])$ can equivalently be written as (see Equation (3.16) and Table 1)

$$E(T[\pi; e, \theta, \tau]) = \sum_{i=1}^{K} \frac{n_{\pi(i)} E V_{\pi(i)} + E\tau_{\pi(i)} + (1 - \rho_{\pi(i)}) E\theta_{\pi(i)}}{1 - \rho_{\pi(i)}} \prod_{r=i+1}^{K} \left(1 + \frac{\rho_{\pi(r)}}{1 - \rho_{\pi(r)}}\right). \tag{4.12}$$

TABLE 2
Core (a_i) and growth (α_i) with swap-in and switch-out times (τ_i, $\theta_i \geqq 0\ \forall i$)

	Exhaustive	Gated
a_i	$\frac{n_i EV_i + E\tau_i}{1-\rho_i} + E\theta_i$	$n_i EV_i + (1+\rho_i)E\tau_i + E\theta_i$
α_i	$\frac{\rho_i}{1-\rho_i}$	ρ_i

Applying the a_i/α_i principle results in $\pi^*(e, \theta, \tau)$. However, for the gated regime, with swap-in and switch-out times (Section 3.4) the 'core' at channel i has the form $a_i = n_i EV_i + (1+\rho_i)E\tau_i + E\theta_i$, and due to the nature of the gating system, the flow of work into channel i per unit time is $\alpha_i = \rho_i$, as each arrival carries only a service requirement and not a busy period. Application of the a_i/α_i principle leads directly to the policy $\pi^*(g, \theta, \tau)$, of Equation (4.9), summarized in Table 2.

As such, it is easy to see that for the case of zero switching times, ($\theta_i = 0 = \tau_i \forall i$), the *true* critical quantity determining π^* is

$$\left(\frac{n_i EV_i}{1-\rho_i}\right) \bigg/ \left(\frac{\rho_i}{1-\rho_i}\right)$$

for the exhaustive regime and $n_i EV_i/\rho_i$ for the gated regime, both leading to the ambiguous quantity n_i/λ_i.

Equation (4.11) is sufficient to prove that our set of optimal policies correspond to the celebrated Dynamic Allocation Index (DAI) or Gittins Index (see e.g., Gittins (1982) and Whittle (1982)) for the multi-armed bandit problem. Our problem however, is not directly a bandit as the channels evolve randomly while not being served, while in the regular multi-armed bandit problem, the processes not being served remain constant, or 'frozen' (but see Whittle's (1982) derivation of an 'open process').

5. Dynamic priority rules

Our attention has been restricted thus far to a single cycle. We may extend our results to the *dynamic* operation of a polling system quite easily. The aforementioned previous research on polling systems dealt with the server continuously following a fixed cyclic template, regardless of whether any customers are in queue or not at any channel. Consequently, the dynamic operation of the system was not dealt with in terms of optimization, and equilibrium results, such as the equilibrium cycle time, $E(c)$, having the evaluation (see Eisenberg (1972), Watson (1984) and

Takagi (1986))

(5.1)
$$E(c) = \frac{\sum_{i=1}^{K} E(\theta_i)}{1 - \sum_{i=1}^{K} \rho_i},$$

(as τ_i is equal to $0\ \forall\ i$ in their model and θ_i is the time to switch from channel i to $i+1$) are central points of analysis, as the evaluation of waiting times is of major importance.

In our model, the server need not cycle endlessly and may remain idle when no customers are in the system. As soon as a customer arrives to the system however, the server must return to work (i.e., swap-in to the channel the customer arrives to) and work thereafter in a manner to be described below until the system is empty once again.

Within this context we define a set of policies, the *pseudo-cyclic* class, whereby the server dichotomizes the channels into two groups, an *unserved* set and a *served* set at every decision epoch on each 'tour': the server must complete all the *unserved* channels on a *tour* (a permutation of the channel indices) before returning to the *served* group to start his next tour. Obviously, a tour starts when the set of unserved channels is full. The server may choose the *path* of each tour consistently with the decision horizon. We are interested in considering the class of pseudo-cyclic policies where the objective of the server is to *minimize the expected length of each tour.* However, for any objective function, the restriction to the pseudo-cyclic class does ensure some degree of 'fairness' to the channels. It also allows the server to poll the system only once in a decision horizon.

To illustrate, consider the *exhaustive* regime with zero switching times ($\theta_i = \tau_i = 0\ \forall i$). As the server is allowed to stop switching when the system is empty, the moment from when a customer arrives to an empty system till the next moment the server becomes idle is finite with probability 1 if $\sum_{i=1}^{K} p_i < 1$ and is distributed as the busy period in an $M/G/1$ queue with arrival rate $\lambda = \sum_{i=1}^{K} \lambda_i$ and service distribution

$$G(x) = \sum_{i=1}^{K} \frac{\lambda_i}{\lambda} G_i(x).$$

We call this period of time the *system busy period*; it has the same evaluation for both the exhaustive and the gated regime as long as switching times are 0. As long as server idleness is not permitted, it is easily shown (see, e.g., Cobham (1954), and the literature on priority queues) that the system busy period has expected duration $\rho/\lambda(1-\rho)$ (where $\rho = \sum_{i=1}^{K} \rho_i$), and the expected number served during this period is $1/(1-\rho)$. These quantities are obviously invariant to the service policy as long as it is work conserving.

Set $t=0$ to be the epoch at which the system busy period starts, and say a customer of type j initiates this busy period, i.e.,

$$Q(0^-)=O, \qquad Q(0^+)=e_j \tag{5.2}$$

where $e_j=(0, 0, \cdots, 0, 1, 0, \cdots, 0)$, a K-vector of 0's with a 1 in the jth position. Let X_j denote the (exhaustive) service sojourn in channel j, so that $E(X_j)=EV_j/(1-\rho_j)$. Then at the completion of channel j the state is

$$Q(X_j)=(N_1(X_j), \cdots, N_{j-1}(X_j), 0, N_{j+1}(X_j), \cdots, N_K(X_j)), \tag{5.3}$$

where $N_j(\cdot)$ denotes a Poisson process with intensity λ_j.

If the server wishes to set the path of the first tour at $t=0$ by using Theorem 1 on the *expected* state $E(Q(X_j))$, it would be set via increasing (in i) values of $E(N_i(X_j))/\lambda_i=E(X_j)$, a constant for every state! It is obvious that by the assumption of Poisson arrivals the rule is *non-anticipative* i.e., one cannot set the route based on an *expected* state, but must rather *wait* for an *actual realization* (the n_i's) and then set the path at time X_j based on that realization. This describes a purely *adaptive* control procedure, i.e., in our example, the server stays at channel j for time X_j, upon his exit from j he is faced with the state $Q(X_j)$, and sets the route at that time ($t=X_j$) by increasing values of the actual realization $N_i(X_j)/\lambda_i$. This implies that now the server may look ahead no further than $K-1$ steps, as there are now at most $K-1$ non-zero channels. If the server now utilizes a $K-1$ step horizon, by our definition of a pseudo-cyclic policy, channel j will always be the last channel served on any tour; in a sense channel j will serve as an 'anchor', as its completion will always coincide with the start of a new tour.

If the server is operating on a 1-step horizon within the context of a pseudo-cyclic policy, then every channel completion affords him a decision epoch, whereby it would be optimal for him to always choose to serve the channel from the unserved group with the minimal value of $n_i(t)/\lambda_i$, where t is a decision epoch within that tour and $n_i(t)$ is the state of channel i at that point (as long as i has not yet been serviced on that tour). Proceeding in this manner, the server can continously choose paths to minimize each pseudo-cycle, or tour. Systems with a general polling table (Baker and Rubin (1987)) can be treated accordingly with only a slight modification of a tour; however, the fairness criteria must be reevaluated. The full effect of this operating policy or dynamic priority rule on waiting times has yet to be explored.

Note however, that once switching times are allowed in the model, the rule *is anticipative* in that if a customer of type j arrives to an empty system—in effect, 'turning-on' the server—and Y_j denotes the total server occupation time at j, i.e.,

$$Q(0^-)=O,$$
$$Q(0^+)=e_j,$$
$$Q(Y_j)=(N_1(Y_j), N_2(Y_j), \cdots, N_{j-1}(Y_j), N_j(\theta_j), N_{j+1}(Y_j), \cdots, N_K(Y_j))$$

for the *exhaustive* regime, and

$$Q(Y_j) = (N_1(Y_j), \cdots, N_{j-1}(Y_j), N_j(S_j - \tau_j), N_{j+1}(Y_j), \cdots, N_K(Y_j))$$

for the *gated* regime.

Then, following a pseudo-cyclic policy, the server may implement the rule based on $E(Q(Y_j))$ at $t = 0$ amounting to fixing the $K - 1$ step-ahead tour via increasing $[E\tau_i + (1 - \rho_i)E\theta_i]/\rho_i$ for the *exhaustive* case and via increasing $[(1 + \rho_i)E\tau_i + E\theta_i]/\rho_i$ for the *gated* regime. Moreover, for this case, necessary conditions under which the system busy period terminates in finite time are not yet known.

Returning to the *exhaustive* regime with zero switching times, consider the consequences of 2-cycle horizon, i.e., setting the path for $2K$ steps ahead at $t = 0$ following a pseudo-cyclic rule (so that each channel can be served at most twice in the next $2K$ steps), but where $Q(0) = (n_1, \cdots, n_K)$. For convenience, assume that we have relabelled the channels via increasing values of n_i/λ_i, so that the optimal path for the first tour is simply $(1, 2, \cdots, K)$.

Anticipating the second cycle, let $S_j^{(i)}$ denote the completion time of the jth channel on the ith tour, $j = 1, \cdots, K$, $i = 1, 2$ (so that $S_K^{(1)}$ is the completion time of the first tour) and let $E(S_j^{(i)}) = Z_j^{(i)}$. Then,

$$Q(S_K^{(1)}) = (N_1(S_K^{(1)} - S_1^{(1)}), \cdots, N_j(S_K^{(1)} - S_j^{(1)}), \cdots, N_{K-1}(S_K^{(1)} - S_{K-1}^{(1)}), 0) \tag{5.4}$$

and for any arbitrary $K - 1$ 'next' sequence, $(i_1, \cdots, i_{K-1})$, let $Z^{(2)}$ denote the expected second tour completion time (this is obviously a partial tour, as channel K is presently 0). It is then seen that for that sequence

$$Z^{(2)} = Z_K^{(1)} + \sum_{j=1}^{K-1} \frac{\rho_{i_j} E(S_K^{(1)} - S_{i_j}^{(1)})}{\prod_{r=j}^{K-1} (1 - \rho_{i_r})} = Z_K^{(1)} + \sum_{j=1}^{K-1} \frac{\rho_{i_j}(Z_K^{(1)} - Z_{i_j}^{(1)})}{\prod_{r=j}^{K-1} (1 - \rho_{i_r})}. \tag{5.5}$$

Applying the a_i/α_i principle (see Equation (4.10)) shows that Equation (5.5) is minimized when $(i_1, \cdots, i_{K-1})$ is chosen by *increasing* values of $Z_K^{(1)} - Z_i^{(1)}$, or equivalently, by *decreasing* values of $Z_i^{(1)}$. However, by our results on the first cycle we know that $Z_1^{(1)} < Z_2^{(1)} < \cdots < Z_K^{(1)}$. Therefore, by expanding our horizon to $2K - 1$ steps ahead, the last $K - 1$ steps are completely determined by the first steps in that the channels will be served in the sequence $(1, 2, \cdots, K - 1, K, K - 1, K - 2, \cdots, 2, 1)$, a *rocking*, or pendulum effect. A similar analysis for both the gated and exhaustive regimes with switching times show that if the decision horizon is expanded past one cycle, the server simply goes back and forth continously across the same sequence. As swap-in and switch-out times exist, the sequence will now look like $(1, 2, \cdots, K - 1, K, K, K - 1, \cdots, 2, 1)$. As such, the *adaptive* control, adapted to the actual realizations during each tour, is preferred.

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