Controlling the GI/M/1 queue by conditional acceptance of customers

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Received June 1979
Revised November 1979

In this paper we consider the problem of controlling the arrival of customers into a GI/M/1 service station. It is known that when the decisions controlling the system are made only at arrival epochs, the optimal acceptance strategy is of a control-limit type, i.e., an arrival is accepted if and only if fewer than n customers are present in the system. The question is whether exercising conditional acceptance can further increase the expected long run average profit of a firm which operates the system. To reveal the relevance of conditional acceptance we consider an extension of the control-limit rule in which the nth customer is conditionally admitted to the queue. This customer may later be rejected if neither service completion nor arrival has occurred within a given time period since the last arrival epoch. We model the system as a semi-Markov decision process, and develop conditions under which such a policy is preferable to the simple control-limit rule.

1. Introduction

Consider an agency that operates a GI/M/1 service station in which the arrival process can be controlled by accepting or rejecting arriving customers. A fixed reward is earned when a customer completes his service and linear holding costs are incurred for the customers waiting in line.

It is well known [5–9] that, when decisions are restricted to arrival instants, the profit-maximizing control policy (known also as 'Social Optimization') is a control limit rule. That is, the agency admits a customer into the queue if and only if fewer than n customers are present in the system.

Consider now an (n, t)-policy under which the nth customer in the queue is *conditionally* accepted to the system. If t units of time elapse without any service completion this last customer is rejected from the queue. The question is whether (or under what conditions) this way of exercising conditional acceptance increases the expected average profit of the agency.

As a tangible example for possible implementation of the (n, t)-policy, consider a telephone congestion system. A customer who calls the station is immediately accepted when the line is idle. A limited number of at most n customers may be waiting to be served on a FIFO basis, whereas calls that find the line busy and all n waiting positions occupied are rejected, (i.e. lost). The implementation of the (n, t)-policy amounts to inspecting the system when the queue has been full for t time units since the last call, and rejecting the last customer.

The problems of optimal acceptance strategies in a queueing system have attracted considerable attention in the literature. Naor [3] was the first to show that in the M/M/1 queue, exercising narrow self-optimization by individual customers does not necessarily optimize public good. Yechiali [7,8] extended Naor's results to the GI/M/1 and GI/M/S queueing systems, and proved that for the infinite horizon, average reward criterion, optimal joining strategies are control-limit rules for both self and social optimization. Several authors then treated the problem under various assumptions and further broadened the results. Comprehensive bibliographies may be found in Stidham [5] in which the reward is random and the holding cost is convex, and in Yechiali [9] where a descriptive survey of the prevailing models is given. It is also worth mentioning the works of Teghem Jr. [6] and Doi [1]. Teghem considers an M/M/1 queue with a removable server and determines the optimal acceptance rules. Doi applies customers' optimization ideas to solve a problem of optimal traffic flow. The problem is solved by considering control-limit policies in the M/M/1 or M/G/1 queueing systems with many input sources.

Considering the conditional acceptance of customers, it might be argued that when the failure rate of the interarrival time distribution is increasing, and t is high enough, the firm would benefit from such a policy, since most probably the nth customer will

North-Holland Publishing Company
European Journal of Operational Research 7 (1981) 77–85

0377-2217/81/0000–0000/$02.50

simply be replaced by the next arrival, and the only effect would be to save some waiting costs in the time interval from t until the following arrival. The fallacy of this reasoning follows from the memoryless property of the service time distribution: no waiting time is actually saved since the distribution of the residual service time remains unaffected. Nevertheless, in this paper we develop conditions under which an (n, t)-policy is preferable to the simple control-limit rule.

The structure of the paper is as follows: In Section 2 we formulate the problem as a semi-Markov decision process. In Section 3 we compare the (n, t)-policy with the simple control limit rule and derive necessary and sufficient conditions for the former to be better than the latter. Special cases are studied in Section 4, whereas Section 5 concludes the paper with few remarks and a conjecture on the optimality of a generalized conditional acceptance rule.

2. The semi-Markov decision process

Consider a GI/M/1 single server station with inter-arrival time distribution $H(\cdot)$ possessing a finite mean $1/\lambda$, and exponentially distributed service times with mean $1/\mu$.

The cost-reward structure faced by the operating agency is composed of four elements:

(i) Upon service completion, the agency obtains a non-negative net fee of g monetary units.

(ii) Each customer residing in the system incurs waiting-time losses at a rate of $c \geqslant 0$ monetary units per unit time.

(iii) Rejecting a customer immediately upon arrival results in a fixed penalty of $l \geqslant 0$ monetary units.

(iv) Rejecting a customer that has been conditionally accepted costs $l_1 \geqslant l$.

We assume that there is no discrimination among customers, and that $g - (c/\mu) \geqslant -l$. Our model is closely related to that of Yechiali [7], which will serve as our main reference.

The GI/M/1 queueing process is usually embedded at instants of arrival to form a Markov chain with states (= number of customers in system) $\{0, 1, 2, 3, ...\}$. It follows from [7] that if the system is controlled at arrival instants, the profit-maximizing acceptance rule belongs to the class of control-limit policies under which a customer is admitted to the queue if and only if the state of the system is less than some number, n, called the control-limit. Under this control-limit policy, the state space is divided into two regions:

(i) states $0, 1, 2, ..., n-1$, where arriving customers are accepted; and

(ii) states $n, n+1, n+2, ...$, where arriving customers are rejected.

An extension of the simple control-limit policy is to allow the service agency to reject the nth customer in the queue if t units of time have elapsed since the last arrival epoch without any service completion. We call this strategy an (n, t)-policy. The implementation of the (n, t)-policy requires the following distinction among states:

(a) States encountered upon arrival:

(i) States $0, 1, 2, ..., n-2$, where customers are accepted unconditionally.

(ii) State $n-1$, where a customer conditionally joins the queue, but may be rejected later.

(iii) States $n, n+1, n+2, ...$, where an arrival is immediately rejected.

(b) State (n, t), which is observed when n customers are present in the system and t units of time have elapsed since the last arrival. Under the (n, t)-policy, whenever the system reaches state (n, t), the last customer is rejected.

In the sequel, we derive conditions under which an (n, t)-policy (with finite t) is preferable to the optimal simple control-limit rule, which we call an (n, ∞)-policy. For that purpose, we compare the (n, t)-policy with the (n, ∞)-policy. The comparison is based on a representation of the problem as a semi-Markov decision process (SMDP).

The probabilistic analysis of the underlying semi-Markov process may be performed by considering the $n+1$ states of the GI/M/1/n queueing process, [7], with the addition of state (n, t). A customer who, upon arrival, finds the system in one of the states $0, 1, 2, ..., n-1$, is admitted. An arrival who finds n customers ahead of him, has to balk. Whenever the process enters state (n, t), the firm faces the decision problem of whether to accept the last customer in the queue or to reject him. Thus, there is more than one possible action only at state (n, t). Although the above $(n+2)$ states suffice for the probabilistic analysis of the process, the associated cost–reward bookkeeping scheme to be described shortly requires that the definition of state n be refined into two distinguished states, depending on the history of the process.

The cost–reward bookkeeping is performed as follows. An arrival who finds the system in state i, $i = 0, 1, 2, ..., n-2$ joins unconditionally and the firm gains an expected net reward of $g - c(i+1)/\mu$. A customer who finds the system in state n is rejected and the firm incurs a penalty l. A customer C who finds the system in state $n-1$ is conditionally accepted. As long as the firm keeps the option of rejecting C, it continuously incurs waiting-time losses. When a decision to reject C is made, the agency suffers an additional penalty l_1. As soon as it is known with certainty that C will stay until his service completion, the reward g, as well as the expected future waiting time losses, to be caused by C, are registered. Observe that the bookkeeping procedure is based on a separate registration of the contribution of each individual customer to the overall profit. Since the total profit is the sum of the individual contributions, this procedure is legitimate.

We turn now to a detailed analysis of the SMDP. Suppose the process is in state i, and action a is taken. Similar to [4], we denote:

$P_{ij}(a)$ = transition probability to state j,
$\bar{\tau}_i(a) = E\,\tau_i(a)$ = expected sojourn time in state i,
$\bar{R}(i, a)$ = expected one-step reward.

When there is only one action possible in state i, we omit the dependence on the action a and write P_{ij}, $\bar{\tau}_i$ and $\bar{R}(i)$. We further define, for $k = 1, 2, 3, \ldots$, $a_k = \int_0^\infty e^{-\mu v}(\mu v)^k/k!\ dH(v)$, the probability of k service completions during an interarrival-time, and $a_k^t = \int_t^\infty e^{-\mu v}[\mu(v-t)]^k/k!\ dH(v)$, the joint probability that neither arrival nor service completion will have occurred by time t, and that k customers will have completed their service during the interarrival-time.

Also, let

$$r_k = \sum_{i=k+1}^{\infty} a_i = 1 - \sum_{i=0}^{k} a_i,$$

and

$$r_k^t = \sum_{i=k+1}^{\infty} a_i^t .$$

The parameters of the SMDP associated with the various states follow.

2.1. States $i = 0, 1, 2, \ldots, n-2$

The transition probabilities are those of the GI/M/1 queue, and are given by

$$P_{ij} = a_{i-j+1}, \quad \text{for } j = 1, 2, \ldots, i+1,$$

$$P_{i0} = r_i . \tag{1}$$

The other transition probabilities, including $P_{i,(n,t)}$, are zero. Since each sojourn time is an interarrival time, we have

$$\bar{\tau}_i = 1/\lambda. \tag{2}$$

Since an arrival is accepted unconditionally, we have

$$\bar{R}(i) = g - c(i+1)/\mu. \tag{3}$$

2.2. State $n-1$

The transition probabilities are:

$$P_{n-1,(n,t)} = \Pr\{\text{neither service completion nor arrival occur by time } t\} = e^{-\mu t}[1 - H(t)]. \tag{4}$$

$$P_{n-1,n} = \Pr\{\text{arrival at some instant } v \in [0, t] \text{ before any service completion}\} = \int_0^t e^{-\mu v}\, dH(v) = a_0 - a_0^t . \tag{5}$$

For $j = 1, 2, 3, \ldots, n-1$:

$$\begin{aligned} P_{n-1,j} &= \Pr\{\text{arrival at some instant } v \leqslant t \text{ and } n-j \text{ service completions by time } v\} \\ &\quad + \Pr\{\text{arrival at } v > t, \text{ first service completion at instant } x < t \text{ and } n-j-1 \text{ service completions during the remaining time interval } (x, v]\} \\ &= \int_0^t e^{-\mu v} \frac{(\mu v)^{n-j}}{(n-j)!}\, dH(v) \\ &\quad + \int_{v=t}^{\infty} \left[\int_{x=0}^{t} \mu e^{-\mu x} e^{-\mu(v-x)} \times \frac{[\mu(v-x)]^{n-j-1}}{(n-j-1)!}\, dx \right] dH(v) \\ &= a_{n-j} - a_{n-j}^t . \end{aligned} \tag{6}$$

Note that the expression for $P_{n-1,j}$ may be explained by the interpretation of a_{n-j} and a_{n-j}^t. By a similar argument, we have

$$P_{n-1,0} = r_{n-1} - r_{n-1}^t . \tag{7}$$

The expected sojourn time in state $n-1$ is the sum of three components:

(i) $\int_0^t v\,\mathrm{d}H(v)$, for the case of arrival before t,

(ii) $t\cdot[1-H(t)]\cdot \mathrm{e}^{-\mu t}$, when the transition is to state (n, t), and

(iii) $(1-\mathrm{e}^{-\mu t})\int_{v=t}^{\infty} v\,\mathrm{d}H(v)$, when service is completed before t, but arrival occurs after t.
Hence,

$$\bar{\tau}(n-1)=\int_0^t v\,\mathrm{d}H(v)+t[1-H(t)]\mathrm{e}^{-\mu t} + (1-\mathrm{e}^{-\mu t})\int_t^{\infty} v\,\mathrm{d}H(v) \tag{8}$$

For the calculation of $\bar{R}(n-1)$, we distinguish between four possibilities:

(i) Arrival at instant $v\leqslant t$, but no service completion by time v. In that case, the system moves to state n and the decision of whether to accept or reject is deferred. Hence, only waiting time losses are considered. The contribution of this case to $\bar{R}(n-1)$ is

$$-c\int_0^t v\,\mathrm{e}^{-\mu v}\,\mathrm{d}H(v).$$

(ii) Neither service completion nor arrival occur by time t. The transition is to state (n, t) and the cost is ct. Thus, the contribution to $\bar{R}(n-1)$ is

$$-ctP_{n-1,(n,t)}=-cte^{-\mu t}[1-H(t)].$$

(iii) Arrival at instant $v\leqslant t$, and at least one service completion by time v. This implies that the arriving customer will find the system in some state $i\leqslant n-1$, and that the customer who was last in the queue during the interarrival-time will stay in the system unconditionally. Denoting the ith service time by x_i, the conditional contribution to $\bar{R}(n-1)$, given that the interarrival-time is v, is

$$\Pr\{x_1\leqslant v\}\cdot\left\{g-cE\left[\sum_{i=1}^{n}x_i\,|\,x_1\leqslant v\right]\right\}= =(1-\mathrm{e}^{-\mu v})(g-cn/\mu)+cve^{-\mu v}.$$

The last equality follows since

$$E\sum_{i=1}^{n}x_i\,|\,x_1\leqslant v\;=(n-1)/\mu+E[x_1\,|\,x_1\leqslant v]$$

and

$$E[x_1\,|\,x_1>v]=v+1/\mu.$$

Hence, by integrating on v, the contribution to $\bar{R}(n-1)$ is given by

$$(g-cn/\mu)\int_0^t(1-\mathrm{e}^{-\mu v})\,\mathrm{d}H(v)+c\int_0^t v\mathrm{e}^{-\mu v}\,\mathrm{d}H(v).$$

(iv) Service completion before time t and arrival after time t. The contribution to $\bar{R}(n-1)$ is

$$\Pr\{x_1\leqslant t\}\,[1-H(t)]\left(g-cE\left[\sum_{i=1}^{n}x_i\,|\,x_1\leqslant t\right]\right)= =[1-H(t)][(g-cn/\mu)(1-\mathrm{e}^{-\mu t})+cte^{-\mu t}].$$

Finally, by summing the contributions and arranging terms, we obtain

$$\bar{R}(n-1)= =(g-cn/\mu)[1-\mathrm{e}^{-\mu t}(1-H(t))-(a_0-a_0^t)]. \tag{9}$$

2.3. States n and $\bar{n}$

Suppose an arrival C joins the queue when there are $(n-1)$ customers in the system. If the following arrival occurs within t units of time and no service has been completed by then, the system moves to state n and is debited only for the waiting time of C. If the same sequence of events is repeated, the system is observed again in state n, and so forth (without traversing state (n, t)). Now consider the case where state (n, t) is reached. If the firm decides to keep C in the system, then, by our bookkeeping scheme, it is immediately endowed with $g-cn/\mu$. Now suppose that no service is completed by the time of the fol-

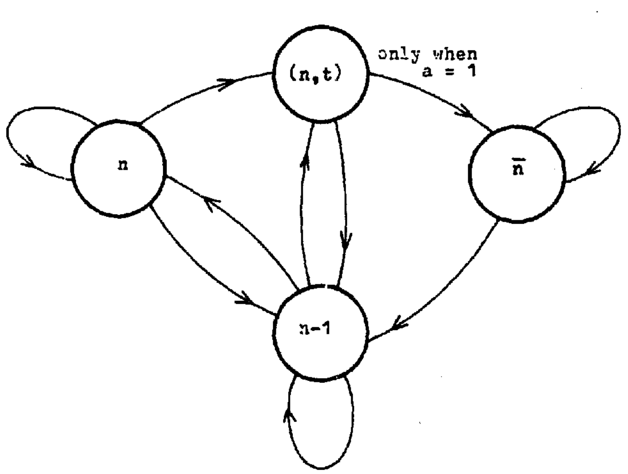

Fig. 1. Transitions among states $(n-1)$, (n, t), n and $\bar{n}$. (Transitions to and from other states are not shown.)

lowing arrival. Had we considered the new state as state n, we might have credited the firm again with the reward that had already been granted! To avoid such malbookkeeping, we define a twin-state, $\bar{n}$, which is observed by an arrival who finds in the system n customers that have already traversed state (n, t) without any customer being rejected. Fig. 1 presents a flow diagram of the possible transitions among states $n-1$, (n, t), n and $\bar{n}$. (Transition from (n, t) to $\bar{n}$ may occur only when C is kept in the system, i.e., $a = 1$.)

2.4. State n

The transition probabilities and the expected sojourn time are identical to those of state $n-1$, as given by eqs. (4)–(8), i.e.,

$$P_{nj} = P_{n-1,j}, \quad \text{for } j = 0, 1, 2, \ldots, n-1, n, (n, t),$$
$$\bar{\tau}(n) = \bar{\tau}(n-1)\,. \tag{10}$$

Since the arrival who finds the system in state n must balk, we have,

$$\bar{R}(n) = \bar{R}(n-1) - l\,. \tag{11}$$

2.5. State $\bar{n}$

State $\bar{n}$ is reached only through state (n, t), after C, the last customer in the queue, has been accepted (unconditionally) to the system. From that moment on, the option of rejecting customer C is irrelevant. That is, state (n, t) is not considered as long as the system is in state $\bar{n}$. We therefore have

$$P_{\bar{n},(n,t)} = P_{\bar{n},n} = 0,$$
$$P_{\bar{n},j} = a_{n-j}, \quad \text{for } j = 1, 2, \ldots, n-1,$$
$$P_{\bar{n},\bar{n}} = a_0, \quad P_{\bar{n},0} = r_{n-1}, \tag{12}$$
$$\bar{\tau}(\bar{n}) = 1/\lambda, \tag{13}$$
$$\bar{R}(\bar{n}) = -l. \tag{14}$$

2.6. State (n, t)

In this state we have two possible actions, i.e., to accept customer C ($a = 1$), or to reject C ($a = 0$). The transition probabilities are:

$$P_{(n,t),j}(0) =$$
$$= \Pr\{n-1-j \text{ service completions from } t \text{ until arrival}\}$$
$$= \int_t^\infty \mathrm{e}^{-\mu(v-t)} \frac{[\mu(v-t)]^{n-j-1}}{(n-j-1)!} \frac{\mathrm{d}H(v)}{1-H(t)}$$
$$= \frac{\mathrm{e}^{\mu t}}{1-H(t)} a^t_{n-j-1}, \quad \text{for } j = 1, 2, \ldots, n-1, \tag{15}$$
$$P_{(n,t),0}(0) = \frac{\mathrm{e}^{\mu t}}{1-H(t)} r^t_{n-2}, \tag{16}$$
$$P_{(n,t),n}(0) =$$
$$= P_{(n,t),(n,t)}(0) = P_{(n,t),\bar{n}}(0) = 0. \tag{17}$$

Similarly,

$$P_{(n,t),j}(1) = \frac{\mathrm{e}^{\mu t}}{1-H(t)} a^t_{n-j}, \quad \text{for } j = 1, 2, \ldots, n-1, \tag{18}$$
$$P_{(n,t),0}(1) = \frac{\mathrm{e}^{\mu t}}{1-H(t)} r^t_{n-1}, \tag{19}$$
$$P_{(n,t),\bar{n}}(1) = \frac{\mathrm{e}^{\mu t}}{1-H(t)} a^t_0, \tag{20}$$
$$P_{(n,t),n}(1) = P_{(n,t),(n,t)}(1) = 0. \tag{21}$$

For both actions,

$$\bar{\tau}_{(n,t)} = \frac{1}{1-H(t)} \int_t^\infty v\, \mathrm{d}H(v) - t. \tag{22}$$

Finally,

$$\bar{R}((n, t), 0) = -l_1, \tag{23}$$
$$\bar{R}((n, t), 1) = g - cn/\mu\,. \tag{24}$$

3. (n, t)-policy versus the control-limit rule

Suppose n is the profit maximizing control-limit. Starting with the (n, ∞)-policy, we use the policy-improvement procedure to reveal the conditions under which an (n, t)-policy is better than the control-limit rule.

The value determination operation [2] for the control-limit rule results in finding $\Theta, v_0, v_1, \ldots, v_{n-1}, v_{(n,t)}, v_n, v_{\bar{n}}$ that satisfy

$$\Theta/\lambda + v_i =$$
$$= g - c(i+1)/\mu + r_i v_0 + \sum_{j=1}^{i+1} a_{i+1-j} v_j,$$
$$i = 0, 1, 2, \ldots, n-2, \tag{25}$$
$$\Theta\bar{\tau}(n-1) + v_{n-1} =$$
$$= \bar{R}(n-1) + (r_{n-1} - r^t_{n-1})v_0 \sum_{j=1}^{n} (a_{n-j} - a^t_{n-j})v_j$$
$$+ \mathrm{e}^{-\mu t}[1 - H(t)]v_{(n,t)}, \tag{26}$$

$$\Theta\bar{\tau}((n,t)) + v_{(n,t)} =$$
$$= g - cn/\mu + \frac{e^{\mu t} r_{n-1}^t}{1-H(t)} v_0 + \sum_{j=1}^{n-1} \frac{e^{\mu t} a_{n-j}^t}{1-H(t)} v_j + \frac{e^{\mu t} a_0^t}{1-H(t)} v_{\bar{n}}, \tag{27}$$

$$\Theta\bar{\tau}(n) + v_n =$$
$$= \bar{R}(n) + (r_{n-1} - r_{n-1}^t) v_0 + \sum_{j=1}^{n} (a_{n-j} - a_{n-j}^t) v_j + e^{-\mu t}[1 - H(t)] v_{(n,t)}, \tag{28}$$

$$\Theta/\lambda + v_{\bar{n}} =$$
$$= -l + r_{n-1} v_0 + \sum_{j=1}^{n-1} a_{n-j} v_j + a_0 v_{\bar{n}}. \tag{29}$$

Multiplying eq. (27) by $e^{-\mu t}[1 - H(t)]$ and adding the resulting equation to eq. (26) yields (after using the expressions for $\bar{\tau}(n-1)$, $\bar{R}(n-1)$ and $\bar{\tau}(n,t)$)

$$\Theta/\lambda + v_{n-1} =$$
$$= [g - cn/\mu][1 - (a_0 - a_0^t)] + r_{n-1} v_0 + \sum_{j=1}^{n} a_{n-j} v_j - a_0^t v_n + a_0^t v_{\bar{n}}. \tag{30}$$

From (26) and (28) we derive

$$v_n = v_{n-1} - l. \tag{31}$$

By subtracting (29) from (30) and using (31), we get (after cancellations)

$$v_{n-1} - v_{\bar{n}} = g - cn/\mu + l, \tag{32}$$

Adding the simple form (32) to (29) results in

$$\Theta/\lambda + v_{n-1} =$$
$$= g - cn/\mu + r_{n-1} v_0 + \sum_{j=1}^{n-1} a_{n-j} v_j + a_0 v_{\bar{n}}. \tag{33}$$

Equations (25), (33) and (29) (in this order) are identical with equation (26) of the Markovian model of [7] where $n_s + 1$ is replaced by an arbitrary control limit $\bar{n}$ and $\phi = \Theta/\lambda$.

Before proceeding with the analysis of the SMDP, we investigate the properties of the v_i's. We correct a flaw in the inductive proof of lemma 5 of [7] and strengthen the results.

Define $\delta_i = v_i - v_{i+1}$ for $i = 0, 1, \ldots, n-2$ and $\delta_{n-1} = v_{n-1} - v_{\bar{n}}$.

Lemma 3.1. *If $\delta_{n-1} \geqslant 0$, then $\delta_i \geqslant 0$, for $i = 0, 1, \ldots, n-2$.*

Proof. By subtracting equation $i+1$ from equation i in (25), and (33) from the last equation of (25), we obtain

$$\delta_i = c/\mu + \sum_{j=0}^{i+1} a_{i+1-j}\delta_j, \quad i = 0, 1, \ldots, n-2. \tag{34}$$

Suppose not all $\delta_i \geqslant 0$. Then, let $k = \min\{i \mid \delta_i < 0\}$. By eq. (34) we have

$$\delta_k = \left(\sum_{j=0}^{\infty} a_j\right)\delta_k$$
$$= c/\mu + \sum_{j=0}^{k-1} a_{k+1-j}\delta_j + a_1\delta_k + a_0\delta_{k+1},$$

or

$$a_0(\delta_k - \delta_{k+1}) =$$
$$= c/\mu + \sum_{j=0}^{k-1} a_{k+1-j}(\delta_j - \delta_k) - \left(\sum_{j=k+2}^{\infty} a_j\right)\delta_k. \tag{35}$$

The second term on the right-hand side of eq. (35) is non-negative since, for $j < k$, $\delta_j \geqslant 0 > \delta_k$. The third term is positive by the definition of k. Hence, $\delta_k - \delta_{k+1} > 0$, so $\delta_{k+1} < \delta_k < 0$. Continuing in the same way for $\delta_{k+1}, \delta_{k+2}, \ldots, \delta_{n-1}$ we obtain $0 > \delta_k > \delta_{k+1} > \ldots > \delta_{n-1}$, which contradicts the assumption that $\delta_{n-1} \geqslant 0$.

Setting $\bar{n} = n_s + 1$ and using the fact that $v_{n_s} - v_{n_s+1} \geqslant 0$, we obtain Lemma 5 of [7].

In our model, let $\bar{n}$ be the profit-maximizing control limit. Using equations (29) and (33), it follows that $\delta_{n-1} = v_{n-1} - v_{\bar{n}} = g - cn/\mu + l$. It follows from theorem 7 of [7] that $\delta_{n-1} \geqslant 0$. Alternatively, one may deduce directly from the optimality of the control limit $\bar{n}$ that $\delta_{n-1} \geqslant 0$, since otherwise joining the queue in state $\bar{n}$ will improve the policy. Hence, $v_0 \geqslant v_1 \geqslant v_2 \geqslant \ldots \geqslant v_{n-1} \geqslant v_{\bar{n}}$. Furthermore, from eq. (34) and Lemma 1, we have:

Corollary 3.2.

$\delta_i \geqslant c/\mu$, *for $i = 0, 1, \ldots, n-2$.*

The interpretation is that starting the process in state i rather than in state $i+1$ is worth at least the expected waiting cost of a single service time.

We now turn to the policy improvement routine,

[2], to compare the initial (n, ∞)-rule with the (n, t)-policy Since the latter policy differs from the former only in the action taken at state (n, t), it suffices to compute the test quantity

$$\Gamma_{(n,t)}(a) = \left[\bar{R}((n, t), a) + \sum_j P_{(n,t),j}(a) \cdot v_j - v_{(n,t)}\right] / \bar{\tau}((n, t), a),$$

for $a = 0, 1$. If $\Gamma_{(n,t)}(1) < \Gamma_{(n,t)}(0)$, it would follow that the (n, t)-policy is preferable. Since $\bar{\tau}((n, t), 0) = \bar{\tau}((n, t), 1) = \bar{\tau}((n, t))$ (given by (22)), we may compare the quantities

$$T(a) = \bar{R}((n, t), a) + \sum_j P_{(n,t),j}(a) v_j, \quad a = 0, 1.$$

For $a = 0$,

$$T(0) = -l_1 + \frac{e^{\mu t}}{1 - H(t)} \left[r^t_{n-2} v_0 + \sum_{j=1}^{n-1} a^t_{n-j-1} v_j\right]. \tag{36a}$$

For $a = 1$,

$$T(1) = g - cn/\mu + \frac{e^{\mu t}}{1 - H(t)} \times \left[r^t_{n-1} v_0 + \sum_{j=1}^{n-1} a^t_{n-j} v_j + a^t_0 v_{\bar{n}}\right]. \tag{36b}$$

Thus, by arranging terms,

$$f(t) \equiv T(0) - T(1) = \frac{e^{\mu t}}{1 - H(t)} \sum_{j=0}^{n-1} a^t_{n-j-1} \delta_j - (g - cn/\mu + l_1). \tag{37}$$

We have proved:

Theorem 3.3. *The (n, t)-policy is better than the optimal control-limit rule if and only if*

$$\frac{e^{\mu t}}{1 - H(t)} \sum_{j=0}^{n-1} a^t_{n-j-1} \delta_j > g - cn/\mu + l_1. \tag{38}$$

It is clear that inequality (38) depends heavily on the interarrival-time distribution $H(\cdot)$, through the function $f(t)$ defined in (37).

Lemma 3.4. $f(0) \leqslant 0$. *The inequality is strict when $l_1 > l$.*

Proof. First consider the case where $l_1 = l$. Then, the $(\bar{n}, 0)$-policy is, in fact, a control limit rule with control limit $\bar{n} - 1$ in the model studied by Yechiali [7], and $f(0)$ is the test-quantity for comparing the control limits $\bar{n}$ and $(\bar{n} - 1)$. Since $\bar{n}$ is the optimal control limit, we obtain $f(0) \leqslant 0$. The result now follows since $f(t)$ as defined by (37) is a decreasing function of l_1.

In the next section, we study a few special cases to reveal the relevance of the (n, t)-policy.

4. Special cases

4.1. Poisson arrival

Let $H(v) = 1 - e^{-\lambda v}$. Here,

$$a^t_k = e^{-(\lambda+\mu)t} \frac{\lambda}{\lambda + \mu}\left(\frac{\mu}{\lambda + \mu}\right)^k, \quad \text{for } k = 0, 1, 2, \ldots.$$

Hence

$$f(t) = \frac{\lambda}{\lambda + \mu} \sum_{j=0}^{n-1} \left(\frac{\mu}{\lambda + \mu}\right)^{n-j-1} \delta_j - (g - cn/\mu + l_1),$$

that is, for Poisson arrival, $f(t)$ is independent of t. It follows from Lemma 3.4 that $f(t) \leqslant 0$ for all t, so, by Theorem 3.3, the (n, t)-policy is not better than the (n, ∞)-policy for all t. Naturally, this result was expected due to the properties of the exponential interarrival-time distribution.

4.2. Deterministic interarrival-time

Let $H(v) = 0$ for $v > 1/\lambda$ and $H(v) = 1$ for $v \geqslant 1/\lambda$. Thus, $f(1/\lambda) = 0$. We shall study the behavior of $f(t)$ for t close to $1/\lambda$ ($t \leqslant 1/\lambda$). Let $\epsilon = 1/\lambda - t$; then,

$$a^t_k = e^{-\mu/\lambda}(\mu\epsilon)^k/k!, \quad k = 0, 1, 2, \ldots$$

Since, for $k = 1, 2, 3, \ldots, a^t_k$ is $o(\epsilon^{k-1})$, and $\delta_{n-1} = g - cn/\mu + l$, we have,

$$\begin{aligned} f(t) &= e^{\mu t}(a^t_0 \delta_{n-1} + a^t_1 \delta_{n-2}) - \delta_{n-1} + l - l_1 + o(\epsilon) \\ &= \mu\epsilon\, e^{-\mu\epsilon} \delta_{n-2} - (1 - e^{-\mu\epsilon})\delta_{n-1} + l - l_1 + o(\epsilon) \\ &= \mu\epsilon(\delta_{n-2} - \delta_{n-1}) + l - l_1 + o(\epsilon). \end{aligned} \tag{39}$$

Hence, if $\delta_{n-2} > \delta_{n-1}$, and l_1 is close enough to l, the (n, t)-policy is an improvement on the control limit rule for t close enough to $1/\lambda$.

The same conclusion may be reached by the following heuristic argument. Suppose the last customer in the queue is rejected when the system reaches state (n, t). Since the probability of more than one service completion before arrival is $o(\epsilon)$, there remain two possibilities:

(i) *One service completion.* The firm then incurs the penalty l and loses the future net reward of $g - cn/\mu$, but the next arrival finds the system in state $n-2$ rather than state $n-1$. Hence, the expected conditional improvement with respect to the control-limit rule is

$$\delta_{n-2} - (g - cn/\mu + l_1) = \delta_{n-2} - \delta_{n-1} + l - l_1.$$

(ii) *No service completion.* In this case, the following arrival simply replaces the rejected customer. By the memoryless property of the service-time distribution, the expected conditional improvement is non-positive and equals $l - l_1$. Note that the remaining waiting-time of the rejected customer (equal to ϵ) is actually *not* saved, since the expected waiting time of the following arrival is still n/μ. Finally, the expected improvement is

$$\mu\epsilon(\delta_{n-2} - \delta_{n-1}) + l - l_1 + o(\epsilon)$$

as found in (39).

We now demonstrate the existence of cases where an (n, t)-policy is better than the optimal control-limit rule. Consider the social-optimization problem studied in [7], under conditions of market equilibrium, and assume $l = l_1$. Then, as shown in [7, pp. 363–365], $n = n_s$. Since $\delta_{n-2} \geq c/\mu$, we have, for small ϵ,

$$f(t) \geq \mu\epsilon\left[\frac{c}{\mu} - (g - cn/\mu + l)\right]$$

$$= -\mu\epsilon[g - c(n_s + 1)/\mu + l] > 0.$$

We have proved:

Theorem 4.1. *There exist (n, t)-policies which are better than any control-limit rule.*

Clearly, the (n, t)-policy is still better than the optimal control limit rule when $l_1 - l$ is positive but small enough.

4.3. General finite-range interarrival time distribution

The considerations applied for the deterministic case may now be extended to any finite-range interarrival-time distribution. Let $H(\cdot)$ be concentrated on a finite interval $[a, b]$, where $0 \leq a < b$, $H(a) = 0$ and $H(b) = 1$. We have

Lemma 4.2. *As $t \to b$ $(t < b)$,*

$$\frac{e^{\mu t}}{1 - H(t)}(a_0^t + a_1^t) = 1 - o(b - t).$$

Proof. Note that $e^{\mu t}a_k^t/[1 - H(t)]$ is the conditional probability of k service completions during an interarrival time, given that neither arrival nor service completion has occurred during (o, t). This is the probability of having k Poisson events during a random time interval which is shorter (with probability 1) than $(b - t)$. Since the probability of having more than one Poisson event in a time interval of length ϵ is $o(\epsilon)$, the result follows.

Using Lemma 4.2, $f(t)$ may be expressed as

$$f(t) = \frac{e^{\mu t}}{1 - H(t)}(a_0^t\delta_{n-1} + a_1^t\delta_{n-2}) - \delta_{n-1} + l - l_1 + o(b - t)$$

$$= \left[\frac{e^{\mu t}}{1 - H(t)}(a_0^t + a_1^t) - 1\right]\delta_{n-1} + \frac{e^{\mu t}}{1 - H(t)}a_1^t(\delta_{n-2} - \delta_{n-1}) + (l - l_1) + o(b - t),$$

where the first expression is $o(b - t)$, in light of Lemma 4.2. Hence, we obtain

$$f(t) = \frac{e^{\mu t}}{1 - H(t)}a_1^t(\delta_{n-2} - \delta_{n-1}) + (l - l_1) + o(b - t). \tag{40}$$

It follows again that if $\delta_{n-2} > \delta_{n-1}$ (e.g., under conditions of market equilibrium in the social-optimization problem) and l_1 is close enough to l, exercising conditional acceptance does increase the long run average profit of the service agency.

5. Conclusion

The aim of this paper was to prove the relevance of conditional acceptance rules in a GI/M/1 queueing system. Thus, in the search for an optimal policy of controlling the arrival process, it is not sufficient to consider only state-dependent policies, as done in the prevailing studies. The elapsed sojourn times are also relevant and ought to be considered. Our conjecture is that the optimal control policy is a generalization of our conditional acceptance rule, which is characterized by a vector of the form $(t_0, t_1, t_2, t_3, \ldots,)$, where $\infty = t_0 \geqslant t_1 \geqslant t_2 \geqslant t_3 \geqslant \ldots$. The significance of this vector is that whenever t_n time units have elapsed since the last arrival, and n customers are present in the system, the last customer in the queue is rejected. The classical control-limit rule is a special case of the form $(\infty, \infty, \ldots, \infty, \infty, 0, 0, \ldots)$, whereas our (n, t)-policy is the (more general) special case $(\infty, \infty, \ldots, \infty, t, 0, 0, \ldots)$.

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