DYNAMIC PRIORITY RULES FOR CYCLIC-TYPE QUEUES

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Abstract
A cyclic service system is composed of $K$ channels (queues) and a single cyclically roving server who typically takes a positive amount of time to switch between channels. Research has previously focused on evaluating and computing performance measures (notably, waiting times) of fixed template routing schemes under three main service disciplines, the exhaustive, gated and limited service regimes.

In this paper, probabilistic results are derived that allow control strategies and optimal policies to be considered for the first time. By concentrating on a new objective function, we are able to derive rules of index form amenable for direct implementation to dynamically control the system at suitably defined decision epochs. These rules utilize current system information, are of an adaptive nature, and are shown to emanate from a general physical principle.

CYCLIC QUEUES; POLLING SYSTEMS; STOCHASTIC SCHEDULING

1. Introduction
A polling system, or cyclic queue, is composed of $K$ queues, (channels) labelled $i = 1, \ldots, K$. Customers arrive to channel $i$ in a Poisson stream of intensity $\lambda_i$, independently of the other channels. There is a single server in the system who moves from channel to channel in a ‘cyclic’ fashion, i.e., the server stays at channel $i(i = 1, \ldots, K)$ for a length of time determined by the queue discipline and then moves to channel $i+1$. Upon ‘completion’ of channel $K$, the server reverts to channel 1 and so on, hence the name ‘cyclic’.

Each customer in channel $i$ carries an independent random service requirement distributed as $V_i$ having distribution function $G_i(\cdot)$, $i = 1, \ldots, K$. The queue discipline determines how many customers are to be served in each channel. The disciplines most often studied are the exhaustive, gated and limited service regimes. To illustrate these regimes, assume the server arrives to channel $i$ to find $n_i$ customers waiting. Under the exhaustive regime, the server must service channel $i$ until it is empty before he is allowed to move on. This amount of time is distributed as the sum of $n_i$ ordinary busy periods in an $M/G/1$ queue. Under the gated regime, the server gates off those customers already present upon his arrival to channel $i$, and serves only them before moving on to channel $i+1$. As such, the total service
time in channel $i$ is distributed as the sum of $n_i$ ordinary service requirements. Under the limited service regimes, the server must serve either 1, at most $k_i$ (where $k_i$ is some fixed integer), or deplete the queue at channel $i$ by 1 (i.e., stay one busy period of $M/G_i/1$ type).

Typically, the server takes a non-negligible amount of time to switch between channels. These switching times are assumed random and complicate the analysis greatly.

Cooper (1970) was the first to explicitly name and study cyclic queues, and Eisenberg (1972) gave a thorough analysis of equilibrium results for both the exhaustive and gated regimes, later improved upon by Baker and Rubin (1987). The exhaustive discipline with zero switching times between channels has been analyzed for $K = 2$ as an 'alternating priority queue' (Avi-Itzhak et al. (1965) and Takács (1968)). For $K = 1$ and positive switching times, the system can be modeled as a single server queue with vacations (Levy and Yechiali (1975)). Konheim and Meister (1974) studied the exhaustive discipline in depth in relation to communications systems. Results on waiting times are notoriously difficult to obtain and (pseudo)-conservation laws are surveyed in Watson (1984). Boxma and Groenendijk (1987) present a probabilistic proof of a pseudo-conservation law allowing a mixture of different service strategies at the various queues, generalizing and unifying previous results. There is an enormous literature on the theory and applications of these models surveyed and presented in Takagi (1986) and (1987).

Previous research has centered on models where the server traverses the same fixed route from cycle to cycle. As such, the switching time from channel to channel need not be functionally indexed only by the channel being exited.

In this paper we seek to determine a routing scheme, or policy, to optimize some measure of system performance. If we no longer hold the server to a fixed cyclic strategy, but allow him to service channel $j$ after channel $i$, $j \neq i + 1$, we must consider a general switching time from channel $i$ to channel $j$, say $r_{ij} \geq 0 \forall i,j$. This of course complicates the analysis of the system by increasing the number of switching variables from $K$ to $K^2$ (or $K(K - 1)$ if $r_{ij} = 0, \forall i$). We assume that $r_{ij}$ may be decomposed into $r_{ij} = \theta_i + \tau_{ij}$, where $\theta_i$ is the time to switch-out of channel $i$ and $\tau_{ij}$ is the time to switch-into channel $j$. Within this framework we will also allow the server to remain idle when no customers are present in the system.

In Section 2 we model the exhaustive regime with zero switching times as a Markov decision process over an unbounded horizon to minimize total weighted waiting time in the system. The difficulties associated with this criterion are illustrated and we are led to consider a 'limited horizon lookahead criterion', minimizing cycle time (to be defined later). To that end, we first introduce in Section 3 some probabilistic results on cycle times for the exhaustive and gated regimes. In Section 4 we define the criteria of minimizing cycle time, and use the results of Section 3 to obtain some surprising rules of a simple form for both the exhaustive and gated regimes with positive switching times. The results are then unified via a
2. Formulation as a Markov decision process

In this section we model the polling system under the exhaustive regime with zero switching times as a (semi) Markov decision process to minimize the expected weighted (by costs) waiting time accruing to the system per unit time. The control variable is simply the index of the next channel to be visited, as we allow the server to choose to serve any channel with a positive number of inhabitants.

The state of the system at any point in time, \( t \), is \( Q(t) = (Q_1(t), \cdots, Q_K(t)) \), where \( Q_i(t) \) is the number of customers in channel \( i \) at time \( t \). Decision epochs occur when the server has emptied a channel and must now move to some other channel with a positive number of occupants (as idleness is not permitted). The state space is therefore \( N^K \) where \( N = \{0, 1, \cdots\} \), and the action space is \( A = \{1, 2, \cdots, K\} \) with generic element \( a \).

By looking at the system only at these decision epochs, we may define an embedded Markov chain over the system. If \( \{t_j\}_{j=0}^{\infty} \) are the completion times of the channels, and \( a_j \) denotes the action taken at the \( j \)th transition (i.e., \( a_j \) denotes the channel served during \( (t_j, t_{j+1}) \)) then

\[
P(Q(t_j) \in S \mid Q(t_{j-1}), a_{j-1}; Q(t_{j-2}), a_{j-2}; \cdots; Q(0), a_0)
= P(Q(t_j) \in S \mid Q(t_{j-1}), a_{j-1}),
\]

for \( S \subseteq N^K \). For this model it is necessary and sufficient to assume \( \sum_{i=1}^{K} \rho_i < 1 \) to ensure the \( t_j \) are well-defined stopping times.

Let \( Q_i(t_i) = n_i, \quad i = 1, \cdots, K \). Note that \( Q_{a_i}(t_i) = n_{a_i} = 0 \) by the exhaustive property. If \( X_{a_i} \) denotes the service sojourn in channel \( a_i \), then it is known (Takács (1967), Cooper (1981), p. 231) that

\[
P(X_{a_i} < t \mid Q(t_i), a_i) = \sum_{n_{a_i}} n_{a_i} \int_0^t \left( \frac{\lambda_{a_i} x}{n_{a_i}} \right)^{n_{a_i}} \frac{e^{-\lambda_{a_i} x}}{n_{a_i}} dG_{a_i}^n(x) = F_{a_i}(t)
\]

where \( G^n \) denotes the \( n \)th convolution of \( G \). Alternatively, conditional upon \( Q(t_i) \), \( X_{a_i} = \sum_{i=1}^{n_{a_i}} B_{a_i} \), where \( B_{a_i} \) is a random variable distributed as an ordinary \( M/G/a_i/1 \) busy period. As such, if for any random variable \( Y \) we let \( Y(s) \) denote its Laplace–Stieltjes transform, we have (where all expectations are conditioned by \( Q(t_i), a_i \))

\[
\tilde{X}_{a_i}(s) = (\tilde{B}_{a_i}(s))^n = \tilde{V}_{a_i}(s + \lambda_{a_i} - \lambda_{a_i} \tilde{B}_{a_i}(s))
\]

leading to

\[
E(X_{a_i}) = n_{a_i} \frac{E(V_{a_i})}{1 - \rho_{a_i}}
\]

\[
E(X_{a_i}^2) = \frac{n_{a_i}}{(1 - \rho_{a_i})^2} \left( \frac{E(V_{a_i}^2)}{1 - \rho_{a_i}^2} + (n_{a_i} - 1)(E(V_{a_i}))^2 \right).
\]
Note too that

\begin{equation}
Q(t_{r+1}) = \begin{cases} n_i + m_i & i \neq a_r \\ 0 & i = a_r \end{cases}
\end{equation}

where \( m_i \) denotes the number of arrivals to channel \( i \) during the service sojourn at \( a_r \), i.e., \( m_i = N_i(X_{a_r}) \), where \( N_i(X_{a_r}) \) is a Poisson random variable with intensity \( \lambda_i \).

Letting \( Q(t_r) = Q \), and \( Q(t_{r+1})(= Q(t_r + X_{a_r})) = Q' \), the process \( Q(\cdot) \) is semi-Markov with kernel

\begin{equation}
P(Q', t_{r+1} - t, \Xi \mid Q, a_r) = \int_0^t \exp \left( \left( - \sum_{i \neq a_r} \lambda_i \right) x \right) \prod_{i \neq a_r} \frac{(\lambda_i x)^{m_i}}{m_i!} dF_{a_r}(x).
\end{equation}

Let

\begin{equation}
P(Q', t_{r+1} - t, \Xi \mid Q, a_r = a)
\end{equation}

so that \( \{P(Q', t_{r+1} - t, \Xi \mid Q, a_r = a)\} \) is the set of one-step transition probabilities for the system.

To complete the formulation as a (semi) Markov decision process, we need the costs associated with a transition. We assume linear holding costs in that the system is charged \( c_l \) per unit time for each type \( i \) customer waiting in the system.

The costs incurred in channel \( i \), \( i \neq a \) is the cost of holding the original \( n_i \) customers through \( X_{a_r} \), i.e. \( c_i n_i X_{a_r} \), plus the cost of holding those arriving during \( X_{a_r} \). Following Yechiali (1976), by Poisson arrivals, the expected total wait of the new arrivals during \( X_{a_r} \) can be written as the product

\begin{equation}
E(\text{wait of an arbitrary arrival}) \cdot E(\text{number of new arrivals}).
\end{equation}

The expected number of arrivals is \( \lambda_i E(X_{a_r}) \), and the expected waiting time of an arbitrary arrival during \( X_{a_r} \) is, by Poisson arrivals, the random modification, or forward recurrence time of \( X_{a_r} \), \( E(X_{a_r}^2)/2E(X_{a_r}) \). Therefore, the total expected costs incurred in the non-served channels is

\begin{equation}
\sum_{i \neq a} c_i \left( n_i E(X_{a_r}) + \frac{\lambda_i^2}{2} E(X_{a_r}^2) \right).
\end{equation}

The total expected waiting time in Channel \( a \) is

\begin{equation}
E(\text{wait of original } n_a) + E(\text{wait of new arrivals during } X_{a_r}).
\end{equation}

Now, once again by Poisson arrivals, the expected wait of new arrivals (under a first-come first-served discipline) during \( X_{a_r} \) is equivalent to

\begin{equation}
E(\text{number of new } a\text{-arrivals during } X_{a_r}) \cdot E(\text{wait for arbitrary new arrival}).
\end{equation}

Treating the service time of the original \( n_a \) customers as a 'vacation' in an \( M/G_a/1 \) queue, we recall Equation (38) of Levy and Yechiali (1975):

\begin{equation}
E(W) = \frac{\lambda E(V^2)}{2(1 - \rho)} + \frac{E(U^2)}{2E(U)},
\end{equation}

where \( E(V) \) is the expected waiting time of an arrival in channel \( a \) and \( E(U) \) is the expected service time.
where $U$ corresponds to the 'vacation period', and $W$ is the average wait (exclusive of service) experienced by an arriving customer in an $M/G/1$ queue with arrival rate $\lambda$, service times distributed as $V$ and a vacation period $U$. As such, we have (where $V_{ak}$ is distributed as $V_a$)

$$U = \sum_{k=1}^{N} V_{ak}$$

(2.11)

$$E(U) = n_a E(V_a)$$

$$E(U^2) = n_a E(V_a^2) + n_a(n_a - 1)(E(V_a))^2.$$ 

From Equations (2.10), (2.11), we may write the expected wait of an arbitrary arrival to channel $a$ during a service sojourn at channel $a$ with $n_a$ initial customers in queue, as

$$\frac{1}{1 - \rho_a} \frac{E(V_a^2)}{2E(V_a)} + \frac{n_a - 1}{2} E(V_a).$$

(2.12)

The expected waiting time for the original $n_a$ customers during $X_a$ is

$$E\left(\sum_{j=1}^{n_a-1} \sum_{k=1}^{j} V_{ak}\right) = \frac{n_a(n_a - 1)}{2} E(V_a).$$

(2.13)

The expected number of arrivals to channel $a$ during $X_a$ is $n_a \rho_a / (1 - \rho_a)$, therefore using (2.9), (2.12), (2.13), the total expected wait incurred at channel $a$ during $X_a$ is

$$\frac{n_a(n_a - 1)}{2} E(V_a) + \frac{n_a \rho_a E(V_a^2)}{2E(V_a)(1 - \rho_a)^2} + \frac{n_a(n_a - 1)\rho_a E(V_a)}{2(1 - \rho_a)}$$

(2.14)

$$= \frac{n_a(n_a - 1)E(V_a)}{2(1 - \rho_a)} + \frac{n_a \lambda_a E(V_a^2)}{2(1 - \rho_a)^2}.$$ 

Using (2.8), (2.3), (2.4) and (2.14) we may now write the total expected cost incurred by a transition from $Q$ during an exhaustive sojourn at channel $a$ as

$$C(Q, a) = \sum_{i \neq a} c_i \left[ n_i n_a E(V_a) + \frac{\lambda_i}{1 - \rho_a} \frac{n_a}{2(1 - \rho_a)^2} \left( \frac{E(V_a^2)}{1 - \rho_a} + (n_a - 1)(E(V_a))^2 \right) \right]$$

(2.15)

$$+ c_a \left( n_a(n_a - 1)E(V_a) + n_a \lambda_a E(V_a^2) \frac{n_a}{2(1 - \rho_a)^2} \right).$$ 

We have now in (2.3), (2.7) and (2.15) a complete characterization of a semi-Markov decision process, with (Ross (1970), Theorem 7.6) optimality equation,

$$h(Q) = \min_a \left( C(Q, a) + \sum_{Q'} P_{Q \to Q'} h(Q') - g E(X_a) \right).$$

(2.16)

Where if a bounded function $h(\cdot)$, and a constant $g$ exist to satisfy Equation (2.16), then a stationary policy, $\pi^*$, exists such that $g$ is the minimal average cost from
\( Q(\forall Q) \) under \( \pi^* \), which prescribes (for each \( Q \)) the action that minimizes the right-hand side of (2.16).

A similar approach can be utilized to derive the optimality equations for the \textit{gated} as well as all forms of the \textit{limited service} disciplines. Unfortunately, Bellman’s curse of dimensionality plagues us, as is obvious from the required search and summation over \( N^{K-1} \) in (2.16). This leads us to consider a truncation to a horizon where optimal policies can be explicitly calculated. To that end, we will examine a set of \textit{’limited lookahead policies’} (see e.g. Bertsekas (1987), p. 149), where we redirect our attention to the \textit{’cyclic’} nature of polling systems. We truncate the time horizon to a \textit{single cycle} and attempt to minimize the cycle time. In the next section we derive probabilistic results for the \textit{cycle times} of both the gated and exhaustive regimes with or without switch-in and switch-out times.

3. Cycle times

We analyze a single cycle where the server traverses the fixed template \((1, 2, \ldots, K)\), denoted \( \pi_0 \), where by a single cycle we mean the total time from the moment the server first enters channel 1 until he first exits channel \( K \) thereafter. Without loss of generality, we set \( 0 \) to be the epoch at which this cycle starts and denote the initial state of the system at the beginning of this cycle as \( Q(0) = (n_1, n_2, \ldots, n_K) \). The reason for this approach will become clear once we study the dynamic evolution of the system from cycle to cycle. Let \( X_i \) denote the service sojourn at channel \( i \) within this cycle, set \( S_i = \sum_{j=1}^{i} X_j, \ Z_i = E(S_i), \) and let \( N_i(t) \) be the number of arrivals to channel \( i \) in \((0, t)\), where \( N_i(t) \) is distributed as a Poisson random variable with intensity \( \lambda_i \).

3.1. Exhaustive regime, zero switching times. Consider first the \textit{exhaustive} discipline with zero switching times. If \( B_i \) denotes a random variable distributed as an \( M/G_i/1 \)-type busy period, with Laplace–Stieltjes transform (LST) \( \tilde{B}_i(s) = \tilde{V}_i(s + \lambda_i) \tilde{B}_i(s) \), then it is immediate that \( X_i(s) = B_i(s) \), and \( E(X_i) = n_i EV_i/(1 - \rho_i) \).

With zero switching times, the state at the start of service in channel 2 is

\[
Q(X_1) = (0, n_2 + N_2(X_1), n_3 + N_3(X_1), \ldots, n_K N_K(X_1)).
\]

Let \( \{B_{jk}\}_{k=1}^{\infty} \) denote a sequence of i.i.d. random variables, all distributed as \( B_j \). Decompose \( X_2 \) as

\[
X_2 = \sum_{j=1}^{n_2} B_{2j} + \sum_{k=1}^{N_2(X_1)} B_{2k},
\]

the collection of busy periods carried by the original \( n_2 \) customers and the busy periods carried by the new arrivals to channel 2 during \( X_1 \).

In general, at the completion of the \((j-1)\)th channel \((2 \leq j \leq K)\) the state of the...
system is
\[
Q(S_{j-1}) = (N_1(S_{j-1} - S_1), N_2(S_{j-1} - S_2), \ldots, N_{j-2}(S_{j-1} - S_{j-2}), 0, n_j + N_1(S_{j-1}), \ldots, n_K + N_K(S_{j-1}))
\]
so that the service sojourn in channel \(j\) can be written as
\[
X_j = \sum_{i=1}^{n_j} B_{ki} + \sum_{k=1}^{N_k(S_{j-1})} B_{jk},
\]
with LST
\[
\bar{X}_j(s) = \bar{B}_j(s)^{n_j(S_{j-1})}(\lambda_j[1 - \bar{B}_j(s)]), \quad j = 1, \ldots, K.
\]
Equation (3.3) leads to
\[
E(X_j) = \frac{n_jEV_j}{1 - \rho_j} + \frac{\rho_j}{1 - \rho_j} \sum_{i=1}^{j-1} E(X_i)
\]
which, using the definition of \(Z_j = E(S_j)\), can be written as a difference equation in \(Z_j\), i.e.,
\[
Z_j - \left(1 - \frac{1}{1 - \rho_j}\right)Z_{j-1} = \frac{n_jEV_j}{1 - \rho_j}, \quad j = 1, \ldots, K, \quad Z_0 = 0.
\]
The solution of (3.5) is
\[
Z_j = \sum_{i=1}^{j} n_iEV_i \prod_{r=1}^{i} \left(1 - \rho_r\right), \quad j = 1, \ldots, K.
\]
Letting \(T[\pi_0; e]\) denote the cycle time (conditional on \(Q(0)\)) under \(\pi_0\) in an exhaustive regime, we have
\[
E(T[\pi_0; e]) = Z_K = \sum_{i=1}^{K} n_iEV_i \prod_{r=1}^{K} \left(1 - \rho_r\right).
\]
Equation (3.7) exhibits quite clearly the evolution of the cycle. Specifically, consider the \(j\)th term in (3.7), \(n_jEV_j/[[\prod_{r=1}^{K} (1 - \rho_r)]\), in which the contribution of the \(n_j\) original customers in channel \(j\) to the cycle time is demonstrated. These \(n_j\) original customers require expected work \(n_jEV_j\) which causes a delay busy period in channel \(j\) of expected duration \(n_jEV_j/(1 - \rho_j)\). In turn, this delay busy period causes a delay busy period in channel \(j + 1\) of expected duration \((n_jEV_j/(1 - \rho_j))/(1 - \rho_{j+1})\), etc. So that the \(j\)th term in (3.7) contains the total expected delay caused to the cycle by the \(n_j\) initial customers in channel \(j\). The expected cycle time is simply the sum of the delays caused by all initial customers present at the start of the cycle.

3.2. Gated regime, zero switching times. The gating discipline, with zero switching times, interestingly, requires only a slight modification to the above analysis. Under the gating regime, if the server traverses policy \(\pi_0\) from the initial state
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\(Q(0) = (n_1, n_2, \ldots, n_K)\), the service sojourn at channel \(j\) is now

\[
X_j = \sum_{i=1}^{n_j} V_i + \sum_{k=1}^{n_j(S_{j-1})} V_{jk}
\]

with LST

\[
\tilde{X}_j(s) = \tilde{V}_j(s)^s S_{j-1}(\lambda_j[1 - \tilde{V}_j(s)]).
\]

as only customers arriving to channel \(j\) prior to time \(S_{j-1}\) are served in this cycle.

Taking expectations on (3.8) gives

\[
E(X_j) = n_j EV_j + \rho_j Z_{j-1}, \quad j = 1, \ldots, K
\]

which can be written as a system of difference equations

\[
Z_j - (1 + \rho_j)Z_{j-1} = n_j EV_j, \quad j = 1, \ldots, K,
\]

\[
Z_0 = 0.
\]

The system (3.10) yields the solution

\[
Z_j = \sum_{i=1}^{j} n_i EV_i \prod_{r=i+1}^{j} (1 + \rho_r), \quad j = 1, \ldots, K.
\]

Letting \(T[\pi_0; g]\) denote the cycle time under gating starting from \(Q(0)\) and utilizing \(\pi_0\), we have

\[
E(T[\pi_0; g]) = Z_K = \sum_{i=1}^{K} n_i EV_i \prod_{r=i+1}^{K} (1 + \rho_r).
\]

The \(j\)th term in Equation (3.12), \(n_j EV_j[\prod_{r=j+1}^{K} (1 + \rho_r)]\), exhibits the contribution of the original \(n_j\) customers found in channel \(j\) to our total cycle time. The \(n_j\) customers cause a delay of expected duration \(n_j EV_j\) in channel \(j\). During this delay in \(j\), we expect \(\lambda_{j+1}(n_j EV_j)\) arrivals to channel \(j + 1\), who will all be served under the gating regime during this cycle. Their expected service requirement is \(\rho_{j+1} n_j EV_j\). So the total delay in channels \(j\) and \(j + 1\) is \(n_j EV_j(1 + \rho_{j+1})\). But this allows an expected \(\lambda_{j+1}[n_j EV_j(1 + \rho_{j+1})]\) customers to enter into channel \(j + 2\), whose service during this cycle will take an expected time \(\rho_{j+2} n_j EV_j(1 + \rho_{j+1})\), so the total delay in channels \(j, j + 1, j + 2\) is now \(n_j EV_j(1 + \rho_{j+1})(1 + \rho_{j+2})\), etc. Once again the cycle is simply the sum of the delays caused by the initial customers in queue.

3.3. Exhaustive regime, swap-in and switch-out times. Consider now the exhaustive discipline where the server must take a random time, \(\tau_i\), to ‘switch into’ channel \(i\), no matter which channel he just completed. One can think of \(\tau_i\) as a set-up time for channel \(i\). We refer to it as the ‘swap-in’ time. After swapping into channel \(i\), the server stays for a service sojourn denoted \(X_i\), after which he must take a random time \(\theta_i\) to switch out of or exit channel \(i\) regardless of which channel he will switch into next. \(\tau_i, \theta_i\) are assumed to be independent of \(X_j, \theta_j\) and \(\tau_j\) for all \(j\) (\(\theta_j\) is also
assumed independent of \( \tau_j, \forall j \). \( X_j \), however is obviously dependent on \( \tau_j \). Let \( Y_j \) denote the ‘total server occupation time with channel \( j \) during our cycle’, and now let \( S_j = \sum_{i=1}^{N(S_j)} Y_i \), \( Z_i = E(S_i) \), i.e., \( S_j \) is the exit time, or completion time of channel \( j \) in our cycle. As such, we may decompose \( Y_j \) as

\[
Y_j = \tau_j + X_j + \theta_j = \tau_j + \sum_{i=1}^{N(S_{j-1})} B_{ji} + \sum_{k=1}^{N(S_j)} B_{jk} + \sum_{r=1}^{N(\tau_i)} B_{jr} + \theta_j
\]

with LST

\[
\hat{Y}_j(s) = \hat{B}_j(s)^n \hat{S}_{j-1}(\lambda_j[1 - \hat{B}_j(s)]) \hat{\tau}_j(s + \lambda_j[1 - \hat{B}_j(s)]) \hat{\theta}_j(s).
\]

Taking expectations we obtain

\[
E(Y_j) = \frac{n_jE\hat{V}_j}{1 - \rho_j} + \frac{\rho_j}{1 - \rho_j} \sum_{i=1}^{j-1} E(Y_i) + \frac{E(\tau_j)}{1 - \rho_j} + E(\theta_j).
\]

That is, the expected total server occupation with channel \( j \) is composed of the delay busy period caused by the original \( n_j \) customers plus the delay busy period caused by the swap-in time \( \tau_j \), plus the delay busy period caused by all the arrivals to channel \( j \) during the cycle so far \((S_{j-1})\), and the switch-out time \( \theta_j \).

We rewrite (3.14) as

\[
Z_j - \left( \frac{1}{1 - \rho_j} \right) Z_{j-1} = \frac{n_jE\hat{V}_j + E(\tau_j) + (1 - \rho_j)E(\theta_j)}{1 - \rho_j}, \quad j = 1, \ldots, K
\]

\[
Z_0 = 0
\]

with the (by now familiar) solution

\[
Z_j = \sum_{i=1}^{j} \frac{n_jE\hat{V}_i + E(\tau_i) + (1 - \rho_i)E(\theta_i)}{\prod_{r=1}^{j-1} (1 - \rho_r)}, \quad j = 1, \ldots, K.
\]

Letting \( T[\pi_0; e, \theta, \tau] \) denote the cycle time under the exhaustive regime with swap-in and switch-out times utilizing \( \pi_0 \), \( E(T[\pi_0; e, \theta, \tau]) = Z_K \) in Equation (3.16).

3.4. Gated regime, swap-in and switch-out times. We assume that customers at channel \( j \) are gated only after the server has switched into channel \( j \), i.e., at time \( S_{j-1} + \tau_j \). For this case, we mimic our results for the exhaustive discipline in that now we decompose \( Y_j \) as

\[
Y_j = \tau_j + \sum_{i=1}^{N(S_{j-1})} V_{ji} + \sum_{k=1}^{N(S_j)} V_{jk} + \sum_{r=1}^{N(\tau_j)} V_{jr} + \theta_j
\]

with LST

\[
\hat{Y}_j(s) = \hat{V}_j(s)^n \hat{S}_{j-1}(\lambda_j[1 - \hat{V}_j(s)]) \hat{\tau}_j(s + \lambda_j[1 - \hat{V}_j(s)]) \hat{\theta}_j(s)
\]

yielding

\[
E(Y_j) = n_jE\hat{V}_j + \rho_j \sum_{i=1}^{j-1} E(Y_i) + (1 + \rho_j)E(\tau_j) + E(\theta_j).
\]
Dynamic priority rules for cyclic-type queues leading to

\[ Z_j - (1 + \rho_j)Z_{j-1} = n_jEV_j + (1 + \rho_j)E(\tau_j) + E(\theta_j), \quad j = 1, \cdots, K \]

\[ Z_0 = 0 \]

so that

\[ Z_j = \sum_{i=1}^{j} [n_iEV_i + (1 + \rho_i)E(\tau_i) + E(\theta_i)] \prod_{r=i+1}^{j} (1 + \rho_r). \] (3.19)

Following our notation, \( E(T[\pi_0; g, \theta, \tau]) = Z_K \) in Equation (3.19).

4. Minimizing cycle times

Many optimization criteria may be considered within this truncated horizon, i.e., looking just \( K \) steps ahead. A classical objective would be to minimize the sum of (weighted) waiting times obtained during a cycle. Unfortunately, this problem appears to be computationally 'hard'. Another (greedy) objective is to minimize the cycle time, where the cycle time is the total time from 0, the epoch when a cycle starts, with \( Q(0) = (n_1, n_2, \cdots, n_K) \), until the server visits every channel exactly once. This allows the server to dispense with the work potentially generated (for the next \( K \) steps) by the current workload in the system as quickly as possible. To expand, we consider an arbitrary, or 'free' cycle where the server faced with \( Q(0) \) may choose the path freely to minimize the expected time of traversing this path. A path or tour is simply a permutation of \( \pi_0 = (1, 2, \cdots, K) \). We first consider the case where the server has only one option, to set the path at the beginning of a cycle for the next \( K \) steps, and shall later explore dynamic aspects.

4.1. Exhaustive and gated regimes with zero switching times. Consider first the exhaustive discipline without switching times. The server’s objective, being to minimize the expected cycle time, cannot be cast in standard stochastic scheduling models as his time spent in each channel is dependent on his entry time into that channel due to accruing customers. We compare the expected cycle time under policy \( \pi_0 \), with the expected cycle time under a policy \( \pi_1 = (1, 2, \cdots, j - 1, j + 1, j, j + 2, \cdots, K) \), i.e., \( \pi_1 \) is the policy \( \pi_0 \) with the \( j \)th and \((j + 1)\)th channels interchanged.

In an obvious notation corresponding to Equation (3.7),

\[ E(T[\pi_1; e]) = \sum_{i=1}^{j-1} n_iEV_i \prod_{r=i}^{K} \left( \frac{1}{1 - \rho_r} \right) \]

\[ + n_{j+1}EV_{j+1} \prod_{r=j+1}^{K} \left( \frac{1}{1 - \rho_r} \right) + n_jEV_j(1 - \rho_{j+1}) \prod_{r=j+1}^{K} \left( \frac{1}{1 - \rho_r} \right) \]

\[ + \sum_{i=j+2}^{K} n_iEV_i \prod_{r=i}^{K} \left( \frac{1}{1 - \rho_r} \right). \] (4.1)
Subtracting Equation (4.1) from (3.7) we find

\[ E(T[\pi_0; e]) - E(T[\pi_1; e]) = \frac{\rho_{j+1} n_j E V_j - \rho_j n_{j+1} E V_{j+1}}{\prod_{r=j}^{K} (1 - \rho_r)} \]

By the stability assumption, the denominator in Equation (4.2) is positive implying

\[ E(T[\pi_0; e]) < E(T[\pi_1; e]) \] if and only if

\[ \rho_{j+1} n_j E V_j < \rho_j n_{j+1} E V_{j+1}, \]

or equivalently, iff

\[ \frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}}. \]

Observation (4.3) yields immediately the form of the optimal policy, derived upon iterating the pairwise interchange, which is to schedule the channels at the initialization of the cycle in an increasing order of the values of \( n_j / \lambda_j \).

Similarly, applying the comparison of \( \pi_0 \) and \( \pi_1 \) to the gating discipline with zero switching times (Equation (3.12)), we may show

\[ E(T[\pi_0; g]) - E(T[\pi_1; g]) = (n_j E V_j \rho_{j+1} - n_{j+1} E V_{j+1} \rho_j) \prod_{r=j+2}^{K} (1 + \rho_r). \]

This results, surprisingly, in the identical criterion as the exhaustive, i.e.

\[ E(T[\pi_0; g]) < E(T[\pi_1; g]) \] iff

\[ \frac{n_j}{\lambda_j} < \frac{n_{j+1}}{\lambda_{j+1}}. \]

Relabel the channels so that lower indices correspond to lower values of \( n_j / \lambda_j \), and call this resulting permutation \( \pi^* \). We summarize the above as follows.

**Theorem 1.** Policy \( \pi^* \) (determined at the initialization of the cycle), by which the server faces with the initial state \( Q(0) = (n_1, \ldots, n_K) \) serves the channels via the path set in increasing order of the values of \( n_j / \lambda_j \) minimizes the expected cycle time in both the exhaustive and gated systems when switching times are zero, i.e., \( E(T[\cdot; e]) \) and \( E(T[\cdot; g]) \) are both minimized under the permutation \( \pi^* \).

4.2. **Discussion.** The policy described in Theorem 1 seems at first counterintuitive as it is independent of the (differing) service times, and is of identical form for both the exhaustive and gated regimes. Before we attempt to gain intuition into Theorem 1 a few remarks are in order.

1. We considered the case where the server chooses the full path at the start of a cycle (a \( K \)-step horizon). By the principle of optimality, however, if the server is able to limit his horizon further, i.e., a \( J \)-step horizon, \( 1 \leq J \leq K \), then at the
completion time of the Jth step, say at time t, it is optimal to utilize again the above strategy based now on $Q(t)$ for the next J steps. In particular, if $J = 1$, $\pi^*$ dictates to always serve next the unserved channel (in that cycle) with minimal value of $n_i(t)/\lambda_i$, where $t$ is a channel completion epoch, and $n_i(t)$ is the number of occupants of channel $i$ at $t$. As such, $\pi^*$ is easily implementable as it makes use only of the on-line information $Q(t)$, and $\lambda$, at any point in time $t$, where $\lambda = (\lambda_1, \ldots, \lambda_K)$. We explore the issue of dynamic implementation in Section 5.

2. We chose $\pi^*$ under the assumption of a 'greedy' server, or system. If, however, economic conditions warrant a longer cycle, e.g., if a cycle is 'expensive' in some sense, and we wish to maximize its expected length, the policy $\pi^*$ with the channels being sequenced via decreasing values of $n_i/\lambda_i$ is optimal. For example, in the symmetric service case, such as the important slotted-time models (see Takagi (1986)), where $EV_i = EV \forall i$, maximizing the expected cycle length is equivalent to maximizing the expected number of customers served during the cycle. This is seen by noting that if $M(\pi, e)$ and $M(\pi, g)$ denote respectively the number of customers served during the cycle starting from $Q$ under the exhaustive and gated regimes following policy $\pi$, then

$$E(M(\pi, \cdot)) = E(T[\pi, \cdot])/EV,$$

which is obviously maximized over $\pi^*$.

4.3. Exhaustive and gated regimes with swap-in and switch-out times. It is of interest to see if the above noted similarities between the exhaustive and gated regimes as well as the lack of dependence on the service times remain when switching times are permitted. Turning to the general cases of Sections 3.3 and 3.4, the analysis of Section 4.1 can be repeated for the exhaustive case (using Equation (3.16)) to show

$$E(T[\pi_0; e, \theta, \tau]) - E(T[\pi_i; e, \theta, \tau])$$

$$= \rho_{i+1}(n_i EV_j + \tau_i (1 - \rho_i)E\theta_j) - \rho_{i+1}(n_{i+1} EV_{j+1} + \tau_{j+1} (1 - \rho_{j+1})E\theta_{j+1}) \prod_{r=j}^{K} (1 - \rho_r).$$

Similarly for the gated case, using Equation (3.19),

$$E(T[\pi_0; g, \theta, \tau]) - E(T[\pi_i; g, \theta, \tau])$$

$$= (\rho_{j+1}[n_j EV_{j+1} + (1 + \rho_j)\tau_{j+1} + E\theta_{j+1}]$$

$$- \rho_{j+1}[n_{j+1} EV_{j+2} + (1 + \rho_{j+1})\tau_{j+2} + E\theta_{j+2}] \prod_{r=j+2}^{K} (1 + \rho_r).$$

Let $\pi^*(e, \theta, \tau)$ and $\pi^*(g, \theta, \tau)$ denote respectively the permutations obtained when the channels are relabelled in an increasing order of the respective values

$$[n_i EV_i + \tau_i (1 - \rho_i)E\theta_i]/\rho_i.$$
and

$$(4.9) \quad [n_i E\tau_i + (1 + \rho_i) E\theta_i + E\theta_i] / \rho_i.$$  

Upon iterating the pairwise interchange, we can readily derive the following result from Equations (4.6) and (4.7).

**Theorem 2.** $E(T[\cdot ; e, \theta, \tau])$ is minimized when calculated over the permutation $\pi^*(e, \theta, \tau)$, and $E(T[\cdot ; g, \theta, \tau])$ is minimized when calculated over the permutation $\pi^*(g, \theta, \tau)$.

**Remark.** We may again define policies $\pi_s(\cdot, \theta, \tau)$ for both the exhaustive and gated regimes that maximize the expected cycle length by following $\pi^*(\cdot, \theta, \tau)$ in reverse order. For the symmetric service case ($EV_i = EV, \forall i$), this policy will maximize the expected number of customers served during the cycle. Letting $m(\pi, \cdot)$ denote the number of customers served in the cycle following policy $\pi$ with swap-in and switch-out times included, we have

$$E(M(\pi, \cdot)) = \frac{E(T[\pi, \cdot] - \sum_{i=1}^{K} E(\theta_i + \tau_i)}}{EV},$$

which is maximized under $\pi_s(\cdot, \theta, \tau)$.

We collect our results in Table 1.

It is obvious from the forms of $\pi^*$ in Table 1 that when any switching times are included service times do play a role in the determination of the optimal policy, and policies are obviously of different forms for the exhaustive and gated regimes.

<table>
<thead>
<tr>
<th>$\pi^*(\cdot)$</th>
<th>Zero-switch times ($\theta_i = 0, \tau_i = 0, \forall i$)</th>
<th>Exhaustive (e)</th>
<th>Gated (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_i$</td>
<td>$\lambda_i$</td>
<td>$n_i$</td>
<td>$\lambda_i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\pi^*(\cdot, \theta)$</th>
<th>Switch-out alone ($\theta_i \geq 0, \tau_i = 0, \forall i$)</th>
<th>$n_i E\tau_i + (1 - \rho_i) E\theta_i$</th>
<th>$n_i E\tau_i + E\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\pi^*(\cdot, \tau)$</th>
<th>Swap-in alone ($\theta_i = 0, \tau_i \geq 0$)</th>
<th>$n_i E\tau_i + E\tau_i$</th>
<th>$n_i E\tau_i + (1 + \rho_i) E\tau_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\pi^*(\cdot, \theta, \tau)$</th>
<th>in-out ($\theta_i \geq 0, \tau_i \geq 0$)</th>
<th>$n_i E\tau_i + E\tau_i + (1 - \rho_i) E\theta_i$</th>
<th>$n_i E\tau_i + (1 + \rho_i) E\tau_i + E\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
<td>$\rho_i$</td>
</tr>
</tbody>
</table>

**Expected Cycle Time**

$$E(T[\pi; e, \theta, \tau]) = \sum_{i=1}^{K} \frac{n_{a(i)} E\tau_{a(i)} + E\tau_{a(i)} + (1 - \rho_{a(i)}) E\theta_{a(i)}}{\prod_{r=1}^{K} (1 - \rho_{a(r)})}$$

$$E(T[\pi; g, \theta, \tau]) = \sum_{i=1}^{K} \left[ (n_{a(i)} E\tau_{a(i)} + (1 + \rho_{a(i)}) E\tau_{a(i)} + E\theta_{a(i)}) \prod_{r=1}^{K} (1 + \rho_{a(r)}) \right]$$
However, a unifying principle does exist by which all the results are clearly generated.

4.4. Unification via a general principle. Consider $K$ tasks that must be sequentially performed in a non-preemptive manner by a single processor. All tasks are available at time $0$ (as is the processor). Task $i$ carries a random initial processing requirement of expected size $a_i$, called its core, but if processing is delayed until $t$, the expected requirement has grown to $a_i + \alpha_i t$ (i.e., $\alpha_i$ is the expected growth per unit time delay in performing task $i$) $i = 1, \ldots, K$. Browne and Yechiali (1987) showed that the dynamics of this process is such that if the policy $\pi = (\pi(1), \pi(2), \ldots, \pi(K))$ is followed, the total time to process all $K$ tasks has expectation

$$
E(T[\pi]) = \sum_{i=1}^{K} a_{\pi(i)} \prod_{r=i+1}^{K} (1 + \alpha_{\pi(r)}),
$$

which is minimized when following the permutation based on increasing values of the critical quantity $a_i/\alpha_i$, the ratio of each task’s core to its growth rate. Call this permutation $\pi^*$. Browne and Yechiali (1987) further showed that within this $\pi^*$

$$
\frac{a_i}{\alpha_j} \leq \frac{\sum_{i=1}^{j-1} a_i \prod_{r=i+1}^{j-1} (1 + \alpha_r)}{\prod_{r=i}^{j-1} (1 + \alpha_r) - 1}, \quad j = 1, \ldots, K.
$$

Consider now the exhaustive regime with both swap-in and switch-out times at the initialization of our cycle from state $Q(0)$. As the server is free to just choose the path for the next $K$ steps but committed to serve all unserviced channels before returning to those serviced previously in the cycle, we may immediately apply the above results. The core at channel $i$ is the expected total server occupation at $i$ if the server chooses $i$ at time $0$. As there are $n_i$ customers in queue there, this is simply the delay busy period caused by those $n_i$ waiting customers plus the delay busy period caused by the swap-in time, plus the switch-out time, i.e.,

$$
a_i = \frac{n_i EV_i}{1 - \rho_i} + \frac{ET_i}{1 - \rho_i} + E\theta_i.
$$

The expected growth in service requirement per unit time delay in channel $i$ is $\alpha_i = \rho_i/(1 - \rho_i)$, as the expected flow of work to channel $i$ per unit time is $\rho_i$, causing a delay busy period there of expected length $\rho_i/(1 - \rho_i)$. Upon application of the above principle, it is seen that scheduling via increasing values of $a_i/\alpha_i$ corresponds to $\pi^*(e, \theta, \tau)$. Note too, that in conjunction with Equation (4.10), $E(T[\pi; e, \theta, \tau])$ can equivalently be written as (see Equation (3.16) and Table 1)

$$
E(T[\pi; e, \theta, \tau]) = \sum_{i=1}^{K} \frac{n_{\pi(i)} EV_{\pi(i)} + ET_{\pi(i)} + (1 - \rho_{\pi(i)})E\theta_{\pi(i)}}{1 - \rho_{\pi(i)}} \prod_{r=i+1}^{K} \left(1 + \frac{\rho_{\pi(r)}}{1 - \rho_{\pi(r)}}\right).
$$
Applying the $a_i/\alpha_i$ principle results in $\pi^*(\epsilon, \theta, \tau)$. However, for the gated regime, with swap-in and switch-out times (Section 3.4) the ‘core’ at channel $i$ has the form $a_i = n_i EV_i + (1 + \rho_i)ET_i + E\theta_i$, and due to the nature of the gating system, the flow of work into channel $i$ per unit time is $\alpha_i = \rho_i$, as each arrival carries only a service requirement and not a busy period. Application of the $a_i/\alpha_i$ principle leads directly to the policy $\pi^*(g, \theta, \tau)$, of Equation (4.9), summarized in Table 2.

As such, it is easy to see that for the case of zero switching times, $(\theta_i = 0 = \tau_i \forall i)$, the true critical quantity determining $\pi^*$ is

$$\left(\frac{n_i EV_i}{1 - \rho_i}\right) / \left(\frac{\rho_i}{1 - \rho_i}\right)$$

for the exhaustive regime and $n_i EV_i / \rho_i$ for the gated regime, both leading to the ambiguous quantity $n_i / \lambda_i$.

Equation (4.11) is sufficient to prove that our set of optimal policies correspond to the celebrated Dynamic Allocation Index (DAI) or Gittins Index (see e.g., Gittins (1982) and Whittle (1982)) for the multi-armed bandit problem. Our problem however, is not directly a bandit as the channels evolve randomly while not being served, while in the regular multi-armed bandit problem, the processes not being served remain constant, or ‘frozen’ (but see Whittle’s (1982) derivation of an ‘open process’).

### 5. Dynamic priority rules

Our attention has been restricted thus far to a single cycle. We may extend our results to the dynamic operation of a polling system quite easily. The aforementioned previous research on polling systems dealt with the server continuously following a fixed cyclic template, regardless of whether any customers are in queue or not at any channel. Consequently, the dynamic operation of the system was not dealt with in terms of optimization, and equilibrium results, such as the equilibrium cycle time, $E(c)$, having the evaluation (see Eisenberg (1972), Watson (1984) and

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**Table 2**

Core ($a_i$) and growth ($\alpha_i$) with swap-in and switch-out times ($\tau_i, \theta_i \geq 0 \forall i$)

<table>
<thead>
<tr>
<th>Exhaustive</th>
<th>Gated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i$</td>
<td>$n_i EV_i + (1 + \rho_i)ET_i + E\theta_i$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>$\rho_i$</td>
</tr>
</tbody>
</table>

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Takagi (1986))

\[ E(c) = \frac{\sum_{i=1}^{K} E(\theta_i)}{1 - \sum_{i=1}^{K} \rho_i}, \]

(as \( \tau_i \) is equal to 0 \( \forall i \) in their model and \( \theta_i \) is the time to switch from channel \( i \) to \( i+1 \)) are central points of analysis, as the evaluation of waiting times is of major importance.

In our model, the server need not cycle endlessly and may remain idle when no customers are in the system. As soon as a customer arrives to the system however, the server must return to work (i.e., swap-in to the channel the customer arrives to) and work thereafter in a manner to be described below until the system is empty once again.

Within this context we define a set of policies, the pseudo-cyclic class, whereby the server dichotomizes the channels into two groups, an unserved set and a served set at every decision epoch on each ‘tour’: the server must complete all the unserved channels on a tour (a permutation of the channel indices) before returning to the served group to start his next tour. Obviously, a tour starts when the set of unserved channels is full. The server may choose the path of each tour consistently with the decision horizon. We are interested in considering the class of pseudo-cyclic policies where the objective of the server is to minimize the expected length of each tour. However, for any objective function, the restriction to the pseudo-cyclic class does ensure some degree of ‘fairness’ to the channels. It also allows the server to poll the system only once in a decision horizon.

To illustrate, consider the exhaustive regime with zero switching times (\( \theta_i = \tau_i = 0 \forall i \)). As the server is allowed to stop switching when the system is empty, the moment from when a customer arrives to an empty system till the next moment the server becomes idle is finite with probability 1 if \( \sum_{i=1}^{K} \rho_i < 1 \) and is distributed as the busy period in an M/G/1 queue with arrival rate \( \lambda = \sum_{i=1}^{K} \lambda_i \) and service distribution

\[ G(x) = \sum_{i=1}^{K} \frac{\lambda_i}{\lambda} G_i(x). \]

We call this period of time the system busy period; it has the same evaluation for both the exhaustive and the gated regime as long as switching times are 0. As long as server idleness is not permitted, it is easily shown (see, e.g., Cobham (1954), and the literature on priority queues) that the system busy period has expected duration \( \rho/\lambda(1 - \rho) \) (where \( \rho = \sum_{i=1}^{K} \rho_i \)), and the expected number served during this period is \( 1/(1 - \rho) \). These quantities are obviously invariant to the service policy as long as it is work conserving.
Set $t = 0$ to be the epoch at which the system busy period starts, and say a customer of type $j$ initiates this busy period, i.e.,

$$Q(0^-) = O, \quad Q(0^+) = e_j$$

where $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, a $K$-vector of 0's with a 1 in the $j$th position. Let $X_j$ denote the (exhaustive) service sojourn in channel $j$, so that $E(X_j) = EV/(1 - \rho_j)$. Then at the completion of channel $j$ the state is

$$Q(X_j) = (N_1(X_j), \ldots, N_{j-1}(X_j), 0, N_{j+1}(X_j), \ldots, N_K(X_j)),$$

where $N_j(\cdot)$ denotes a Poisson process with intensity $\lambda_j$.

If the server wishes to set the path of the first tour at $t = 0$ by using Theorem 1 on the expected state $E(Q(X_j))$, it would be set via increasing (in $i$) values of $E(N_i(X_j))/\lambda_i = E(X_j)$, a constant for every state! It is obvious that by the assumption of Poisson arrivals the rule is non-anticipative i.e., one cannot set the route based on an expected state, but must rather wait for an actual realization (the $n_i$'s) and then set the path at time $X_j$ based on that realization. This describes a purely adaptive control procedure, i.e., in our example, the server stays at channel $j$ for time $X_j$, upon his exit from $j$ he is faced with the state $Q(X_j)$, and sets the route at that time ($t = X_j$) by increasing values of the actual realization $N_i(X_j)/\lambda_i$. This implies that now the server may look ahead no further than $K - 1$ steps, as there are now at most $K - 1$ non-zero channels. If the server now utilizes a $K - 1$ step horizon, by our definition of a pseudo-cyclic policy, channel $j$ will always be the last channel served on any tour; in a sense channel $j$ will serve as an 'anchor', as its completion will always coincide with the start of a new tour.

If the server is operating on a 1-step horizon within the context of a pseudo-cyclic policy, then every channel completion affords him a decision epoch, whereby it would be optimal for him to always choose to serve the channel from the unserved group with the minimal value of $n_i(t)/\lambda_i$, where $i$ is a decision epoch within that tour and $n_i(t)$ is the state of channel $i$ at that point (as long as $i$ has not yet been serviced on that tour). Proceeding in this manner, the server can continously choose paths to minimize each pseudo-cycle, or tour. Systems with a general polling table (Baker and Rubin (1987)) can be treated accordingly with only a slight modification of a tour; however, the fairness criteria must be reevaluated. The full effect of this operating policy or dynamic priority rule on waiting times has yet to be explored.

Note however, that once switching times are allowed in the model, the rule is anticipative in that if a customer of type $j$ arrives to an empty system—in effect, 'turning-on' on the server—and $Y_j$ denotes the total server occupation time at $j$, i.e.,

$$Q(0^-) = O, \quad Q(0^+) = e_j, \quad Q(Y_j) = (N_1(Y_j), N_2(Y_j), \ldots, N_{j-1}(Y_j), N_j(\theta_j), N_{j+1}(Y_j), \ldots, N_K(Y_j))$$
for the exhaustive regime, and
\[ Q(Y_t) = (N_1(Y_t), \ldots, N_{j-1}(Y_t), N_j(S_t - \tau_j), N_{j+1}(Y_t), \ldots, N_K(Y_t)) \]
for the gated regime.

Then, following a pseudo-cyclic policy, the server may implement the rule based on
\[ E(Q(Y_t)) = \rho(1 - \rho)E\theta \] for the exhaustive case and via increasing \[ [(1 + \rho)E\tau + E\theta]/\rho \] for the gated regime. Moreover, for this case, necessary conditions under which the system busy period terminates in finite time are not yet known.

Returning to the exhaustive regime with zero switching times, consider the consequences of 2-cycle horizon, i.e., setting the path for 2K steps ahead at \( t = 0 \) following a pseudo-cyclic rule (so that each channel can be served at most twice in the next 2K steps), but where \( Q(0) = (n_1, \ldots, n_K) \). For convenience, assume that we have relabelled the channels via increasing values of \( n_i/\lambda_i \), so that the optimal path for the first tour is simply \((1, 2, \ldots, K)\).

Anticipating the second cycle, let \( S^{(1)}_j \) denote the completion time of the \( j \)-th channel on the \( i \)-th tour, \( j = 1, \ldots, K, i = 1, 2 \) (so that \( S^{(1)}_K \) is the completion time of the first tour) and let \( E(S^{(1)}_j) = Z^{(1)}_j \). Then,
\[ Q(S^{(1)}_K) = (N_1(S^{(1)}_K - S^{(1)}_1), \ldots, N_j(S^{(1)}_K - S^{(1)}_j), \ldots, N_{K-1}(S^{(1)}_K - S^{(1)}_{K-1}), 0) \]
and for any arbitrary \( K-1 \) 'next' sequence, \((i_1, \ldots, i_{K-1})\), let \( Z^{(2)} \) denote the expected second tour completion time (this is obviously a partial tour, as channel \( K \) is presently 0). It is then seen that for that sequence,
\[ Z^{(2)} = Z^{(1)}_K + \sum_{j=1}^{K-1} \frac{\rho_j E(S^{(1)}_K - S^{(1)}_j)}{\prod_{r=j}^{K-1} (1 - \rho_r)} = Z^{(1)}_K + \sum_{j=1}^{K-1} \frac{\rho_j (Z^{(1)}_K - Z^{(1)}_j)}{\prod_{r=j}^{K-1} (1 - \rho_r)}. \]

Applying the \( a_i/a_i \) principle (see Equation (4.10)) shows that Equation (5.5) is minimized when \((i_1, \ldots, i_{K-1})\) is chosen by increasing values of \( Z^{(1)}_K - Z^{(1)}_j \), or equivalently, by decreasing values of \( Z^{(1)}_K \). However, by our results on the first cycle we know that \( Z^{(1)}_1 < Z^{(1)}_2 < \cdots < Z^{(1)}_K \). Therefore, by expanding our horizon to \( 2K - 1 \) steps ahead, the last \( K - 1 \) steps are completely determined by the first steps in that the channels will be served in the sequence \((1, 2, \ldots, K-1, K, K-1, K-2, \ldots, 2, 1)\), a rocking, or pendulum effect. A similar analysis for both the gated and exhaustive regimes with switching times show that if the decision horizon is expanded past one cycle, the server simply goes back and forth continuously across the same sequence. As swap-in and switch-out times exist, the sequence will now look like \((1, 2, \ldots, K-1, K, K, K-1, \ldots, 2, 1)\). As such, the adaptive control, adapted to the actual realizations during each tour, is preferred.

Acknowledgement

This paper is based on parts of the first author’s dissertation carried out under the supervision of the second.
References


