On the Batch Size and the Busy Period in the Finite 'Israeli Queue'

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Abstract

The Israeli Queue is a polling-type system with N groups (queues) and un-limited size batch service, where the next group to be served is the one with the most senior customer. For such a system we derive (i) the mean size of each group, and (ii) the Laplace-Stieltjes Transform and mean of the length of a busy period starting with $n \geq 1$ groups. Numerical calculations are presented and the parameters' effect is discussed.

Keywords Israeli Queue · Unlimited-size batch service · Group size · Busy period

Mathematics Subject Classification 60K25 · 90B22

1 Introduction

The 'Israeli Queue' model was introduced in Boxma, van der Wal and Yechiali [3, Section 5] when studying a multi-queue single-server polling system with unlimited-size batch service [2, 3], where the next queue to be served is the one with the most senior customer (the customer who has been waiting for the longest time among all present customers). The term 'Israeli Queue' originates from a real waiting line of individuals formed in order to buy tickets for a show. The associated queueing system is comprised of heads of groups, where each head can buy an unrestricted number of tickets for the group of individuals he ("he" stands for "she" as well) represents, while the purchasing time is assumed to be independent of the number of tickets bought. A new arrival either joins one of the existing groups, if he knows the group's head, or creates a new group, acting as its leader.

Unlimited batch-service was studied by van der Wal and Yechiali [12] when analyzing a computer tape-reading problem in a system where large amounts of information are stored on tapes. It is assumed that the time to mount, read and dismount the tape is independent of the amount of information read from the tape. The problem was formulated as a polling system and optimal visiting rules of the server were studied. Unlimited batch-service models were also considered in the literature as application to videotex, telex and TDMA (Time Division Multiple Access) systems (see e.g. [1], [4] and [6]). Van Oyen and Teneketzis [13] formulated a central data base system and an Automated Guided Vehicle (AGV) as a polling system with an infinite capacity batch service.

Recently, Perel and Yechiali [8] extended the Israeli Queue model to the case where there is no bound on the number of different groups that can be present simultaneously in the system. They analyzed single-server models with finite and infinite number of groups, as well as models with multiple servers, and derived various performance measures. Perel and Yechiali [9] further studied a two-class single-server preemptive priority queueing model in which the high priority customers form a classical $M/M/1$ queue, while the low priority (class 2) customers form the unlimited-size batch service Israeli Queue with a finite number of groups. They calculated various performance measures, such as the mean number of low priority groups in the system along with the mean size of a class-2 group; the covariance between the number of high priority customers and the number of low priority groups; sojourn times of a class-2 group leader, as well as of an arbitrary class-2 customer. In addition, Perel and Yechiali [10] studied an Israeli Queue model with retrials in which the system is comprised of a 'main' queue and an orbit queue. The main queue consists of at most M groups, where a new arrival enters the main queue either by joining one of the existing groups, or by creating a new group. If an arrival can not join one of the groups in the main queue, he goes to a retrial (orbit) queue. The orbit queue dispatches orbiting customers back to the main queue at a constant rate. Various performance measures were derived, such as the mean number of groups in the main queue, the mean number of orbiting customers, the mean size of each group standing in the main queue, and the mean number of bypasses made by an arriving customer.

In this work we consider a $M/M^{Batch}/1$ Israeli Queue, where the number of groups present in the system is at most N. Each group has a "leader" or a "head" - the first one of the group to arrive to the system. A new arrival sees only the head of each existing group, and the probability that he knows a group leader is p, independent of the group size. A new arrival joins the group of the first leader he knows. That is, if there are $1 \leq n \leq N-1$ groups in the system (including the one in service), then the probability that a new arrival joins the $k-th$ group is $(1-p)^{k-1}p$, for $1 \leq k \leq n$, while the probability that he creates a new group (the $(n+1)$-st) is $(1-p)^n$. We assume that an arriving customer can also join the group that is being served. Also, if N groups are present and a new arrival does not join any of the first $N-1$ groups, he will necessarily join the last group (in the N-th position). The arrival process is Poisson with rate λ, and the service is given, as indicated, in unlimited-size batches. That is, it takes one (random) service duration to serve a group, independent of its size. We assume that a service duration of each group is exponentially distributed with parameter μ.

Previous results. Define X as the total number of different groups in the system in steady state, where $0 \leq X \leq N$. Let $\pi_n = P(X = n)$, for $0 \leq n \leq N$. In [8] we showed that

$$\begin{aligned} \pi_n &= \left(\frac{\lambda}{\mu}\right)^n (1-p)^{\frac{n(n-1)}{2}} \pi_0, \quad 1 \leq n \leq N, \\ \pi_0 &= \left(\sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n (1-p)^{\frac{n(n-1)}{2}}\right)^{-1}. \end{aligned} \tag{1.1}$$

In addition, let $D^{(k)}$ denote the total size of the group standing at the k-th position $(1 \leq k \leq N)$, *an instant after a service completion.* We have, for $1 \leq k \leq N$ (see [8]),

$$\mathbb{E}\left[D^{(k)}\right] = \frac{\lambda}{\mu}(1-p)^{k-1} + \frac{\pi_k}{\sum_{j=k}^{N} \pi_j}. \tag{1.2}$$

In this work we derive (Section 2) the mean value of the group sizes right after a moment of service completion or an arrival. The Laplace-Stieltjes Transform of the length of a busy period, starting with $n \geq 1$ groups, is obtained in Section 3 and its first moment is calculated. Numerical results are presented in Section 4.

2 The Size of a Group

Define a 'Poissonian event' as an instant where there is either an arrival or a service completion. Let L_k^m denote the number of customers present in the k-th group ($k = 1, 2, \ldots, N$) immediately after the m-th Poissonian event took place, for $m \geq 1$, and let $\vec{L}^m = (L_1^m, L_2^m, \ldots, L_N^m)$. We observe the system at two successive Poissonian events, m and $m+1$. Note that, if the system is not empty, the time elapsing until the next Poissonian event is exponentially distributed with mean $\frac{1}{\lambda+\mu}$, whereas, if the system is empty, the time elapsing until the next Poissonian event is exponentially distributed with mean $\frac{1}{\lambda}$.

2.1 Number of groups before a Poissonian event

Let $\{Y_m, m \geq 1\}$ be the number of groups in the system (state of the process Y) a moment before the m-th Poissonian event occurs. $\{Y_m, m \geq 1\}$ defines a finite (semi) Markov chain with one-step transition probabilities $\nu_{ij} = \mathbb{P}(Y_{m+1} = j | Y_m = i)$, for $i, j = 0, 1, \ldots, N$. Let $Q = [\nu_{ij}]_{i,j}$ be the one step transition probability matrix of the process $\{Y_m, m \geq 1\}$. Then, Q is given by

$$Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda p}{\lambda+\mu} & \frac{\lambda(1-p)}{\lambda+\mu} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-(1-p)^2)}{\lambda+\mu} & \frac{\lambda(1-p)^2}{\lambda+\mu} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-(1-p)^{N-1})}{\lambda+\mu} & \frac{\lambda(1-p)^{N-1}}{\lambda+\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix}.$$

Let $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_N)^T$ denote the limiting distribution of $Y = \lim_{m\to\infty} Y_m$, i.e. $\sigma_k = \mathbb{P}(Y = k)$, $\vec{\sigma}Q = \vec{\sigma}$, and $\sum_{k=0}^{N} \sigma_k = 1$. The derivation of σ_k, for $k = 0, 1, \ldots, N$ is as follows. First, we have

$$\sigma_0 = \frac{\mu}{\lambda+\mu}\sigma_1, \tag{2.1}$$

which gives

$$\sigma_1 = \sigma_0 \frac{\lambda + \mu}{\mu}. \tag{2.2}$$

Second,

$$\sigma_1 = \sigma_0 + \frac{\lambda p}{\lambda + \mu}\sigma_1 + \frac{\mu}{\lambda + \mu}\sigma_2,$$

and after using equation (2.2) and rearranging terms, we get

$$\sigma_2 = \sigma_0 (\lambda + \mu) \frac{\lambda}{\mu^2}(1 - p). \tag{2.3}$$

Continuing further, we obtain

$$\sigma_2 = \frac{\lambda(1 - p)}{\lambda + \mu}\sigma_1 + \frac{\lambda(1 - (1 - p)^2)}{\lambda + \mu}\sigma_2 + \frac{\mu}{\lambda + \mu}\sigma_3,$$

and after using equations (2.2), (2.3) and rearranging terms, we get

$$\sigma_3 = \sigma_0 (\lambda + \mu) \frac{\lambda^2}{\mu^3}(1 - p)^3. \tag{2.4}$$

It can be verified that for $k = 1, 2, \ldots, N$, σ_k is given by

$$\sigma_k = \sigma_0 (\lambda + \mu) \frac{\lambda^{k-1}}{\mu^k}(1 - p)^{\frac{k(k-1)}{2}}, \tag{2.5}$$

where σ_0 is obtained from the normalization equation, $\sum_{k=0}^{N} \sigma_k = 1$. We thus have

$$\sigma_0 = \left(1 + (\lambda + \mu) \sum_{k=1}^{N} \frac{\lambda^{k-1}}{\mu^k}(1 - p)^{\frac{k(k-1)}{2}}\right)^{-1}. \tag{2.6}$$

As Q is a semi-Markov process of $\{Y_m\}$ in steady state, and σ_k is the fraction of visits of process Y at state k, then the proportion of time that there are k groups in the system ($\{X = k\}$) is given by (see [11])

$$\pi_0 = \frac{\frac{\sigma_0}{\lambda}}{\frac{\sigma_0}{\lambda} + \frac{1}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j},$$

$$\pi_k = \frac{\frac{\sigma_k}{\lambda+\mu}}{\frac{\sigma_0}{\lambda} + \frac{1}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j}, \quad k = 1, 2, \ldots, N. \tag{2.7}$$

Indeed, substituting in equation (2.7) the expressions for σ_k $(0 \le k \ \le N)$ given in equations (2.5) and (2.6), results in equation (1.1).

The Makovian law of motion of the process $\left(\vec{L}^m\right)_{m=1}^{\infty}$ is given by (see explanation below equation (2.8))

$$
\left(L_1^{m+1}, L_2^{m+1}, \ldots, L_N^{m+1}\right) = \begin{cases}
(1,0,0,\ldots,0) & \text{w.p.} \quad \sigma_0 \\
(L_1^m+1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_1 \\
(L_1^m,1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)}{\lambda+\mu}\sigma_1 \\
(0,0,\ldots,0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_1 \\
(L_1^m+1,L_2^m,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_2 \\
(L_1^m,L_2^m+1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)p}{\lambda+\mu}\sigma_2 \\
(L_1^m,L_2^m,1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)^2}{\lambda+\mu}\sigma_2 \\
(L_2^m,0,\ldots,0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_2 \\
(L_1^m+1,L_2^m,L_3^m,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_3 \\
(L_1^m,L_2^m+1,L_3^m,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)p}{\lambda+\mu}\sigma_3 \\
(L_1^m,L_2^m,L_3^m+1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)^2p}{\lambda+\mu}\sigma_3 \\
(L_1^m,L_2^m,L_3^m,1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)^3}{\lambda+\mu}\sigma_3 \\
(L_2^m,L_3^m,0,\ldots,0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_3 \\
\vdots & \vdots \\
(L_1^m+1,L_2^m,\ldots,L_k^m,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_k \\
\vdots & \vdots \\
(L_1^m,L_2^m,\ldots,L_k^m+1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)^{k-1}p}{\lambda+\mu}\sigma_k \\
(L_1^m,L_2^m,\ldots,L_k^m,1,0,\ldots,0) & \text{w.p.} \quad \frac{\lambda(1-p)^k}{\lambda+\mu}\sigma_k \\
(L_2^m,L_3^m,\ldots,L_k^m,0,\ldots,0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_k \\
\vdots & \vdots \\
\left(L_1^m+1,L_2^m,\ldots,L_{N-1}^m,0\right) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_{N-1} \\
\vdots & \vdots \\
\left(L_1^m,L_2^m,\ldots,L_{N-1}^m+1,0\right) & \text{w.p.} \quad \frac{\lambda(1-p)^{N-2}p}{\lambda+\mu}\sigma_{N-1} \\
\left(L_1^m,L_2^m,\ldots,L_{N-1}^m,1\right) & \text{w.p.} \quad \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}\sigma_{N-1} \\
\left(L_2^m,L_2^m,\ldots,L_{N-1}^m,0,0\right) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_{N-1} \\
(L_1^m+1,L_2^m,\ldots,L_N^m) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_N \\
\vdots & \vdots \\
(L_1^m,L_2^m,\ldots,L_N^m+1) & \text{w.p.} \quad \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}\sigma_N \\
(L_2^m,L_2^m,\ldots,L_N^m,0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_N
\end{cases} \tag{2.8}
$$

Explanation: When the system is empty (with probability σ_0), the next Poissonian event will be an arrival, so that the first group will contain a single customer. Next, assume that only one group is in the system (with probability σ_1). Then, if the next event is an arrival (with probability $\frac{\lambda}{\lambda+\mu}$), then the new customer will join the single group with probability p or will create a new (second) group (with probability $(1-p)$). The third option is that this group completes its service before an arrival occurs, so the system becomes empty. This occurs with probability $\frac{\mu}{\lambda+\mu}$. In this manner, we consider all possible group vectors of customers and all possible events. So, if k groups are present (with probability σ_k), an arriving customer may join each of these groups, or create a new group (with the corresponding probabilities). In all cases, when the system is not empty, a service completion before an arrival causes each group to move one step forward towards the server.

2.2 First moment of L_k

We now use equation (2.8) in order to derive $\mathbb{E}[L_k]$, for $k = 1, 2, \ldots, N$.

For $k = 1$ we have,

$$\mathbb{E}\left[L_1^{m+1}\right] = \sigma_0 + \frac{\lambda p}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j + \mathbb{E}\left[L_1^m\right]\frac{\lambda}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j + \mathbb{E}\left[L_2^m\right]\frac{\mu}{\lambda+\mu}\sum_{j=2}^{N}\sigma_j. \tag{2.9}$$

When $k = 2$, we get

$$\mathbb{E}\left[L_2^{m+1}\right] = \frac{\lambda(1-p)}{\lambda+\mu}\sigma_1 + \frac{\lambda(1-p)p}{\lambda+\mu}\sum_{j=2}^{N}\sigma_j + \mathbb{E}\left[L_2^m\right]\frac{\lambda}{\lambda+\mu}\sum_{j=2}^{N}\sigma_j + \mathbb{E}\left[L_3^m\right]\frac{\mu}{\lambda+\mu}\sum_{j=3}^{N}\sigma_j. \tag{2.10}$$

In general, for $k = 2, 3, \ldots, N-1$ we obtain

$$\mathbb{E}\left[L_k^{m+1}\right] = \frac{\lambda(1-p)^{k-1}}{\lambda+\mu}\sigma_{k-1} + \frac{\lambda(1-p)^{k-1}p}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j + \mathbb{E}\left[L_k^m\right]\frac{\lambda}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j + \mathbb{E}\left[L_{k+1}^m\right]\frac{\mu}{\lambda+\mu}\sum_{j=k+1}^{N}\sigma_j. \tag{2.11}$$

Finally, for $k = N$ we get

$$\mathbb{E}\left[L_N^{m+1}\right] = \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}(\sigma_{N-1} + \sigma_N) + \mathbb{E}\left[L_N^m\right]\frac{\lambda}{\lambda+\mu}\sigma_N. \tag{2.12}$$

Define:

$$\alpha_k = \frac{\lambda}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j, \qquad k = 1, 2, \ldots, N,$$

$$\beta_k = \frac{\mu}{\lambda+\mu}\sum_{j=k+1}^{N}\sigma_j, \quad k = 1, 2, \ldots, N-1,$$

$$q_1 = \sigma_0 + \frac{\lambda p}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j,$$

$$q_k = \frac{\lambda(1-p)^{k-1}}{\lambda+\mu}\sigma_{k-1} + \frac{\lambda(1-p)^{k-1}p}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j, \quad k = 2, 3, \ldots, N-1,$$

$$q_N = \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}(\sigma_{N-1} + \sigma_N).$$

In addition, we now observe the system in steady state where $L_k^m \to L_k$ when $m \to \infty$. We then have $E[L_k^m] = E[L_k]$. Therefore, equations (2.9), (2.11) and (2.12) can be written as

$$\mathbb{E}[L_k] = q_k + \alpha_k \mathbb{E}[L_k] + \beta_k \mathbb{E}[L_{k+1}], \quad k = 1, 2, \ldots, N-1, \tag{2.13}$$

$$\mathbb{E}[L_N] = q_N + \alpha_N \mathbb{E}[L_N], \tag{2.14}$$

or equivalently,

$$\mathbb{E}[L_k] = \frac{\beta_k}{1-\alpha_k}\mathbb{E}[L_{k+1}] + \frac{q_k}{1-\alpha_k}, \quad k = 1, 2, \ldots, N-1, \tag{2.15}$$

$$\mathbb{E}[L_N] = \frac{q_N}{1-\alpha_N}. \tag{2.16}$$

Iterating equation (2.15) and using (2.16) lead to the following closed-form expression,

$$\mathbb{E}[L_k] = \frac{q_N}{1-\alpha_N}\prod_{j=0}^{N-k-1}\frac{\beta_{k+j}}{1-\alpha_{k+j}} + \sum_{j=0}^{N-k-1}\frac{q_{k+j}}{1-\alpha_{k+j}}\prod_{i=0}^{j-1}\frac{\beta_{k+i}}{1-\alpha_{k+i}}, \quad k = 1, 2, \ldots, N, \tag{2.17}$$

where we define $\prod_{j=0}^{-1}(\cdot) \triangleq 1$, and $\sum_{j=0}^{-1}(\cdot) \triangleq 0$.

3 The Busy Period

Let Θ_n $(n = 1, 2, \ldots, N)$ denote the time from the first moment when there are n groups in the system until the first moment thereafter when no groups are present. Then the busy period, i.e. the period of time during which the server is working continuously, is Θ_1. In this section we derive the Laplace Stieltjes Transform (LST) of Θ_n as well as a closed-form expression for $\mathbb{E}[\Theta_n]$.

3.1 The LST of Θ_n

Let $\widetilde{\Theta}_n(s)$ denote the LST of Θ_n, and let $Exp(\lambda)$ denote an exponential distribution with parameter λ. We derive $\{\Theta_n\}_{n=0}^{N}$ by solving a set of N linear equations, as follows. First, we have that

$$\Theta_1 \stackrel{d}{=} Exp(\lambda(1-p)+\mu) + \begin{cases} 0 & w.p. \ \frac{\mu}{\lambda(1-p)+\mu} \\ \Theta_2 & w.p. \ \frac{\lambda(1-p)}{\lambda(1-p)+\mu} \end{cases},$$

which yields

$$\widetilde{\Theta}_1(s) = \frac{\mu}{\lambda(1-p)+\mu+s} + \frac{\lambda(1-p)}{\lambda(1-p)+\mu+s}\widetilde{\Theta}_2(s). \tag{3.1}$$

Second, for $n = 2, 3, \ldots, N-1$,

$$\Theta_n \stackrel{d}{=} Exp(\lambda(1-p)^n+\mu) + \begin{cases} \Theta_{n-1} & w.p. \ \frac{\mu}{\lambda(1-p)^n+\mu} \\ \Theta_{n+1} & w.p. \ \frac{\lambda(1-p)^n}{\lambda(1-p)^n+\mu} \end{cases}, \tag{3.2}$$

which leads to

$$\widetilde{\Theta}_n(s) = \frac{\mu}{\lambda(1-p)^n+\mu+s}\widetilde{\Theta}_{n-1}(s) + \frac{\lambda(1-p)^n}{\lambda(1-p)^n+\mu+s}\widetilde{\Theta}_{n+1}(s). \tag{3.3}$$

Last, for $n = N$,

$$\Theta_N \stackrel{d}{=} Exp(\mu) + \Theta_{N-1}. \tag{3.4}$$

That is,

$$\widetilde{\Theta}_N(s) = \frac{\mu}{\mu+s}\widetilde{\Theta}_{N-1}(s). \tag{3.5}$$

Equations (3.1) - (3.5) comprise a set of N linear equations which can be written in the following matrix form:

$$A(s) \cdot \vec{\Theta}(s) = \vec{b}, \tag{3.6}$$

where

$$A(s) = \begin{pmatrix} \lambda(1-p)+\mu+s & -\lambda(1-p) & 0 & \cdots & \cdots & \cdots & 0 \\ -\mu & \lambda(1-p)^2+\mu+s & -\lambda(1-p)^2 & 0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\mu & \lambda(1-p)^{N-1}+\mu+s & -\lambda(1-p)^{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & -\mu & \mu+s \end{pmatrix},$$

$\vec{\Theta}(s) = \left(\widetilde{\Theta}_1(s), \widetilde{\Theta}_2(s), \ldots, \widetilde{\Theta}_N(s)\right)^T$ is a column vector of the desired LST's, and $\vec{b} = (\mu, 0, 0, \ldots, 0)^T$. The solution for (3.6) is given by $\vec{\Theta}(s) = (A(s))^{-1}\vec{b}$, and since $\vec{b}$ is all zeros except from its first coordinate (which equals μ), we have that $\vec{\Theta}(s)$ equals the first column of $(A(s))^{-1}$ multiplied by μ. Note that $A(s)$ is a tridiagonal matrix. There is an increasing interest in tridiagonal matrices in many different theoretical fields, in which inversions of these matrices are necessary. Examples for recent works that present explicit formula for the elements of the inverse of a general tridiagonal matrix are Mallik [7] and Kiliç [5], and references there. Thus, once the inverse of $A(s)$ is calculated, the vector $\vec{\Theta}(s)$ is fully obtained.

An alternative approach to derive the LST of the busy period, $\widetilde{\Theta}_1(s)$, is by using continued fractions, which often provide good representations for transcendental functions. A finite continued fraction is denoted by

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{\cdots + \frac{a_n}{b_n}}}}$$

or equivalently by

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n},$$

where the a_n's and the b_n's are real or complex numbers.

From equation (3.1) we obtain

$$\widetilde{\Theta}_1(s) = \frac{\mu}{\lambda(1-p) + \mu + s - \lambda(1-p)\frac{\widetilde{\Theta}_2(s)}{\widetilde{\Theta}_1(s)}}. \tag{3.7}$$

Equation (3.3) can be written as

$$\frac{\widetilde{\Theta}_n(s)}{\widetilde{\Theta}_{n-1}(s)} = \frac{\mu}{\lambda(1-p)^n + \mu + s - \lambda(1-p)^n \frac{\widetilde{\Theta}_{n+1}(s)}{\widetilde{\Theta}_n(s)}}, \quad n = 2, 3, \ldots, N-1, \tag{3.8}$$

and equation (3.5) gives

$$\frac{\widetilde{\Theta}_N(s)}{\widetilde{\Theta}_{N-1}(s)} = \frac{\mu}{\mu + s}. \tag{3.9}$$

Iterating (3.8) and using (3.9) results in

$$\frac{\widetilde{\Theta}_n(s)}{\widetilde{\Theta}_{n-1}(s)} = \frac{\mu}{\lambda(1-p)^n + \mu + s -} \frac{\lambda(1-p)^n \mu}{\lambda(1-p)^{n+1} + \mu + s -} \cdots \frac{\lambda(1-p)^{N-2}\mu}{\lambda(1-p)^{N-1} + \mu + s -} \frac{\lambda(1-p)^{N-1}\mu}{\mu + s}. \tag{3.10}$$

By substituting equation (3.10) for $n = 2$ in equation (3.7) we get a continued fraction representation for $\widetilde{\Theta}_1(s)$,

$$\widetilde{\Theta}_1(s) = \frac{\mu}{\lambda(1-p) + \mu + s -} \frac{\lambda(1-p)\mu}{\lambda(1-p)^2 + \mu + s -} \cdots \frac{\lambda(1-p)^{N-2}\mu}{\lambda(1-p)^{N-1} + \mu + s -} \frac{\lambda(1-p)^{N-1}\mu}{\mu + s}. \tag{3.11}$$

Note that the approach to derive $\widetilde{\Theta}_1(s)$ via continued fractions is in fact much related to algorithms which are used to calculate the inverse of a tridiagonal matrix, as shown, for example, in [5].

3.2 Calculation of $\mathbb{E}[\Theta_n]$

As mentioned, Θ_1 is the period of time during which the server is working continuously. Since the idle time of the server is $\text{Exp}(\lambda)$, we get

$$\frac{\mathbb{E}[\Theta_1]}{\frac{1}{\lambda} + \mathbb{E}[\Theta_1]} = 1 - \pi_0,$$

resulting in

$$\mathbb{E}[\Theta_1] = \frac{1-\pi_0}{\lambda\pi_0}. \tag{3.12}$$

We now calculate $\mathbb{E}[\Theta_n]$ for all $n = 1, 2, \ldots, N$. From equation (3.4) we get

$$\mathbb{E}[\Theta_{N-1}] = \mathbb{E}[\Theta_N] - \frac{1}{\mu}. \tag{3.13}$$

Calculating the expectation in both sides of equation (3.2) gives

$$\mathbb{E}[\Theta_n] = \frac{1}{\lambda(1-p)^n + \mu} + \frac{\lambda(1-p)^n}{\lambda(1-p)^n + \mu}\mathbb{E}[\Theta_{n+1}] + \frac{\mu}{\lambda(1-p)^n + \mu}\mathbb{E}[\Theta_{n-1}],$$

or equivalently

$$(\lambda(1-p)^n + \mu)\mathbb{E}[\Theta_n] = 1 + \lambda(1-p)^n\mathbb{E}[\Theta_{n+1}] + \mu\mathbb{E}[\Theta_{n-1}]. \tag{3.14}$$

Substituting $n = N-1$ in equation (3.14) leads to

$$\mathbb{E}[\Theta_{N-1}] = \frac{1}{\lambda(1-p)^{N-1} + \mu} + \frac{\lambda(1-p)^{N-1}}{\lambda(1-p)^{N-1} + \mu}\mathbb{E}[\Theta_N] + \frac{\mu}{\lambda(1-p)^{N-1} + \mu}\mathbb{E}[\Theta_{N-2}].$$

Using the expression for $\mathbb{E}[\Theta_{N-1}]$ given in (3.13) and rearranging terms give

$$\mathbb{E}[\Theta_{N-2}] = \mathbb{E}[\Theta_N] - \frac{\lambda(1-p)^{N-1}}{\mu^2} - \frac{2}{\mu} \tag{3.15}$$

Continuing further, substituting $n = N-2$ in equation (3.14) gives

$$\mathbb{E}[\Theta_{N-2}] = \frac{1}{\lambda(1-p)^{N-2} + \mu} + \frac{\lambda(1-p)^{N-2}}{\lambda(1-p)^{N-2} + \mu}\mathbb{E}[\Theta_{N-1}] + \frac{\mu}{\lambda(1-p)^{N-2} + \mu}\mathbb{E}[\Theta_{N-3}].$$

Using the expression for $\mathbb{E}[\Theta_{N-2}]$ given in (3.15) and for $\mathbb{E}[\Theta_{N-1}]$ given in (3.13), and rearranging terms give

$$\mathbb{E}[\Theta_{N-3}] = \mathbb{E}[\Theta_N] - \frac{\lambda^2}{\mu^3}(1-p)^{N-1}(1-p)^{N-2} - \frac{\lambda}{\mu^2}\Big((1-p)^{N-1} + (1-p)^{N-2}\Big) - \frac{3}{\mu}. \tag{3.16}$$

In the same manner, we get

$$\begin{aligned}\mathbb{E}[\Theta_{N-4}] = \mathbb{E}[\Theta_N] &- \frac{\lambda^3}{\mu^4}(1-p)^{N-1}(1-p)^{N-2}(1-p)^{N-3} \\ &- \frac{\lambda^2}{\mu^3}\Big((1-p)^{N-1}(1-p)^{N-2} + (1-p)^{N-2}(1-p)^{N-3}\Big) \\ &- \frac{\lambda}{\mu^2}\Big((1-p)^{N-1} + (1-p)^{N-2} + (1-p)^{N-3}\Big) - \frac{4}{\mu}. \end{aligned} \tag{3.17}$$

Continuing, the structure of equations (3.13), (3.15)-(3.17), leads to the following general solution,

$$\mathbb{E}[\Theta_{N-j}] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{j-1} \frac{\lambda^i}{\mu^{i+1}} \sum_{k=1}^{j-i} (1-p)^{Ni-\frac{i(i+2k-1)}{2}} - \frac{j}{\mu}, \quad j = 0,1,\dots,N-1,$$

and by setting $n = N - j$ and rewriting the power of the term $(1-p)$ we get

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{N-n-1} \frac{\lambda^i}{\mu^{i+1}} \sum_{k=1}^{N-n-i} (1-p)^{\frac{i(2N-2k-i+1)}{2}} - \frac{N-n}{\mu}, \quad n = 1,2,\dots,N, \tag{3.18}$$

where we define $\sum\limits_{i=1}^{-1}(\cdot) = \sum\limits_{i=1}^{0}(\cdot) = 0$.

Now, the second summation appearing in equation (3.18) is

$$\begin{aligned}\sum_{k=1}^{N-n-i} (1-p)^{\frac{i(2N-2k-i+1)}{2}} &= (1-p)^{\frac{i(2N-i+1)}{2}} \sum_{k=1}^{N-n-i} (1-p)^{-ik} \\ &= (1-p)^{\frac{i(2N-i+1)}{2}} \frac{(1-p)^{i(i-N+n)}-1}{1-(1-p)^i} = \frac{(1-p)^{\frac{i(2n+i+1)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}}{1-(1-p)^i},\end{aligned}$$

so that equation (3.18) becomes

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{N-n-1} \frac{\lambda^i \Big((1-p)^{\frac{i(2n+i+1)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}\Big)}{\mu^{i+1}(1-(1-p)^i)} - \frac{N-n}{\mu}, \quad n = 1,2,\dots,N. \tag{3.19}$$

The following lemma shows the validity of equation (3.19).

Lemma 1. *For $n = 1,2,\dots,N$, equation (3.19) satisfies the recursion in (3.14).*

Proof. Let us substitute (3.19) in (3.14). We then need to check whether the following equality holds:

$$
\begin{aligned}
&(\lambda(1-p)^n+\mu)\left(\mathbb{E}[\Theta_N]-\sum_{i=1}^{N-n-1}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+1)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}-\frac{N-n}{\mu}\right)\\
&=1+\lambda(1-p)^n\left(\mathbb{E}[\Theta_N]-\sum_{i=1}^{N-n-2}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+3)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}-\frac{N-n-1}{\mu}\right)\\
&+\mu\left(\mathbb{E}[\Theta_N]-\sum_{i=1}^{N-n}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i-1)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}-\frac{N-n+1}{\mu}\right)
\end{aligned}
$$

After moving all terms to the left-hand side and dropping some terms we get

$$
\begin{aligned}
&\frac{\lambda(1-p)^n}{\mu}+\lambda(1-p)^n\left(\sum_{i=1}^{N-n-1}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+1)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}-\sum_{i=1}^{N-n-2}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+3)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}\right)\\
&+\mu\left(\sum_{i=1}^{N-n-1}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+1)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}-\sum_{i=1}^{N-n}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i-1)}{2}}-(1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)}\right)\\
&=\frac{\lambda(1-p)^n}{\mu}+\lambda(1-p)^n\left(\sum_{i=1}^{N-n-2}\frac{\lambda^i}{\mu^{i+1}(1-(1-p)^i)}\left((1-p)^{\frac{i(2n+i+1)}{2}}-(1-p)^{\frac{i(2n+i+3)}{2}}\right)\right)\\
&+\frac{\lambda^{N-n}(1-p)^n}{\mu^{N-n}(1-(1-p)^{N-n-1})}\left((1-p)^{\frac{(N-n-1)(N+n)}{2}}-(1-p)^{\frac{(N-n-1)(N+n+2)}{2}}\right)\\
&+\mu\left(\sum_{i=1}^{N-n-1}\frac{\lambda^i}{\mu^{i+1}(1-(1-p)^i)}\left((1-p)^{\frac{i(2n+i+1)}{2}}-(1-p)^{\frac{i(2n+i-1)}{2}}\right)\right)\\
&-\frac{\lambda^{N-n}}{\mu^{N-n}(1-(1-p)^{N-n})}\left((1-p)^{\frac{(N-n)(N+n-1)}{2}}-(1-p)^{\frac{(N-n)(N+n+1)}{2}}\right). \qquad (3.20)
\end{aligned}
$$

Continuing the algebra, equation (3.20) can be rewritten as

$$
\begin{aligned}
&\frac{\lambda(1-p)^n}{\mu}+\lambda(1-p)^n\sum_{i=1}^{N-n-2}\frac{\lambda^i}{\mu^{i+1}(1-(1-p)^i)}(1-p)^{\frac{i(2n+i+1)}{2}}(1-(1-p)^i)\\
&+\frac{\lambda^{N-n}(1-p)^n}{\mu^{N-n}(1-(1-p)^{N-n-1})}(1-p)^{\frac{(N-n-1)(N+n)}{2}}(1-(1-p)^{N-n-1})\\
&-\sum_{i=1}^{N-n-1}\frac{\lambda^i}{\mu^i(1-(1-p)^i)}(1-p)^{\frac{i(2n+i-1)}{2}}(1-(1-p)^i)\\
&-\frac{\lambda^{N-n}}{\mu^{N-n}(1-(1-p)^{N-n})}(1-p)^{\frac{(N-n)(N+n-1)}{2}}(1-(1-p)^{N-n}), \qquad (3.21)
\end{aligned}
$$

which leads to

$$\frac{\lambda(1-p)^n}{\mu} - \frac{\lambda^{N-n}}{\mu^{N-n}}(1-p)^{\frac{(N-n)(N+n-1)}{2}} + \sum_{i=1}^{N-n-1} \frac{\lambda^{i+1}}{\mu^{i+1}}(1-p)^{n+\frac{i(2n+i+1)}{2}} - \sum_{i=1}^{N-n-1} \frac{\lambda^i}{\mu^i}(1-p)^{\frac{i(2n+i-1)}{2}}.$$

Now, noting that

$$\sum_{i=1}^{N-n-1} \frac{\lambda^{i+1}}{\mu^{i+1}}(1-p)^{n+\frac{i(2n+i+1)}{2}} - \sum_{i=1}^{N-n-1} \frac{\lambda^i}{\mu^i}(1-p)^{\frac{i(2n+i-1)}{2}} = -\frac{\lambda(1-p)^n}{\mu} + \frac{\lambda^{N-n}}{\mu^{N-n}}(1-p)^{\frac{(N-n)(N+n-1)}{2}},$$

completes the proof. □

Finally, substituting $n = 1$ in equation (3.19), and using the expression for $\mathbb{E}[\Theta_1]$ given in equation (3.12), yield an expression for $\mathbb{E}[\Theta_N]$ in terms of π_0,

$$\mathbb{E}[\Theta_N] = \frac{1-\pi_0}{\lambda\pi_0} + \sum_{i=1}^{N-2} \frac{\lambda^i\left((1-p)^{\frac{i(i+3)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)} + \frac{N-1}{\mu}. \tag{3.22}$$

Thus, in view of (3.19), $\mathbb{E}[\Theta_n]$ is completely determined for all $1 \le n \le N$.

4 Numerical Results

In this section some numerical results are presented, for the cases $N = 5$ and $N = 10$.

In Table 1 ($N = 5$) we calculate the first moment of L_k and of $D^{(k)}$, $k = 1, 2, \ldots, 5$, as well as the first moment of Θ_n, $n = 1, 2, \ldots, 5$. Different values of λ and p are considered, and $\mu = 1$ in all calculations. The results show that $\mathbb{E}[L_1]$, the mean size of the group standing in the first position (the one being served) increases with p, since for larger values of p, most customers concentrate in the first group. The size of the group in the second position behaves different for various values of p. Meaning that, when p increases from 0.01 to 0.2, $\mathbb{E}[L_2]$ slightly increases, while when p grows from 0.2 to 0.6, $\mathbb{E}[L_2]$ significantly decreases. Furthermore, $\mathbb{E}[L_3]$, $\mathbb{E}[L_4]$ and $\mathbb{E}[L_5]$ decrease as p increases. We also observe that $\mathbb{E}\left[D^{(k)}\right]$ is larger than $\mathbb{E}[L_k]$. This follows since $\mathbb{E}\left[D^{(k)}\right]$ is calculated after a service completion, when assuming that the k-th group exists. So, $\mathbb{E}\left[D^{(k)}\right]$ contains all the customers that join this group during a single service period, while $\mathbb{E}[L_k]$ is calculated right after a Poissonian event (arrival or service completion). In addition, Table 1 shows that for all n, $\mathbb{E}[\Theta_n]$ drops radically with the enlargement of p.

Table 2 presents results for $\mathbb{E}[L_k]$ when $N = 10$. As expected, the values of $\mathbb{E}[L_k]$ decrease as the group's index k grows. However, when $p = 0.01$, the mean size of the last group is slightly greater than the mean sizes of the groups in front of it. Also, for $p = 0.01$, the differences between the values of $\mathbb{E}[L_k]$ are not as great as in other values of p.

In Table 3 the values for $\mathbb{E}\left[D^{(k)}\right]$ are presented, when $N = 10$. When $\frac{\lambda}{\mu}$ is large, π_0 is very small, and therefore, from equation (1.2), $\mathbb{E}\left[D^{(1)}\right]$ is very close to $\frac{\lambda}{\mu}$.

Table 4 exhibits the values of $\mathbb{E}[\Theta_n]$ when $N = 10$. Since large values for λ are considered, it turns out that when p is small, $\mathbb{E}[\Theta_n]$ is extremely large. However, when increasing the value of p from 0.2 to 0.6, $\mathbb{E}[\Theta_n]$ drops, so that the influence of p on $\mathbb{E}[\Theta_n]$ is clearly shown.

We indicate that a regular $M/M/1$ queue with $\mu = 1$ and $\lambda \geq 1$, would collapse.

Table 1: Numerical results for $N = 5$, $\mu = 1$.

	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
Value of p	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[L_1]$	0.3224	0.5317	0.7901	1.6689	2.6655	3.1171	6.6855	9.5628	11.5512
$\mathbb{E}[L_2]$	0.2165	0.2505	0.1481	1.6221	1.6941	0.6519	6.5358	6.5652	2.8826
$\mathbb{E}[L_3]$	0.1583	0.1174	0.0182	1.5927	0.9955	0.1255	6.3916	4.1884	0.4367
$\mathbb{E}[L_4]$	0.1283	0.0505	0.0011	1.6486	0.5819	0.0213	6.3008	2.4301	0.0687
$\mathbb{E}[L_5]$	0.1826	0.0242	$2.73 \cdot 10^{-5}$	2.2565	0.5505	0.0023	7.0287	1.7942	0.0129
$\mathbb{E}[D^{(1)}]$	1.2081	1.3729	1.6811	5.0014	5.0085	5.1931	15.0000	15.0000	15.0247
$\mathbb{E}[D^{(2)}]$	1.2501	1.2758	1.2543	4.9569	4.0343	2.4787	14.8503	12.0019	6.1518
$\mathbb{E}[D^{(3)}]$	1.3246	1.2208	1.0984	4.9348	3.3136	1.5348	14.7059	9.6179	2.8294
$\mathbb{E}[D^{(4)}]$	1.4803	1.2244	1.0390	5.0238	2.8881	1.2065	14.6194	7.8199	1.6825
$\mathbb{E}[D^{(5)}]$	1.9606	1.4096	1.0256	5.8030	3.0480	1.1280	15.4089	7.1440	1.3840
$\mathbb{E}[\Theta_1]$	4.8062	2.6815	1.4682	171.131	117.677	5.1775	49196.0	6448.75	40.5324
$\mathbb{E}[\Theta_2]$	8.6508	4.7834	2.6387	856.996	146.846	7.2663	52508.8	6986.07	47.1212
$\mathbb{E}[\Theta_3]$	11.5531	6.5051	3.7043	886.149	155.649	8.6273	52734.1	7041.93	49.4498
$\mathbb{E}[\Theta_4]$	13.5137	7.9147	4.7299	891.952	158.697	9.7553	52749.5	7049.08	50.8338
$\mathbb{E}[\Theta_5]$	14.5137	8.9147	5.7299	892.952	159.697	10.7553	52750.5	7050.08	51.8338

Table 2: Numerical results for $\mathbb{E}[L_k]$, where $N = 10$, $\mu = 1$.

	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
Value of p	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[L_1]$	0.2644	0.5308	0.7902	1.7420	3.2775	3.1175	6.8794	12.1098	11.5783
$\mathbb{E}[L_2]$	0.1908	0.2508	0.1481	1.6920	2.2837	0.6521	6.7294	9.1099	2.9039
$\mathbb{E}[L_3]$	0.1446	0.1179	0.0182	1.6425	1.5083	0.1255	6.5809	6.7099	0.4432
$\mathbb{E}[L_4]$	0.1135	0.0509	0.0010	1.5936	0.9311	0.0214	6.4339	4.7905	0.0699
$\mathbb{E}[L_5]$	0.0914	0.0191	$2.71 \cdot 10^{-5}$	1.5423	0.5381	0.0022	6.2883	3.2585	0.0118
$\mathbb{E}[L_6]$	0.0748	0.0059	$2.74 \cdot 10^{-7}$	1.4983	0.2992	$1.07 \cdot 10^{-4}$	6.1443	2.0494	0.0014
$\mathbb{E}[L_7]$	0.0621	0.0015	$1.12 \cdot 10^{-9}$	1.4555	0.1656	$2.16 \cdot 10^{-6}$	6.0019	1.1388	$7.84 \cdot 10^{-5}$
$\mathbb{E}[L_8]$	0.0519	$3.14 \cdot 10^{-4}$	$1.83 \cdot 10^{-12}$	1.4306	0.0916	$1.76 \cdot 10^{-8}$	5.8649	0.5415	$1.88 \cdot 10^{-6}$
$\mathbb{E}[L_9]$	0.0448	$5.21 \cdot 10^{-5}$	$1.2 \cdot 10^{-15}$	1.4899	0.0493	$5.76 \cdot 10^{-11}$	5.7839	0.2467	$1.83 \cdot 10^{-8}$
$\mathbb{E}[L_{10}]$	0.0696	$7.65 \cdot 10^{-6}$	$3.14 \cdot 10^{-19}$	2.0807	0.0357	$7.54 \cdot 10^{-14}$	6.5099	0.2244	$7.20 \cdot 10^{-11}$

Table 3: Numerical results for $\mathbb{E}[D^{(k)}]$, where $N = 10$, $\mu = 1$.

	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
Value of p	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[D^{(1)}]$	1.1168	1.3665	1.6811	5.0000	5.0014	5.1930	15.0000	15.0000	15.0242
$\mathbb{E}[D^{(2)}]$	1.1209	1.2629	1.2543	4.9500	4.0054	2.4784	14.8500	12.0000	6.1485
$\mathbb{E}[D^{(3)}]$	1.1277	1.1916	1.0984	4.9005	3.2175	1.5336	14.7015	9.6000	2.8186
$\mathbb{E}[D^{(4)}]$	1.1383	1.1418	1.0388	4.8516	2.6057	1.2013	14.5545	7.6803	1.6512
$\mathbb{E}[D^{(5)}]$	1.1546	1.1064	1.0154	4.8033	2.1461	1.0783	14.4089	6.1458	1.2436
$\mathbb{E}[D^{(6)}]$	1.1799	1.0809	1.0062	4.7566	1.8166	1.0309	14.2649	4.9242	1.0937
$\mathbb{E}[D^{(7)}]$	1.2209	1.0624	1.0025	4.7154	1.5949	1.0123	14.1226	3.9679	1.0372
$\mathbb{E}[D^{(8)}]$	1.2936	1.0498	1.0009	4.6978	1.4649	1.0049	13.9859	3.2622	1.0148
$\mathbb{E}[D^{(9)}]$	1.4453	1.0494	1.0004	4.7933	1.4373	1.0019	13.9092	2.8484	1.0059
$\mathbb{E}[D^{(10)}]$	1.9135	1.1342	1.0002	5.5676	1.6711	1.0013	14.7028	3.0133	1.0039

Table 4: Numerical results for $\mathbb{E}[\Theta_n]$, where $N = 10$, $\mu = 1$.

	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
Value of p	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[\Theta_1]$	8.563	2.728	1.468	1589601.4	734.695	5.181	$2.63808 \cdot 10^{10}$	2974866.9	41.399
$\mathbb{E}[\Theta_2]$	16.203	4.888	2.638	1910732.8	918.118	7.271	$2.81573 \cdot 10^{10}$	3222772.4	48.132
$\mathbb{E}[\Theta_3]$	22.977	6.701	3.704	1976262.9	975.126	8.634	$2.82781 \cdot 10^{10}$	3248595.8	50.521
$\mathbb{E}[\Theta_4]$	28.928	8.289	4.730	1989769.9	997.004	9.769	$2.82864 \cdot 10^{10}$	3251958.1	51.968
$\mathbb{E}[\Theta_5]$	34.083	9.724	5.741	1992581.9	1007.198	10.822	$2.8287016 \cdot 10^{10}$	3252505.2	53.131
$\mathbb{E}[\Theta_6]$	38.451	11.052	6.744	1993173.5	1012.809	11.842	$2.828705 \cdot 10^{10}$	3252616.3	54.194
$\mathbb{E}[\Theta_7]$	42.029	12.302	7.746	1993298.5	1016.328	12.851	$2.828706 \cdot 10^{10}$	3252644.3	55.219
$\mathbb{E}[\Theta_8]$	44.795	13.492	8.747	1993325.2	1018.729	13.854	$2.828706 \cdot 10^{10}$	3252652.9	56.229
$\mathbb{E}[\Theta_9]$	46.708	14.626	9.747	1993330.7	1020.401	14.855	$2.828706 \cdot 10^{10}$	3252655.9	57.233
$\mathbb{E}[\Theta_{10}]$	47.708	15.626	10.747	1993331.7	1021.401	15.855	$2.828706 \cdot 10^{10}$	3252656.9	58.233

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