A NOTE ON A STOCHASTIC PRODUCTION-MAXIMIZING TRANSPORTATION PROBLEM

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ABSTRACT

A stochastic production-maximizing problem with transportation constraints is considered where the production rates, $R_{ij}$, of man $i$—job $j$ combinations are random variables rather than constants. It is shown that for the family of Weibull distributions (of which the Exponential is a special case) with scale parameters $\lambda_{ij}$ and shape parameter $\beta$, the plan that maximizes the expected rate of the entire line is obtained by solving a deterministic fixed charge transportation problem with no linear costs and with "set-up" cost matrix $||\lambda_{ij}||$.

The Time-Minimizing Transportation Problem (TMTP) was treated by Barsov (1959) [2] and again by Hammer (1969) [3] and may be stated as follows: Given a set of $m$ origins and $n$ destinations, where there are $a_i$ ($i=1, 2, \ldots, m$) units available at the $i$th origin and $b_j$ ($j=1, 2, \ldots, n$) units required at the $j$th destination (such that, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$), find a set of nonnegative variables $x_{ij}$ ($i=1, 2, \ldots, m; j=1, 2, \ldots, n$) satisfying the (classical transportation-type) constraints

\[ \sum_{j=1}^{n} x_{ij} = a_i \quad (i=1, 2, \ldots, m) \]

\[ \sum_{i=1}^{m} x_{ij} = b_j \quad (j=1, 2, \ldots, n) \]

and minimizing the greatest of the given nonnegative numbers $t_{ij}$ for which $x_{ij} > 0$.

The $t_{ij}$ may be interpreted as the time required to transport a positive load $x_{ij}$ (however big or small) from the $i$th origin to the $j$th destination. Thus the problem is to find a transportation plan which makes the most time-consuming trip as short as possible.

In a production context the analogous problem will be to consider the rate of production, $R_{ij}$, instead of the time $t_{ij}$, and to seek a production plan $X = \{x_{ij}\}$ satisfying (1) and maximizing the smallest of the given nonnegative rates $R_{ij}$ for which $x_{ij} > 0$. The $R_{ij}$'s are now interpreted as the rate of production (on a production line, say) of a man belonging to group (origin) $i$ when he is assigned to job (destination) $j$. As above, there are $a_i$ men available in the $i$th group and $b_j$ men required for the $j$th job.
For any production plan $X$ that satisfies (1) let $A_X = \{(ij) | x_{ij} > 0\}$. For any such plan, the rate of production of the entire line, $R$, will be given by

\[(2) \quad R = \min_{(ij) \in A_X} \{R_{ij}\} \]

and the problem is then to find a plan for which $R$ is as large as possible.

Now, suppose that for each man $i$–job $j$ combination, the corresponding $R_{ij}$ is not a constant, but a continuous nonnegative random variable with distribution function $F_{ij}(\cdot)$. This implies that, for any plan $X$, $R$ (as given by (2)) is also a random variable. Our objective then is to find a production plan that will maximize the expected rate of production of the line, i.e., we seek a plan $X$ satisfying (1) so as to achieve

\[(3) \quad \max_X \{E[R]\} = \max_X \{E[\min_{(ij) \in A_X} \{R_{ij}\}]\}. \]

Assuming that the $R_{ij}$'s are independent random variables with finite means, the distribution function of $R$, $F_R(\cdot)$, is found to be

\[F_R(r) = 1 - \prod_{(ij) \in A_X} \left[1 - F_{ij}(r)\right], \]

and the expected rate of production is given by

\[E[R] = \int_0^\infty \left\{ \prod_{(ij) \in A_X} \left[1 - F_{ij}(r)\right]\right\} dr. \]

Now consider the family of Weibull distributions where $R_{ij}$ has a scale parameter $\lambda_{ij} > 0$ and shape parameter $\beta > 0$ (equal for all man-job combinations). In this case, the distribution function of $R_{ij}$ is

\[(4) \quad F_{ij}(r) = 1 - \exp \left(-\lambda_{ij} r^\beta\right), \quad r \geq 0. \]

We consider also the following “Fixed Charge Transportation Problem” (FCTP) with no linear costs: Given $\lambda_{ij} > 0 (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ find a plan $X$ satisfying (1) so as to achieve

\[(5) \quad \min \left\{ \sum_{(ij) \in A_X} \lambda_{ij} \right\}. \]

We now show the following:

**THEOREM:** The solution to the stochastic production-maximizing transportation problem (3) is given by the solution of the FCTP (5).
PROOF: For any given plan $X$ we obtain:

$$
E[R] = \int_0^\infty \exp \left\{ - \left( \sum_{(i,j) \in A_X} \lambda_{ij} \right)^{\beta} \right\} dr \\
= \left( \frac{1}{\sum_{(i,j) \in A_X} \lambda_{ij}} \right)^{1/\beta} \Gamma\left( \frac{1}{\beta} + 1 \right),
$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is clear that $E[R]$ in (6) is maximized whenever $\sum_{(i,j) \in A_X} \lambda_{ij}$ is minimized. This completes the proof.

By letting $\beta = 1$ in (4) it is readily seen that the exponential family of distributions is a special case of the family of Weibull distributions. Note also that if we let $m = n$ and $a_i = b_j = 1$ for all $i$ and $j$ then the deterministic and stochastic production-maximizing problems are transformed, respectively, into the classical [3] and stochastic [6] bottleneck assignment problems, whereas the FCTP [1] is transformed into the assignment problem.

In general, fixed charge problems have proven difficult to solve, primarily because each extreme point (here a basic solution of (1)) of the convex set of feasible solutions is a local optima. In our case, however, a direct way to solve the FCTP would be to enlarge it into an assignment problem of order

$$
\left( \sum_{i=1}^n a_i \right) \times \left( \sum_{j=1}^n b_j \right).
$$

Another approach could be to formulate the FCTP as an all-integer linear program [1]. A third method would employ a branch and bound algorithm as presented in [5]; however, for large problems all of the above methods would eventually become inefficient, and an approximative procedure (such as the one suggested in [1]) seems to be more practical. Additional references for approximative methods may be found in [5].

In summary, we have shown that the plan that maximizes the expected production rate of the entire line in a randomized production-maximizing transportation problem can be found by solving a deterministic fixed charge transportation problem with no linear costs and with fixed-charge cost matrix $\|\lambda_{ij}\|$ whose entries are the scale parameters of the random variables $R_{ij}$.

REFERENCES