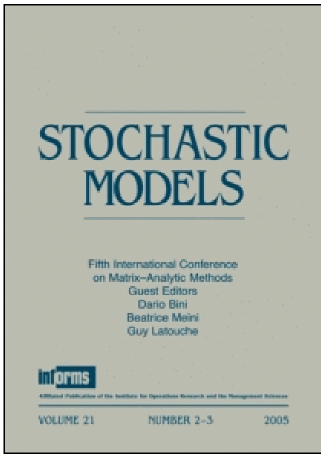


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# Multi-Server Queues with Intermediate Buffer and Delayed Information on Service Completions

V. Kitsio [a]; U. Yechiali [a]
[a] Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

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# MULTI-SERVER QUEUES WITH INTERMEDIATE BUFFER AND DELAYED INFORMATION ON SERVICE COMPLETIONS

**V. Kitsio and U. Yechiali**

*Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel*

□ *A controller with an unlimited buffer receives messages to be dispatched to c servers downstream in the network. However, the queue sizes at the individual servers are not known exactly, since information on each service completion reaches the controller only after some random delay. The controller can dispatch messages as soon as they arrive, in a cyclic manner, to the c servers. Alternatively, s/he can wait until full information is gained and dispatch a waiting message to a server only when s/he is sure that the server is free. Another strategy is to maintain a limited intermediate buffer in front of the servers, and forward messages to this buffer when information on service completion reaches the controller. If a server completes a job and the intermediate buffer is non-empty, it starts serving a job from this buffer with no delay.*

*Such situations are common in many real life processes (such as passport control procedures, or at large waiting rooms in public offices) where customers wait, in front of c servers, to be served. A customer walks (=delay) to the next idle server when s/he sees her/his "waiting number" flashing on the screen.*

*We analyze this model when the underlying process is the $M/M/c$ queue and the information delay is exponential. We use both:* i) *probability generating functions of the multi-dimensional state space to calculate the boundary probabilities, and* ii) *matrix geometric approach to derive the stability condition of the system. We show that the intermediate buffer scheme reduces queue sizes and waiting times. Numerical examples are presented.*

**Keywords** Delayed information; Intermediate buffers; Multi-server queues.

**Mathematics Subject Classification** Primary 60K25; Secondary 60M20, 90B22.

## 1. INTRODUCTION

The goal of this work is to construct, analyze and solve a model for reducing servers (and customers) idle (and waiting) times in multi-server

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Address correspondence to V. Kitsio, Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel; E-mail: vkitsio@mlm.iai.co.il

queues with delayed information on service completion. This model extends previous works (Refs.[1,3]) by introducing an intermediate buffer between the controller and the servers. In general, analysis and control of queuing system with delayed information, either on service completion or on arrivals, are complex issues that have been studied very little in the literature. These issues are critical in high-speed networks where routing decisions have to be made based on delayed information on the actual state of down stream nodes. The lack of full information makes the problem of optimal routing of jobs extremely difficult.

The "delayed information" phenomenon is also common to many real life situations at large waiting rooms in public offices where a "delay" is equivalent to the time it takes for a customer to walk from her/his seating location to the server's window.

Another practical example for intermediate buffer is the loading process of a certain commodity on trucks in time of a relatively high load. A fleet of trucks is waiting outside the warehouse and in order to save time, a few trucks wait near the loading platform.

Consider a network with $c$ parallel channels (servers) and a single controller. Variable-length messages (jobs) arrive randomly, and the controller has to route them to the various channels. If the controller has full information on the state of each server, then holding a central common buffer for all queues and assigning a job to a server as soon as the latter becomes available, is the best policy in terms of minimizing queue lengths and waiting times. However, if the information about the actual state of each queue reaches the controller only after some considerable delay, then the problem of optimal management of the queue becomes much more complex.

Suppose, indeed, that the information about each service completion reaches the controller only after some random delay. To improve the performance of such systems we consider a finite-size intermediate buffer in front of the servers, such that, whenever a service is completed, a waiting job in this buffer can enter service with no further delay. In this case, if there are jobs in the intermediate buffer, a server can start serving immediately without waiting for the controller to dispatch a job (which will be done only after the delay time). In such a way the server's idle time, as well as customers sojourn times, are reduced. A server will be idle only in the event when the intermediate buffer is empty. Clearly, if the intermediate buffer size becomes unlimited, the model reduces to a regular $c$-server.

The underlying queueing model in this work is the $M/M/c$ queue, where the information on each service completion reaches the controller only after exponentially distributed time. In Litvak and Yechiali[3] two routing policies by the controller were studied: (i) the controller dispatches jobs as soon as they arrive, in a round-robin mechanism, to the various down-stream servers, without knowledge of the actual queue size in front of each server. This implies that the controller maintains no buffer and all buffers are in front of the servers. (ii) the controller holds all arriving jobs in its buffer and dispatches a job to a server only when the information about service completion by that server reaches the controller. It has been shown in Litvak and Yechiali[3] that the former policy is better if the mean delay on service completion is greater than some (calculated) threshold, and vise versa. We propose in this work an improvement on the above policies by introducing an intermediate buffer in front of the group of servers and show, by a numerical example (based on our analytical results) that this policy leads to a significant improvement.

The structure of the work is as follows: in Section 2 we present the general description of the model, along with a set of assumptions, definitions and notation used throughout the work. In Section 3 we analyze the model with only a single server. We derive closed-form expressions for the so called "boundary" probabilities determining the probability generating functions (PGFs) of the system states. In addition, we directly derive the stability condition of the system. Furthermore, we use a Matrix-Analytic method to derive the same stability condition. In Section 4 we study the two-server case and present a numerical example, showing the reduction in the mean queue size (compared to the result in Ref.[3]). In Section 5 we analyze the general model with $c \geq 2$ servers. The stability condition is derived by using matrix–geometric analysis. A relationship to a machine-repair problem is indicated.

## 2. THE MODEL

Consider an $M/M/c$-type queue with Poisson arrival rate $\lambda$, exponential service times with parameter $\mu$, and a controller with an unlimited buffer. However, in contrast to a regular $M/M/c$ queue, the information on each service completion reaches the controller only after some random duration, exponentially distributed with parameter $\gamma$. There is an intermediate buffer of size $c$ in front of the $c$ servers. When the controller gets the (delayed) information that a service has been completed, and there are less than $c$ waiting customers in the intermediate buffer, s/he dispatches a waiting customer (if any) from its buffer to the intermediate buffer. When a server completes service of customer and the intermediate buffer is non-empty, it starts serving one of the customers there with no further delay. We denote such a system by $M(\boldsymbol{\lambda})/intermediate : M(\boldsymbol{\mu}) + M(\boldsymbol{\gamma})/c$. As indicated in the Introduction, the purpose of having an intermediate buffer is to reduce the idle time of the servers so as to reduce queue sizes and customers waiting times. We define the state of the system as a triplet $(N, J, X)$ where $N$ denotes the number of customers waiting in the controller's overall buffer, $X$ counts the combined number

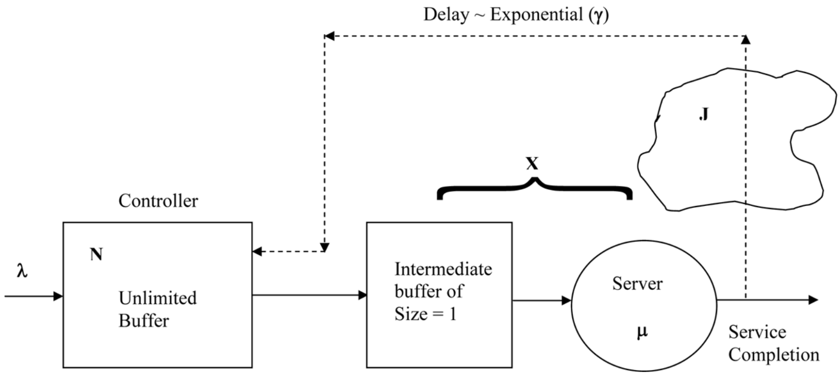


**FIGURE 1** $M/M/1$-type queue with an intermediate buffer of size = 1.

of customers in intermediate buffer or in service, and $J$ denotes the total number of customers 'satellite', i.e. customers whose service has already been completed, but this information hasn't reached the controller yet.

The system is depicted in Figure 1.

## 3. THE SINGLE SERVER CASE

We start our analysis with the single server M/M/1-type queue with an intermediate buffer of size $c = 1$. This implies the following: $x = 0$ denotes an idle server; $x = 1$ indicates that the server is busy giving service; when $x = 2$, one customer is being served and another is waiting in the intermediate buffer. Note, that the controller always knows the sum $X + J \leq 2$, but without precise knowledge of the specific values of $X$ or $J$.

### 3.1. Balance Equations

Investigating the structure of the transition rate diagram (Figure 2) we see that, for each $n \geq 0$, the 3 states for which $x + j = 2$ (namely, $(n, 2, 0)$, $(n, 1, 1)$ and $(n, 0, 2)$) repeat themselves. The states, (0, 0, 0), (0, 0, 1) and (0, 1, 0) together with the state (0, 2, 0) are different and denoted as "boundary states".

Let $P_{n,j,x} = \text{Prob}(N = n, J = j, X = x)$, $n = 0, 1, 2, 3, \ldots$; $j = 0, 1, 2$; $x = 0, 1, 2$. Then, for each value of $N = n$, the set of balance equations is the following:

For $n = 0$, the equations involving the first 3 boundary states are given by

$$\begin{cases} \lambda P_{0,0,0} = \gamma P_{0,1,0} & (j = x = 0), \\ (\lambda + \mu) P_{0,0,1} = \lambda P_{0,0,0} + \gamma P_{0,1,1} & (j = 0, x = 1), \\ (\lambda + \gamma) P_{0,1,0} = \mu P_{0,0,1} + 2\gamma P_{0,2,0} & (j = 1, x = 0). \end{cases} \tag{3.1-1}$$

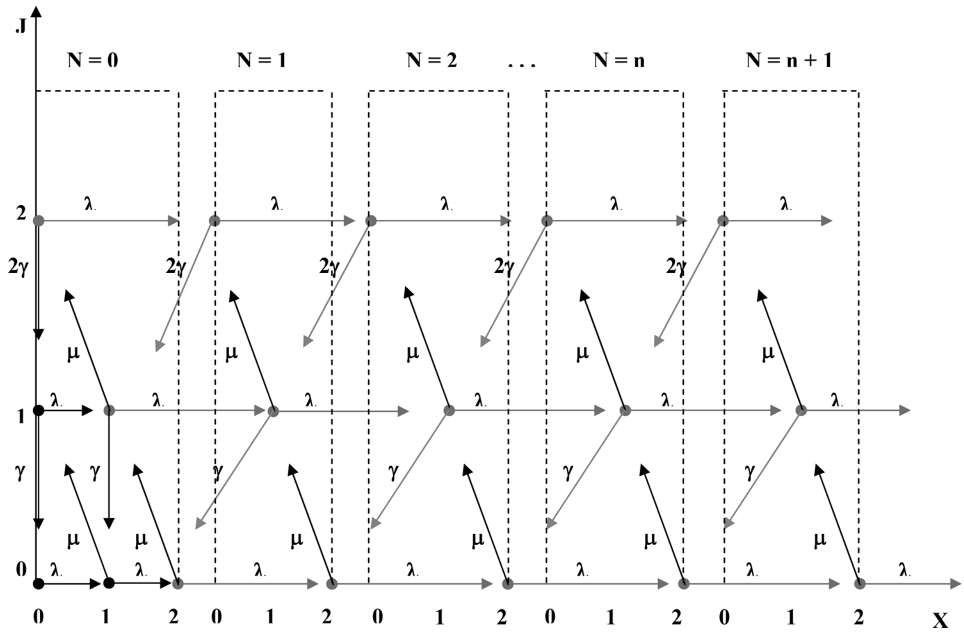


**FIGURE 2** Transition rate diagram for the single server case.

For $n = 0$ and $j + x = 2$,

$$\begin{cases} (\lambda + 2\gamma)P_{0,2,0} = \mu P_{0,1,1} & (j = 2, x = 0), \\ (\lambda + \mu + \gamma)P_{0,1,1} = \lambda P_{0,1,0} + \mu P_{0,0,2} + 2\gamma P_{1,2,0} & (j = 1, x = 1), \\ (\lambda + \mu)P_{0,0,2} = \lambda P_{0,0,1} + \gamma P_{1,1,1} & (j = 0, x = 2). \end{cases} \tag{3.1-2}$$

In general, for $N = n \geq 1$ (where $j + x = 2$),

$$\begin{cases} (\lambda + 2\gamma)P_{n,2,0} = \lambda P_{n-1,2,0} + \mu P_{n,1,1} & (j = 2, x = 0), \\ (\lambda + \mu + \gamma)P_{n,1,1} = \lambda P_{n-1,1,1} + \mu P_{n,0,2} + 2\gamma P_{n+1,2,0} & (j = 1, x = 1), \\ (\lambda + \mu)P_{n,0,2} = \lambda P_{n-1,0,2} + \gamma P_{n+1,1,1} & (j = 0, x = 2). \end{cases} \tag{3.1-3}$$

### 3.2. Partial Generation Functions

For each level of the satellite customers $J = 0, 1, 2$ we define the corresponding (partial) generating function (PGF) as follows:

$$G_0(z) = (P_{0,0,0} + P_{0,0,1})z^0 + \sum_{n=0}^{\infty} P_{n,0,2}z^n, \tag{3.2-1}$$

$$G_1(z) = P_{0,1,0}z^0 + \sum_{n=0}^{\infty} P_{n,1,1}z^n, \tag{3.2-2}$$

$$G_2(z) = \sum_{n=0}^{\infty} P_{n,2,0}z^n. \tag{3.2-3}$$

Now, from the balance equations, for $j = 2$, we obtain

$$(\lambda + 2\gamma)\sum_{n=0}^{\infty} P_{n,2,0}z^n = \lambda z\sum_{n=0}^{\infty} P_{n,2,0}z^n + \mu\sum_{n=0}^{\infty} P_{n,1,1}z^n, \tag{3.2-4}$$

implying that

$$\mu G_1(z) - (\lambda(1-z) + 2\gamma)G_2(z) = \mu P_{0,1,0}. \tag{3.2-5}$$

Similarly, when $j = 1$, we get

$$\begin{aligned}(\lambda + \mu + \gamma)z\sum_{n=0}^{\infty} P_{n,1,1}z^n &= \lambda z^2\sum_{n=0}^{\infty} P_{n,1,1}z^n + \mu z\sum_{n=0}^{\infty} P_{n,0,2}z^n \\ &\quad + 2\gamma\sum_{n=0}^{\infty} P_{n,2,0}z^n - 2\gamma P_{0,2,0} + \lambda z P_{0,1,0}.\end{aligned} \tag{3.2-6}$$

That is,

$$\begin{aligned}&-\mu z G_0(z) + (\lambda(1-z) + \mu + \gamma)zG_1(z) - 2\gamma G_2(z) \\ &\quad = (\lambda + \mu + \gamma)zP_{0,1,0} - \lambda z^2 P_{0,1,0} - \mu z(P_{0,0,0} + P_{0,0,1}) - 2\gamma P_{0,2,0} + \lambda z P_{0,1,0}.\end{aligned} \tag{3.2-7}$$

Finally, for $j = 0$,

$$(\lambda + \mu)z\sum_{n=0}^{\infty} P_{n,0,2}z^n = \lambda z^2\sum_{n=0}^{\infty} P_{n,0,2}z^n + \gamma\sum_{n=0}^{\infty} P_{n,1,1}z^n - \gamma P_{0,1,1} + \lambda z P_{0,0,1}, \tag{3.2-8}$$

leading to

$$\begin{aligned}&(\lambda(1-z) + \mu)zG_0(z) - \gamma G_1(z) \\ &\quad = (\lambda(1-z) + \mu)z(P_{0,0,0} + P_{0,0,1}) + \lambda z P_{0,0,1} - \gamma(P_{0,1,0} + P_{0,1,1}).\end{aligned} \tag{3.2-9}$$

Equations (3.2-5), (3.2-7) and (3.2-9) define a set of linear equations with unknowns $G_0(z)$, $G_1(z)$ and $G_2(z)$, depending on the 3 "boundary" probabilities as well as the probabilities $P_{0,2,0}$ and $P_{0,1,1}$. Knowledge of these probabilities fully determines the PGFs.

We proceed now to calculate the above five probabilities.

Consider the state where $j + x = 2$. Then for every $j = 0, 1, 2$ we defined, respectively, the following probabilities:

$$P0 \equiv \sum_{n=0}^{\infty} P_{n,0,2}, \quad P1 \equiv \sum_{n=0}^{\infty} P_{n,1,1}, \quad P2 \equiv \sum_{n=0}^{\infty} P_{n,2,0}. \tag{3.2-10}$$

Clearly, the sum of those "total" probabilities and the 3 "boundary" probabilities equals 1.

That is,

$$P0 + P1 + P2 + P_{0,0,0} + P_{0,0,1} + P_{0,1,0} = 1. \tag{3.2-11}$$

We now have 8 unknown probabilities: the 6 probabilities appearing in equation (3.2-11) together with $P_{0,2,0}$ and $P_{0,1,1}$.

We'll create a set of 8 independent linear equations in those 8 unknowns probabilities.

We use a "diagonal" cut in Figure 2 between the states $n$ and $n + 1$, where $j + x = 2$, to get

$$\lambda(P_{n,2,0} + P_{n,1,1} + P_{n,0,2}) = 2\gamma P_{n+1,2,0} + \gamma P_{n+1,1,1} \quad (n = 0, 1, 2, \ldots). \tag{3.2-12}$$

Summing over $n$ we obtain

$$\lambda(P2 + P1 + P0) = 2\gamma P2 + \gamma P1 - 2\gamma P_{0,2,0} - \gamma P_{0,1,1}. \tag{3.2-13}$$

Now "cutting" between levels $J = 1$ and $J = 2$ we get

$$2\gamma P2 = \mu P1. \tag{3.2-14}$$

Similarly, "cutting" between levels $J = 0$ and $J = 1$ yields

$$\gamma P_{0,1,0} + \gamma P1 = \mu P_{0,0,1} + \mu P0. \tag{3.2-15}$$

Clearly, equations (3.2-14) and (3.2-15) can be directly obtained by setting $z = 1$ in equations (3.2-5) and (3.2-7), respectively.

Equations (3.2-11), (3.2-13), (3.2-14) and (3.2-15) together with the 3 boundary equations defined in (3.1-1) and the equation $(\lambda + 2\gamma)P_{0,2,0} = \mu P_{0,1,1}$ comprise a set of 8 equations in the probabilities $P0, P1, P2, P_{0,0,0}, P_{0,0,1}, P_{0,1,0}, P_{0,2,0}$ and $P_{0,1,1}$.

### 3.3. Explicit Solution

Using the 2 equations from (3.1-1) for which $j + x = 1$, together with equation (3.2-11), and substituting in (3.2-13), we get, after some algebra,

$$2\gamma P2 + \gamma P1 = \lambda - 2\lambda P_{0,0,0} + \gamma P_{0,1,0} = \lambda - \gamma P_{0,1,0}, \tag{3.3-1}$$

where the last step follows by using the first equation of (3.1-1).

Now, substituting equation (3.2-14) in (3.3-1) we get

$$P1 = \frac{\lambda - \gamma P_{0,1,0}}{\mu + \gamma} \text{ and} \tag{3.3-2}$$

$$P2 = \frac{\mu(\lambda - \gamma P_{0,1,0})}{2\gamma(\mu + \gamma)}. \tag{3.3-3}$$

Substituting the value of P1 from (3.3-2) into (3.2-15) yields

$$P0 = \frac{\lambda\gamma - \mu(\mu + \gamma)P_{0,0,1} + \mu\gamma P_{0,1,0}}{\mu(\mu + \gamma)}. \tag{3.3-4}$$

Substituting (3.3-2), (3.3-3), (3.3-4) in (3.2-11) and using $\lambda P_{0,0,0} = \gamma P_{0,1,0}$ we get

$$P_{0,0,0} = \frac{2\gamma(\mu + \gamma)(\mu - \lambda) - \lambda\mu^2}{\mu\{2\gamma(\mu + \gamma) + \lambda(\mu + 2\gamma)\}} \tag{3.3-5}$$

and

$$P_{0,1,0} = \frac{\lambda\{2\gamma(\mu + \gamma)(\mu - \lambda) - \lambda\mu^2\}}{\mu\gamma\{2\gamma(\mu + \gamma) + \lambda(\mu + 2\gamma)\}}. \tag{3.3-6}$$

It remains to find $P_{0,0,1}$.

Using the equation from (3.1-2) for which $j = 2, x = 0$, and substituting in (3.1-1) for $j = 1, x = 0$, we get

$$\mu P_{0,0,1} = (\lambda + \gamma)P_{0,1,0} - \frac{2\mu\gamma P_{0,1,1}}{\lambda + 2\gamma}. \tag{3.3-7}$$

Using the equation from (3.1-1) for which $j = 0, x = 1$, and substituting in (3.3-7) we obtain

$$\mu P_{0,0,1} = (\lambda + \gamma)P_{0,1,0} - \frac{2\mu(\lambda + \mu)P_{0,0,1} - \lambda P_{0,0,0}}{\lambda + 2\gamma}. \tag{3.3-8}$$

Now, substituting the equation $\lambda P_{0,0,0} = \gamma P_{0,1,0}$ in (3.3-8) we get, after some algebra,

$$P_{0,0,1} = \frac{P_{0,1,0}(\lambda^2 + 3\lambda\gamma + 2\mu\gamma + 2\gamma^2)}{\mu(3\lambda + 2\mu + 2\gamma)}. \tag{3.3-9}$$

Finally, substituting (3.3-6) in (3.3-9) we obtain

$$P_{0,0,1} = \frac{\lambda\{2\gamma(\mu+\gamma)(\mu-\lambda) - \lambda\mu^2\}(\lambda^2 + 3\lambda\gamma + 2\mu\gamma + 2\gamma^2)}{\mu^2\gamma\{2\gamma(\mu+\gamma) + \lambda(\mu+2\gamma)\}(3\lambda + 2\mu + 2\gamma)}. \tag{3.3-10}$$

Clearly, the values of $P1, P2$, and $P0$ are now explicitly derived from (3.3-2), (3.3-3) and (3.3-4).

Finally, $P_{0,2,0}$ and $P_{0,1,1}$ are obtained from the 2 boundary equations (3.1-1) for which $j + x = 1$.

The above "direct" solution holds only for the single server case. For models with $c > 1$ servers we need a more elaborate approach, which we present in the next section.

### 3.4. Matrix Representation

The set of equations (3.2-5), (3.2-7) and (3.2-9) can be represented in a matrix form as

$$\begin{pmatrix} (\lambda(1-z)+\mu)z & -\gamma & 0 \\ -\mu z & (\lambda(1-z)+\mu+\gamma)z & -2\gamma \\ 0 & \mu & -(\lambda(1-z)+2\gamma) \end{pmatrix} \times \begin{pmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{pmatrix} = \begin{pmatrix} b_0(z) \\ b_1(z) \\ b_2(z) \end{pmatrix}, \tag{3.4-1}$$

i.e., $A(z)G(z) = b(z)$, where $G(z) = (G_0(z), G_1(z), G_2(z))^T$ and $b(z) = (b_0(z), b_1(z), b_2(z))^T$ are column vectors for which

$$b_0(z) = (\lambda(1-z)+\mu)z(P_{0,0,0} + P_{0,0,1}) - \lambda z P_{0,0,1} - \gamma(P_{0,1,0} + P_{0,1,1}), \tag{3.4-2}$$

$$b_1(z) = (\lambda+\mu+\gamma)zP_{0,1,0} - \lambda z^2 P_{0,1,0} - \mu z(P_{0,0,0} + P_{0,0,1}) - 2\gamma P_{0,2,0} + \lambda z P_{0,1,0}, \tag{3.4-3}$$

where $b_2(z) = \mu P_{0,1,0}$. (3.4-4)

The PGFs $G_j(z), j = 0, 1, 2$, are positive and bounded for $0 \le z \le 1$.

By Cramer's rule

$$G_j(z) = \frac{|A_j(z)|}{|A(z)|}, \tag{3.4-5}$$

where $A_j(z)$ is obtained from $A(z)$ by replacing the $j$th column with the vector $b(z)$.

After tedious calculations of the various determinants involved we derive explicit solution for the PGF's in terms of the probabilities $P_{0,0,0}$, $P_{0,0,1}$, $P_{0,1,0}$ and the expressions $b_0(z)$, $b_1(z)$, $b_2(z)$. The results are the following:

$$G_0(z) = \frac{\lambda^2 b_2(z)(1-z) + \lambda(\mu+\gamma)b_2(z) + 2\lambda\gamma b_2(z) + 2\lambda\gamma^2 P_{0,1,0} - 2\mu\gamma^2(P_{0,0,0}+P_{0,0,1})}{-\lambda\mu\gamma + \lambda^3 z(1-z)^2 + \lambda^2\mu z(1-z)}$$
$$+ \frac{2\lambda\gamma^2 z(P_{0,0,0}+P_{0,0,1}) - 2\lambda\gamma^2 P_{0,0,1} + \lambda\gamma(\lambda+\mu+\gamma)P_{0,1,0} - \lambda^2\gamma P_{0,1,0} - \lambda\mu\gamma(P_{0,0,0}+P_{0,0,1})}{\lambda^2 z(\mu+\gamma)(1-z) + \lambda\mu(\mu+\gamma)z} \quad (3.4\text{-}6)$$
$$+ \frac{\lambda^2\gamma P_{0,1,0} - 2\mu\gamma(\lambda(1-z)+\mu)(P_{0,0,0}+P_{0,0,1}) - 2\lambda\mu\gamma P_{0,0,1}}{2\lambda^2\gamma z(1-z) + 2\lambda\gamma(\mu+\gamma)z - 2\mu\gamma(\mu+\gamma)}.$$

$$G_1(z) = \frac{-2\lambda\gamma b_0(z) + \lambda\mu b_2(z) + \lambda^2(1-z)b_1(z) + \lambda\mu b_1(z) + 2\lambda\gamma b_1(z)}{-\lambda\mu\gamma + \lambda^3 z(1-z)^2 + \lambda^2\mu z(1-z) + \lambda^2 z(\mu+\gamma)(1-z)}$$
$$+ \frac{2\mu\gamma\{\lambda z(P_{0,0,0}+P_{0,0,1}+P_{0,1,0}) - (\lambda+\mu+\gamma)P_{0,1,0} - \lambda P_{0,0,1}\}}{\lambda\mu(\mu+\gamma)z + 2\lambda^2\gamma z(1-z) + 2\lambda\gamma(\mu+\gamma)z - 2\mu\gamma(\mu+\gamma)}. \quad (3.4\text{-}7)$$

$$G_2(z) = \frac{\lambda\mu b_1(z) - b_0(z)z\lambda^2(1-z) - \lambda b_0(z)z(\mu+\gamma) - \lambda\mu b_0(z)z + \mu\gamma b_0(z)}{-\lambda\mu\gamma + \lambda^3 z(1-z)^2 + \lambda^2\mu z(1-z) + \lambda^2 z(\mu+\gamma)(1-z)}$$
$$+ \frac{\mu^2 b_0(z) + \mu^2\{\lambda z(P_{0,0,0}+P_{0,0,1}+P_{0,1,0}) - (\lambda+\mu+\gamma)P_{0,1,0} - \lambda P_{0,0,1}\}}{\lambda\mu(\mu+\gamma)z + 2\lambda^2\gamma z(1-z) + 2\lambda\gamma(\mu+\gamma)z - 2\mu\gamma(\mu+\gamma)}. \quad (3.4\text{-}8)$$

Note that $|A(z)|$ is a polynomial of degree 5. Nevertheless, it can be shown that in the interval $z \in [0,1]$, $|A(z)| = 0$ only for $z=0$ and $z=1$. This is in fact the reason why we were able to obtain a direct explicit solution for the above-unknown probabilities. The case with $c > 1$ will require finding the roots of $|A(z)| = 0$.

### 3.5. $E[J]$, $E[X]$ and Stability Condition for the Single Server Model

The marginal distribution of $J$ is derived as follows:

For $J = 0$,

$$P(J=0) = G_0(1) = P_{0,0,0} + P_{0,0,1} + \sum_{n=0}^{\infty} P_{n,0,2} = P_{0,0,0} + P_{0,0,1} + P0.$$

Using (3.3-4), (3.3-5) and (3.3-10) we derive $P(J = 0)$:

$$G_0(1) = \frac{\mu P_{0,0,0}(\lambda(\lambda+\mu) - 2\gamma(\mu+\gamma))}{\lambda\mu^2 + 2\gamma(\mu+\gamma)(\lambda-\mu)} = \frac{2\gamma(\mu+\gamma) - \lambda(\lambda+\mu)}{2\gamma(\mu+\gamma) + 2\lambda\gamma + \lambda\mu}. \quad (3.5\text{-}1)$$

For $J = 1$,

$$P(J=1) = G_1(1) = P_{0,1,0} + \sum_{n=0}^{\infty} P_{n,1,1} = P_{0,1,0} + P1.$$

Utilizing (3.3-6) and (3.3-2) yields

$$G_1(1) = \frac{\mu P_{0,1,0}(\lambda\mu - 2\gamma(\mu+\gamma))}{\lambda\mu^2 + 2\gamma(\mu+\gamma)(\lambda-\mu)} = \frac{\lambda(2\gamma(\mu+\gamma) - \lambda\mu)}{\gamma(2\gamma(\mu+\gamma) + 2\lambda\gamma + \lambda\mu)}. \quad (3.5\text{-}2)$$

Finally, for $J = 2, P(J = 2) = G_2(1) = \sum_{n=0}^{\infty} P_{n,2,0} = P2$. Substituting (3.3-6) in (3.3-3) we obtain the value of $P(J = 2)$:

$$G_2(1) = \frac{-\lambda\mu P_{0,1,0}(\mu+\gamma)}{\lambda\mu^2 + 2\gamma(\mu+\gamma)(\lambda-\mu)} = \frac{\lambda^2(\mu+\gamma)}{\gamma(2\gamma(\mu+\gamma) + 2\lambda\gamma + \lambda\mu)}. \quad (3.5\text{-}3)$$

Clearly, $E[J] = \sum_{j=0}^{2} jP(J = j)$.

The distribution of $X$ is derived with the aid of Figure 2. Examining the state points there, we readily write

$$\begin{aligned} P(X=0) &= P(J=2) + P_{0,0,0} + P_{0,1,0}, \\ P(X=1) &= P(J=1) - P_{0,1,0} + P_{0,0,1}, \\ P(X=2) &= P(J=0) - P_{0,0,0} - P_{0,0,1}. \end{aligned} \quad (3.5\text{-}4)$$

Clearly, $1 = \sum_{x=0}^{2} P(X = x) = \sum_{j=0}^{2} P(J = j)$.

Finally, $E(X) = \sum_{x=1}^{2} xP(X = x)$.

Next we derive the stability condition for the system.

Since $G_2(1) > 0$ we must have

$$\lambda\mu^2 + 2\gamma(\mu+\gamma)(\lambda-\mu) < 0. \quad (3.5\text{-}5)$$

This implies, from (3.5-3), that

$$\mu P_{0,1,0}(\lambda\mu - 2\gamma(\mu+\gamma)) < 0 \quad (3.5\text{-}6)$$

and, from (3.5-2),

$$\mu P_{0,0,0}(\lambda(\lambda+\mu) - 2\gamma(\mu+\gamma)) < 0. \quad (3.5\text{-}7)$$

Since $P_{0,1,0} > 0$, equation (3.5-5) implies that

$$2\gamma(\mu+\gamma) - \lambda\mu > 0. \tag{3.5-8}$$

Similarly, from (3.5-6),

$$2\gamma(\mu+\gamma) - \lambda(\lambda+\mu) > 0. \tag{3.5-9}$$

Clearly, (3.5-8) implies (3.5-7).

We now show that condition (3.5-4) implies condition (3.5-8).

From (3.5-4) it follows that $\lambda < \mu$ and $2\gamma(\mu+\gamma) > \frac{\lambda\mu^2}{\mu-\lambda}$.

The last inequalities leads to (3.5-8) since

$$2\gamma(\mu+\gamma) - \lambda(\lambda+\mu) > \frac{\lambda\mu^2}{\mu-\lambda} - \lambda(\lambda+\mu) = \frac{\lambda^3}{\mu-\lambda} > 0.$$

That is, the condition for stability is

$$2\gamma(\mu+\gamma)(\mu-\lambda) - \lambda\mu^2 > 0. \tag{3.5-10}$$

Define $\rho = \frac{\lambda}{\mu}$ and $\theta = \frac{\gamma}{\mu}$. Then, from (3.5-9), by dividing by $\mu^3$, we get, after some calculations,

$$\rho < 1 - \frac{1}{1+2\theta+2\theta^2} = 1 - \frac{1}{1+2\frac{\gamma}{\mu}+2\left(\frac{\gamma}{\mu}\right)^2}. \tag{3.5-11}$$

That is, $\lambda < \mu$ is *in sufficient* for stability.

For the case when $\gamma \to \infty$, i.e. $\frac{1}{\gamma} \to 0$ (no delay), the above condition reduces to the usual $M/M/1$ stability condition $\rho = \frac{\lambda}{\mu} < 1$.

Moreover, if $\gamma \to \infty$, then $P_{0,1,0} \to 0$. That is, there is never a satellite customer. Also, $P_{0,0,0} \to \frac{\mu-\lambda}{\mu} = 1 - \frac{\lambda}{\mu}$ and $P_{0,0,1} \to \rho(1-\rho)$, giving the fraction of idle time and the probability of a single customer present, respectively, in the regular M/M/1 queue.

In case $\mu \to \infty$, $X \to 0$ and the system converges to a M/M/2-type queue with arrival rate $\lambda$ and 'service' rate $\gamma$. The non-zero states are (0, 0, 0), (0, 1, 0) and $(n, 2, 0)$ for $n \geq 0$.

A straight calculation of the $M/M/2$ queue leads to $P_{0,0,0} = \frac{2\gamma-\lambda}{2\gamma+\lambda} = \frac{1-\rho}{1+\rho}$ for $\rho = \frac{\lambda}{2\gamma}$.

In the case where the mean delay gets large, it is clear that the intermediate buffer will be empty most of the time and the controller's queue will behave close to a $M/G/1$ ($M/G/c$) queue with service time equal to the sum of $\frac{1}{\mu} + \frac{1}{\gamma}$. Suppose $\gamma = \alpha\mu$ with $\alpha < 1$. Then the stability condition is $\rho < 1 - \frac{1}{1+2\alpha+2\alpha^2} \equiv \beta$. Clearly, $\beta \to 0$ when $\alpha \to 0$, implying that the system becomes unstable for any positive $\lambda$.

### 3.6. A Matrix–Geometric Approach

The QBD queing system with intermediate buffer and delayed information can also be analyzed via Neuts's matrix–geometric approach. We define a two-dimensional continuous-time Markov chain with state space $(n, (j, x))$. We arrange the states in the following order: $\{(0, (0, 0)), (0, (1, 0)), (0, (0, 1))\}, \dots, \{(n, (2, 0)), (n, (1, 1)), (n, (0, 2))\}$, where $n = 0, 1, 2, \dots$ This order yields a block-diagonal transition matrix (see Chap. 6, in Latouche and Ramaswami[4]), as follows:

$$Q = \begin{bmatrix} B & B_0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & . \end{bmatrix},$$

where each block is of dimension $3 \times 3$, and

$$B = \begin{bmatrix} -\lambda & 0 & \lambda \\ \gamma & -(\lambda+\gamma) & 0 \\ 0 & \mu & -(\lambda+\mu) \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2\gamma & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+2\gamma) & 0 & 0 \\ \mu & -(\lambda+\mu+\gamma) & 0 \\ 0 & \mu & -(\lambda+\mu) \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The stability condition for such a system is given in Neuts[5] as

$$\pi A_2 e > \pi A_0 e, \tag{3.6-1}$$

where $e$ is the unit column vector, $\pi$ is the stationary probability vector of the matrix

$$A = A_2 + A_1 + A_0 = \begin{bmatrix} -2\gamma & 2\gamma & 0 \\ \mu & -(\mu+\gamma) & \gamma \\ 0 & \mu & -\mu \end{bmatrix}, \quad \text{and} \quad \pi = (\pi_0, \pi_1, \pi_2).$$

The vector $\pi$ satisfies

$$\begin{cases} \pi A = 0, \\ \pi_0 + \pi_1 + \pi_2 = 1. \end{cases} \tag{3.6-2}$$

It then follows that, setting $\theta = \frac{\gamma}{\mu}$,

$$\pi_0 = \frac{1}{s}, \quad \pi_1 = \frac{2\theta}{s}, \quad \pi_2 = \frac{2\theta^2}{s} \quad \text{and} \quad s = 1 + 2\theta + 2\theta^2. \tag{3.6-3}$$

Now, $\pi A_2 e = \gamma(2\pi_0 + \pi_1)$, $\pi A_0 e = \lambda$. Substituting this expression into (3.6-1), the stability condition is $\lambda < \frac{2\gamma(1+\theta)}{1+2\theta+2\theta^2}$.

Dividing by $\mu$ and setting $\rho = \frac{\lambda}{\mu}$ leads to

$$\rho < \frac{2\theta + 2\theta^2}{1 + 2\theta + 2\theta^2} = 1 - \frac{1}{1 + 2\theta + 2\theta^2}. \tag{3.6-4}$$

Indeed, condition (3.6-4) is nothing but condition (3.5-11).

## 4. THE CASE $c = 2$

In this section we analyze the two-server $M/M/2$-type queue with intermediate buffer of size $c = 2$.

The model is depicted in Figure 3.

Again, the state space is $\{N, J, X\}$ with the following interpretation:

$x = 0$ denotes idle servers; $x = 1$ indicates that only one server is busy; $x = 2$ indicates that both servers are busy; when $x = 3$, two customers are being served and another is waiting in the intermediate buffer; when $x = 4$, two customers are being served and two are waiting. Note, that the controller only knows the sum $X + J \leq 4$, without having a precise knowledge of the specific values of $X$ or $J$.

A transition-rate diagram is depicted in Figure 4.

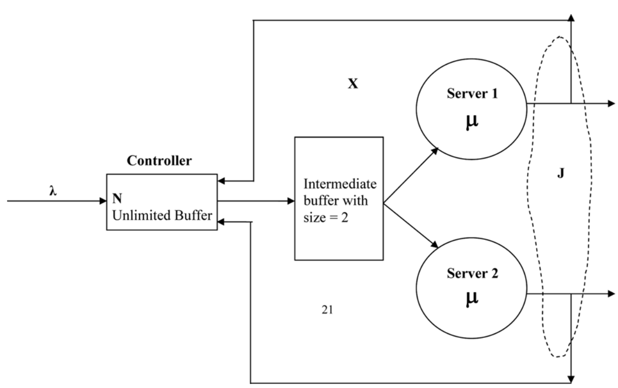


**FIGURE 3** A two-server model with intermediate buffer of size = 2.

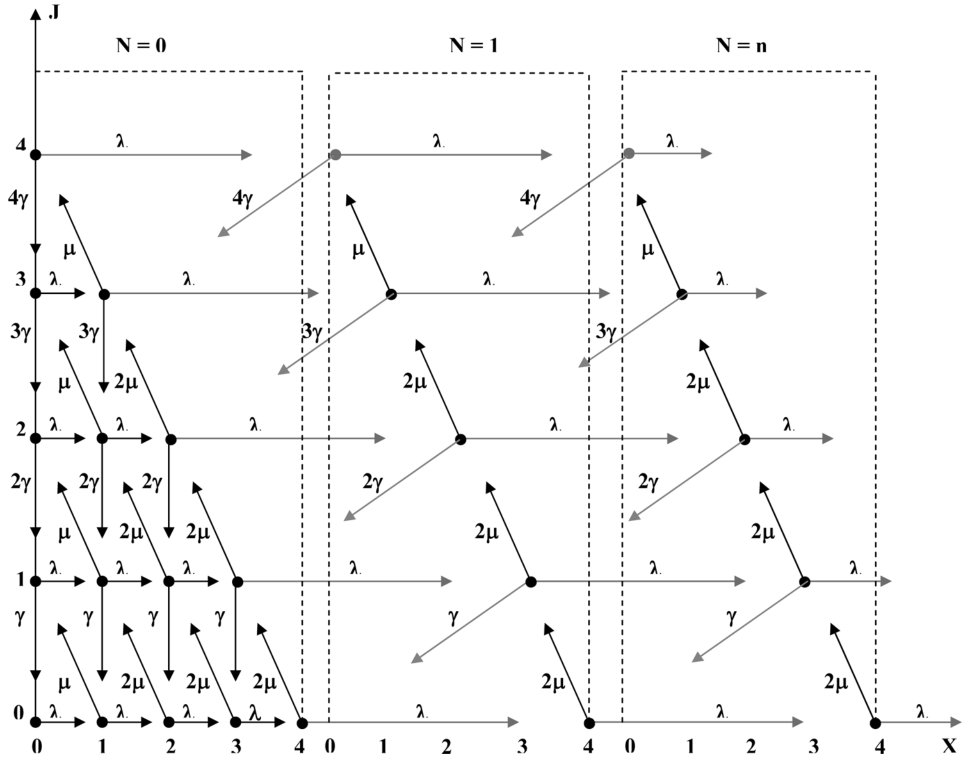


**FIGURE 4** Transition-rate diagram for the two-server case.

### 4.1. Balance Equation

Investigating the structure of the transition-rate diagram we see that, for each $n \geq 0$, the 5 states for which $x + j = 4$ (namely, $(n, 0, 4), (n, 1, 3), (n, 2, 2), (n, 3, 1), (n, 4, 0)$) repeat themselves. The 10 states (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1) and (0, 3, 0) are different. Together with (0, 4, 0) are denoted as 'boundary states'.

$$\text{Let } P_{n,j,x} = \text{Prob}(N = n, J = j, X = x),$$
$$n = 0, 1, 2, 3 \ldots; \;\; j = 0, 1, 2, 3, 4; \;\; x = 0, 1, 2, 3, 4.$$

Then, for each value of $N$ $(N = 0, 1, 2, \ldots, n, \ldots)$, the set of balance equations is the following:

For $n = 0$, the equations for the first 10 boundary states satisfy

$$\begin{cases} \lambda P_{0,0,0} = \gamma P_{0,1,0} & j = x = 0, \\ (\lambda + \mu)P_{0,0,1} = \lambda P_{0,0,0} + \gamma P_{0,1,1} & j = 0, x = 1, \\ (\lambda + 2\mu)P_{0,0,2} = \lambda P_{0,0,1} + \gamma P_{0,1,2} & j = 0, x = 2, \\ (\lambda + 2\mu)P_{0,0,3} = \lambda P_{0,0,2} + \gamma P_{0,1,3} & j = 0, x = 3, \\ (\lambda + \gamma)P_{0,1,0} = \mu P_{0,0,1} + 2\gamma P_{0,2,0} & j = 1, x = 0, \\ (\lambda + \mu + \gamma)P_{0,1,1} = \lambda P_{0,1,0} + 2\mu P_{0,0,2} + 2\gamma P_{0,2,1} & j = x = 1, \\ (\lambda + 2\mu + \gamma)P_{0,1,2} = \lambda P_{0,1,1} + 2\mu P_{0,0,3} + 2\gamma P_{0,2,2} & j = 1, x = 2, \\ (\lambda + 2\gamma)P_{0,2,0} = \mu P_{0,1,1} + 3\gamma P_{0,3,0} & j = 2, x = 0, \\ (\lambda + \mu + 2\gamma)P_{0,2,1} = \lambda P_{0,2,0} + 2\mu P_{0,1,2} + 3\gamma P_{0,3,1} & j = 2, x = 1, \\ (\lambda + 3\gamma)P_{0,3,0} = \mu P_{0,2,1} + 4\gamma P_{0,4,0} & j = 3, x = 0. \end{cases} \tag{4.1-1}$$

Note that all equations in (4.1-1) include only probabilities for which $n = 0$ in both sides of each equation. Such property holds also fore the last equation in the following set (4.1-2).

For $n = 0$ and $j + x = 4$,

$$\begin{cases} (\lambda + 2\mu)P_{0,0,4} = \lambda P_{0,0,3} + \gamma P_{1,1,3} & j = 0, x = 4, \\ (\lambda + 2\mu + \gamma)P_{0,1,3} = \lambda P_{0,1,2} + 2\mu P_{0,0,4} + 2\gamma P_{1,2,2} & j = 1, x = 3, \\ (\lambda + 2\mu + 2\gamma)P_{0,2,2} = \lambda P_{0,2,1} + 2\mu P_{0,1,3} + 3\gamma P_{1,3,1} & j = x = 2, \\ (\lambda + \mu + 3\gamma)P_{0,3,1} = \lambda P_{0,3,0} + 2\mu P_{0,2,2} + 4\gamma P_{1,4,0} & j = 3, x = 1, \\ (\lambda + 4\gamma)P_{0,4,0} = \mu P_{0,3,1} & j = 4, x = 0. \end{cases} \tag{4.1-2}$$

In general, for $N = n \geq 1$ (where $j + x = 4$),

$$\begin{cases} (\lambda + 2\mu)P_{n,0,4} = \lambda P_{n-1,0,4} + \gamma P_{n+1,1,3} & j = 0, x = 4, \\ (\lambda + 2\mu + \gamma)P_{n,1,3} = \lambda P_{n-1,1,3} + 2\mu P_{n,0,4} + 2\gamma P_{n+1,2,2} & j = 1, x = 3, \\ (\lambda + 2\mu + 2\gamma)P_{n,2,2} = \lambda P_{n-1,2,2} + 2\mu P_{n,1,3} + 3\gamma P_{n+1,3,1} & j = x = 2, \\ (\lambda + \mu + 3\gamma)P_{n,3,1} = \lambda P_{n-1,3,1} + 2\mu P_{n,2,2} + 4\gamma P_{n+1,4,0} & j = 3, x = 1, \\ (\lambda + 4\gamma)P_{n,4,0} = \lambda P_{n-1,4,0} + \mu P_{n,3,1} & j = 4, x = 0. \end{cases} \tag{4.1-3}$$

### 4.2. Partial General Functions

For each level of the satellite customers $J = 0, 1, 2, 3, 4$, we define the corresponding (partial) generating function (PGF) as follows:

$$G_0(z) = P_{0,0,0} + P_{0,0,1} + P_{0,0,2} + P_{0,0,3} + \sum_{n=0}^{\infty} P_{n,0,4} z^n. \tag{4.2-1}$$

$$G_1(z) = P_{0,1,0} + P_{0,1,1} + P_{0,1,2} + \sum_{n=0}^{\infty} P_{n,1,3} z^n. \tag{4.2-2}$$

$$G_2(z) = P_{0,2,0} + P_{0,2,1} + \sum_{n=0}^{\infty} P_{n,2,2} z^n. \tag{4.2-3}$$

$$G_3(z) = P_{0,3,0} + \sum_{n=0}^{\infty} P_{n,3,1} z^n. \tag{4.2-4}$$

$$G_4(z) = \sum_{n=0}^{\infty} P_{n,4,0} z^n. \tag{4.2-5}$$

Now, for $j = 0$ and $x = 4$, we obtain from the balance equations

$$(\lambda + 2\mu) \sum_{n=0}^{\infty} P_{n,0,4} z^n = \lambda P_{0,0,3} + \lambda \sum_{n=0}^{\infty} P_{n,0,4} z^{n+1} + \frac{\gamma}{z} \sum_{n=0}^{\infty} P_{n,1,3} z^n. \tag{4.2-6}$$

This implies

$$\begin{aligned} &z(\lambda(1-z) + 2\mu) G_0(z) - \gamma G_1(z) \\ &\quad = z(P_{0,0,0} + P_{0,0,1} + P_{0,0,2} + P_{0,0,3})(\lambda(1-z) + 2\mu) \\ &\qquad + \lambda z P_{0,0,3} - \gamma(P_{0,1,0} + P_{0,1,1} + P_{0,1,2} + P_{0,1,3}). \end{aligned} \tag{4.2-7}$$

Similarly, when $j = 1$ and $x = 3$, we get

$$\begin{aligned} &(\lambda + 2\mu + \gamma) \sum_{n=0}^{\infty} P_{n,1,3} z^n \\ &\quad = \lambda P_{0,1,2} + \lambda \sum_{n=0}^{\infty} P_{n,1,3} z^{n+1} + 2\mu \sum_{n=0}^{\infty} P_{n,0,4} z^n + 2\gamma \sum_{n=1}^{\infty} P_{n,2,2} z^{n-1}. \end{aligned} \tag{4.2-8}$$

That is,

$$\begin{aligned} &-2\mu z G_0(z) + (\lambda(1-z) + 2\mu + \gamma) z G_1(z) - 2\gamma G_2(z) \\ &\quad = z(P_{0,1,0} + P_{0,1,1} + P_{0,1,2})(\lambda(1-z) + 2\mu + \gamma) + \lambda z P_{0,1,2} \\ &\qquad - 2\mu z(P_{0,0,0} + P_{0,0,1} + P_{0,0,2} + P_{0,0,3}) - 2\gamma(P_{0,2,0} + P_{0,2,1} + P_{0,2,2}). \end{aligned} \tag{4.2-9}$$

When $j = 2$ and $x = 2$, we obtain

$$
\begin{aligned}
&(\lambda + 2\mu + 2\gamma)\sum_{n=0}^{\infty} P_{n,2,2}z^n \\
&\quad = \lambda P_{0,2,1} + \lambda \sum_{n=0}^{\infty} P_{n,2,2}z^{n+1} + 2\mu \sum_{n=0}^{\infty} P_{n,1,3}z^n + 3\gamma \sum_{n=1}^{\infty} P_{n,3,1}z^{n-1}. \quad (4.2\text{-}10)
\end{aligned}
$$

The above leads to

$$
\begin{aligned}
&-2\mu z G_1(z) + (\lambda(1-z) + 2\mu + 2\gamma) z G_2(z) - 3\gamma G_3(z) \\
&\quad = z(P_{0,2,0} + P_{0,2,1})(\lambda(1-z) + 2\mu + 2\gamma) + \lambda z P_{0,2,1} \\
&\qquad - 2\mu z(P_{0,1,0} + P_{0,1,1} + P_{0,1,2}) - 3\gamma(P_{0,3,0} + P_{0,3,1}). \quad (4.2\text{-}11)
\end{aligned}
$$

When $j = 3$ and $x = 1$, we get

$$
\begin{aligned}
&(\lambda + 2\mu + 3\gamma)\sum_{n=0}^{\infty} P_{n,3,1}z^n \\
&\quad = \lambda P_{0,3,0} + \lambda \sum_{n=0}^{\infty} P_{n,3,1}z^{n+1} + 2\mu \sum_{n=0}^{\infty} P_{n,2,2}z^n + 4\gamma \sum_{n=1}^{\infty} P_{n,4,0}z^{n-1}, \quad (4.2\text{-}12)
\end{aligned}
$$

leading to

$$
\begin{aligned}
&-2\mu z G_2(z) + (\lambda(1-z) + \mu + 3\gamma) z G_3(z) - 4\gamma G_4(z) \\
&\quad = z P_{0,3,0}(\lambda(1-z) + \mu + 3\gamma) + \lambda z P_{0,3,0} \\
&\qquad - 2\mu z(P_{0,2,0} + P_{0,2,1}) - 4\gamma P_{0,4,0}. \quad (4.2\text{-}13)
\end{aligned}
$$

Finally, for $j = 4$, we obtain

$$
(\lambda + 4\gamma)\sum_{n=0}^{\infty} P_{n,4,0}z^n = \lambda \sum_{n=0}^{\infty} P_{n,4,0}z^{n+1} + \mu \sum_{n=0}^{\infty} P_{n,3,1}z^n. \quad (4.2\text{-}14)
$$

That is,

$$
-\mu G_3(z) + (\lambda(1-z) + 4\gamma) G_4(z) = -\mu P_{0,3,0}. \quad (4.2\text{-}15)
$$

Equations (4.2-7), (4.2-9), (4.2-11), (4.2-13) and (4.2-15) define a set of linear equations with unknowns $G_0(z)$, $G_1(z)$, $G_2(z)$, $G_3(z)$ and $G_4(z)$, depending on the 11 boundary probabilities as well as on the probabilities $P_{0,1,3}$, $P_{0,2,2}$ and $P_{0,3,1}$. Knowledge of these 14 probabilities fully determines the PGFs.

However, we have only 11 equations in those 14 probabilities: 10 equations are given by the set (4.1-1) and the 11th by the 5th equation in (4.1-2).

In order to solve for the unknown probabilities we proceed, similarly to Sections 3.2 and 3.3, as follows:

It can be shown that repeating the 'cutting' method used successfully in Sections 3.2 and 3.3 yields only 17 equations with 19 unknown probabilities, implying that a closed form result is un-attainable for the case of $c \geq 2$ servers, when using this method. We therefore turn to utilize the set of 5 equations derived for the PGFs $G_j(z)$ for $j = 0, 1, 2, 3, 4$.

## 4.3. Matrix Representation

Considering the right-hand side of (4.2-7) we define

$$b_0(z) = z(P_{0,0,0} + P_{0,0,1} + P_{0,0,2} + P_{0,0,3})(\lambda(1-z) + 2\mu) + \lambda z P_{0,0,3} - \gamma(P_{0,1,0} + P_{0,1,1} + P_{0,1,2} + P_{0,1,3}). \tag{4.3-1}$$

Using (4.2-9) we define

$$b_1(z) = z(P_{0,1,0} + P_{0,1,1} + P_{0,1,2})(\lambda(1-z) + 2\mu + \gamma) - 2\gamma(P_{0,2,0} + P_{0,2,1} + P_{0,2,2}) + \lambda z P_{0,1,2} - 2\mu z(P_{0,0,0} + P_{0,0,1} + P_{0,0,2} + P_{0,0,3}). \tag{4.3-2}$$

From (4.2-11) we have

$$b_2(z) = z(P_{0,2,0} + P_{0,2,1})(\lambda(1-z) + 2\mu + 2\gamma) - 3\gamma(P_{0,3,0} + P_{0,3,1}) + \lambda z P_{0,2,1} - 2\mu z(P_{0,1,0} + P_{0,1,1} + P_{0,1,2}). \tag{4.3-3}$$

From (4.2-13)

$$b_3(z) = zP_{0,3,0}(\lambda(1-z) + \mu + 3\gamma) + \lambda z P_{0,3,0} - 2\mu z(P_{0,2,0} + P_{0,2,1}) - 4\gamma P_{0,4,0}. \tag{4.3-4}$$

Finally, from (4.2-15)

$$b_4(z) = -\mu P_{0,3,0}. \tag{4.3-5}$$

Combining equations (4.2-7), (4.2-9), (4.2-11), (4.2-13) and (4.2-15) with (4.3-1) to (4.3-5), we obtain the following system of linear equations in the unknowns $G_j(z)$:

$$\begin{cases} z(\lambda(1-z)+2\mu)G_0(z) - \gamma G_1(z) = b_0(z), \\ -2\mu z G_0(z) + (\lambda(1-z)+2\mu+\gamma)zG_1(z) - 2\gamma G_2(z) = b_1(z), \\ -2\mu z G_1(z) + (\lambda(1-z)+2\mu+2\gamma)zG_2(z) - 3\gamma G_3(z) = b_2(z), \\ -2\mu z G_2(z) + (\lambda(1-z)+\mu+3\gamma)zG_3(z) - 4\gamma G_4(z) = b_3(z), \\ -\mu G_3(z) + (\lambda(1-z)+4\gamma)G_4(z) = b_4(z). \end{cases} \tag{4.3-6}$$

For compactness we set

$$a_0(z) = \lambda(1-z) + 2\mu, \tag{4.3-7}$$

$$a_1(z) = \lambda(1-z) + 2\mu + \gamma, \tag{4.3-8}$$

$$a_2(z) = \lambda(1-z) + 2\mu + 2\gamma, \tag{4.3-9}$$

$$a_3(z) = \lambda(1-z) + \mu + 3\gamma, \tag{4.3-10}$$

$$a_4(z) = \lambda(1-z) + 4\gamma. \tag{4.3-11}$$

The system (4.3-6) can now be presented in a matrix form, similarly to (3.4-1), as

$$\begin{pmatrix} za_0(z) & -\gamma & 0 & 0 & 0 \\ -2\mu z & za_1(z) & -2\gamma & 0 & 0 \\ 0 & -2\mu z & za_2(z) & -3\gamma & 0 \\ 0 & 0 & -2\mu z & za_3(z) & -4\gamma \\ 0 & 0 & 0 & -\mu & a_4(z) \end{pmatrix} \begin{pmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \\ G_3(z) \\ G_4(z) \end{pmatrix} = \begin{pmatrix} b_0(z) \\ b_1(z) \\ b_2(z) \\ b_3(z) \\ b_4(z) \end{pmatrix}. \tag{4.3-12}$$

Define

$$A(z) = \begin{pmatrix} za_0(z) & -\gamma & 0 & 0 & 0 \\ -2\mu z & za_1(z) & -2\gamma & 0 & 0 \\ 0 & -2\mu z & za_2(z) & -3\gamma & 0 \\ 0 & 0 & -2\mu z & za_3(z) & -4\gamma \\ 0 & 0 & 0 & -\mu & a_4(z) \end{pmatrix}. \tag{4.3-13}$$

Then (4.3-12) is written as $A(z)G(z) = b(z)$, where $G(z)$ and $b(z)$ are each a 5-dimensinal column vector. By Cramer's rule we can write

$$G_j(z) = \frac{|A_j(z)|}{|A(z)|} \quad \text{for } j = 0, 1, 2, 3, 4, \tag{4.3-14}$$

where, as before, $A_j(z)$ is obtained from $A(z)$ by replacing the $j$th column by $b(z)$.

Therefore, if there exists $z_0 \in [0,1]$ such that $|A(z_0)| = 0$, then $|A_0(z_0)|$, $|A_1(z_0)|$, $|A_2(z_0)|$, $|A_3(z_0)|$ and $|A_4(z_0)|$ must equal 0 as well.

A very tedious determinant's calculations, together with an interlacing analysis of the roots of

$$|A(z)| \text{ shows that there exists a } \textit{unique} \text{ solution } z_0 \in (0,1) \text{ for which } |A(z_0)| = 0 \tag{4.3-15}$$

(see also Section 5 where $c > 2$).

Hence, $|A_j(z_0)| = 0$ yields an additional equation in the unknown probabilities. Note that, the equations $|A_j(z_0)| = 0, j = 0,1,2,3,4,$ are linearly dependent (Levy and Yechiali [2]), and therefore yield only one equation.

Going back to $|A(z)|$, it can be shown that $|A(z)| = (1-z)z^2D(z)$, (4.3-16)

where $D(z)$ is the following polynom of degree 6 (calculated with the aid of Maple8).

$$\begin{aligned} D(z) = & -20\lambda^2\mu^3z^3 + 20\lambda^2\mu^3z^2 + 8\lambda\mu^4z^2 - 32\mu^4\gamma z + 18\lambda^3\mu^2z^4 - 7\lambda^4\mu z^5 \\ & + 35\lambda^3\gamma^2z^4 - 48\mu\gamma^4z - 144\mu^3\gamma^2z - 21\lambda^4\mu z^3 + 24\lambda\gamma^4z^2 + 50\lambda^2\gamma^3z^2 \\ & - 160\mu^2\gamma^3z + 7\lambda^4\mu z^2 + 36\lambda\mu^2\gamma^2 - 70\lambda^3\gamma^2z^3 + 18\lambda^3\mu^2z^2 + 35\lambda^3\gamma^2z^2 \\ & - 30\lambda^4\gamma z^3 + 30\lambda^4\gamma z^4 - 50\lambda^2\gamma^3z^3 + 21\lambda^4\mu z^4 + 10\lambda^4\gamma z^2 - 36\lambda^3\mu^2z^3 \\ & - 10\lambda^4\gamma z^5 - 4\lambda^5z^5 - 4\lambda^5z^3 + 48\mu^3\gamma^2 + 64\mu^2\gamma^3 + 6\lambda^5z^4 + \lambda^5z^6 + \lambda^5z^2 \\ & - 138\lambda^3\mu\gamma z^3 - 132\lambda^2\mu^2\gamma z^3 - 174\lambda^2\mu\gamma^2z^3 + 61\lambda^3\mu\gamma z^4 + 280\lambda\mu^2\gamma^2z^2 \\ & + 116\lambda\mu^3\gamma z^2 + 198\lambda^2\mu^2\gamma z^2 + 262\lambda^2\mu\gamma^2z^2 + 180\lambda\mu\gamma^3z^2 + 93\lambda^3\mu\gamma z^2 \\ & - 268\lambda\mu^2\gamma^2z - 84\lambda\mu^3\gamma z - 66\lambda^2\mu^2\gamma z - 88\lambda^2\mu\gamma^2z - 132\lambda\mu\gamma^3z - 16\lambda^3\mu\gamma z. \end{aligned} \tag{4.3-17}$$

Clearly, $|A(z)|$ has a root at $z = 1$ and double roots at $z = 0$. Therefore, according to (4.3-14), there exist polynoms $D_j(z)$, $j = 0,1,2,3,4,$ satisfying

$$|A_j(z)| = (1-z)z^2D_j(z). \tag{4.3-18}$$

Considering $|A_0(z)|$, we write $A_0^1(z) = \frac{|A_0(z)|}{z(1-z)} = zD_0(z)$. Now, for $z = 0$, $zD_0(z) = 0$, so that $A_0^1(z)|_{z=0} = 0$. Calculating $A_0^1(z)$, it follows that $A_0^1(0) = 0$

if and only if the following condition holds:

$$\begin{aligned}&-21\lambda\gamma P_{0,3,1}-3\lambda^2P_{0,3,1}-12\lambda\gamma P_{0,2,1}+3\lambda^2P_{0,3,0}-4\lambda\mu P_{0,2,1}\\&\quad-3\lambda\mu P_{0,3,1}+10\lambda\mu P_{0,2,2}+8\mu^2P_{0,2,2}-8\mu^2P_{0,1,3}+12\gamma^2P_{0,1,3}\\&\quad+24\gamma^2P_{0,2,2}-12\lambda\gamma P_{0,0,3}-12\lambda\gamma P_{0,1,2}+32\mu\gamma P_{0,2,2}=0.\end{aligned}\tag{4.3-19}$$

Clearly, result (4.3-19) gives another equation in the unknown probabilities.

## 4.4. Solving the Model

We now have 13 equations in the 14 unknown probabilities, where 10 equations are given by the set (4.1-1); the 11th is the 5th equation in (4.1-2) where $n=0, j=4$ and $x=0$; the 12th is given by (4.3-19); and the 13th is $|A_j(z_0)|=0$. Adding the 'total probability' equation $\sum_{j=0}^{4}G_j(1)=1$, where $G_j(1)=\frac{|A_j(1)|}{|A(1)|}$, we have a set of 14 linear equations as follows:

$$\begin{cases}\lambda P_{0,0,0}=\gamma P_{0,1,0},\\(\lambda+\mu)P_{0,0,1}=\lambda P_{0,0,0}+\gamma P_{0,0,1},\\(\lambda+2\mu)P_{0,0,2}=\lambda P_{0,0,1}+\gamma P_{0,1,2},\\(\lambda+2\mu)P_{0,0,3}=\lambda P_{0,0,2}+\gamma P_{0,1,3},\\(\lambda+\gamma)P_{0,1,0}=\mu P_{0,0,1}+2\gamma P_{0,2,0},\\(\lambda+\mu+\gamma)P_{0,1,1}=\lambda P_{0,1,0}+2\mu P_{0,0,2}+2\gamma P_{0,2,1},\\(\lambda+2\mu+\gamma)P_{0,1,2}=\lambda P_{0,1,1}+2\mu P_{0,0,3}+2\gamma P_{0,2,2},\\(\lambda+2\gamma)P_{0,2,0}=\mu P_{0,1,1}+3\gamma P_{0,3,0},\\(\lambda+\mu+2\gamma)P_{0,2,1}=\lambda P_{0,2,0}+2\mu P_{0,1,2}+3\gamma P_{0,3,1},\\(\lambda+3\gamma)P_{0,3,0}=\mu P_{0,2,1}+4\gamma P_{0,4,0},\\(\lambda+4\gamma)P_{0,4,0}=\mu P_{0,3,1},\\-21\lambda\gamma P_{0,3,1}-3\lambda^2P_{0,3,1}-12\lambda\gamma P_{0,2,1}+3\lambda^2P_{0,3,0}-4\lambda\mu P_{0,2,1}\\\quad-3\lambda\mu P_{0,3,1}+10\lambda\mu P_{0,2,2},\\+8\mu^2P_{0,2,2}-8\mu^2P_{0,1,3}+12\gamma^2P_{0,1,3}+24\gamma^2P_{0,2,2}-12\lambda\gamma P_{0,0,3}\\\quad-12\lambda\gamma P_{0,1,2}+32\mu\gamma P_{0,2,2}=0,\\|A_0(z_0)|=0,\\\dfrac{1}{|A(1)|}\left[\sum_{j=0}^{4}|A_j(1)|\right]=1.\end{cases}\tag{4.4-1}$$

Note that $|A(1)|$ contains only parameters (see (4.3-13)) while the $|A_j(1)|$ determinants contain the unknown probabilities appearing in the first 11 equations in (4.4-1).

### 4.5. A Matrix–Geometric Representation

Similarly to Section 3.6 the stability condition is $\pi A_2 e > \pi A_0 e$. The matrix $A$ is given by

$$A = \begin{bmatrix} -4\gamma & 4\gamma & 0 & 0 & 0 \\ \mu & -(\mu+3\gamma) & 3\gamma & 0 & 0 \\ 0 & 2\mu & -(2\mu+2\gamma) & 2\gamma & 0 \\ 0 & 0 & 2\mu & -(2\mu+\gamma) & \gamma \\ 0 & 0 & 0 & 2\mu & -2\mu \end{bmatrix}.$$

The stationary vector $\pi$ is given by

$$\pi_0 = \frac{1}{M}, \quad \pi_1 = \frac{4\theta}{M}, \quad \pi_2 = \frac{6\theta^2}{M}, \quad \pi_3 = \frac{6\theta^3}{M}, \quad \pi_4 = \frac{3\theta^4}{M}, \quad \text{where}$$
$$M = 1 + 4\theta + 6\theta^2 + 6\theta^3 + 3\theta^4.$$

Then the stability condition $\pi A_2 e > \pi A_0 e$ results in

$$\lambda < \gamma(4\pi_0 + 3\pi_1 + 2\pi_2 + \pi_3) = \frac{\gamma(4 + 12\rho + 12\rho^2 + 6\rho^3)}{M}. \qquad (4.5\text{-}1)$$

### 4.6. Numerical Example

We take $\lambda = 1, \mu = 1, \gamma = 2$ (which satisfy (4.5-1)).
Substituting the above values in (4.3-17) yields

$$z^6 - 31z^5 + 367z^4 - 2507z^3 + 5420z^2 - 5500z + 848 = 0. \qquad (4.6\text{-}1)$$

Solving equation (4.4-3) by using Maple8 gives 6 roots:

$$z_0 = 0.1859203482, \quad z_1 = 1.983296038, \quad z_2 = 4.504664998,$$
$$z_3 = 6.386321371, \quad z_4 = 8.249933184, \quad z_5 = 9.6898606. \qquad (4.6\text{-}2)$$

Substituting $z_0 = 0.1859203482$ in the next to last equation of the system (4.4-1), we obtain the set of equations leading to the following explicit solution:

$$P_{0,0,0} = 0.2011092708, \quad P_{0,0,1} = 0.2013286358, \quad P_{0,1,0} = 0.1005546354,$$
$$P_{0,1,1} = 0.1007740004, \quad P_{0,0,2} = 0.1009616386, \quad P_{0,1,2} = 0.05077814,$$

$$P_{0,2,1} = 0.02515452226, \quad P_{0,2,0} = 0.0250838176, \quad P_{0,0,3} = 0.05074499283,$$
$$P_{0,1,3} = 0.02563666995, \quad P_{0,3,1} = 0.00404783935, \quad P_{0,3,0} = 0.0041075146,$$
$$P_{0,4,0} = 0.0004497599277, \quad P_{0,2,2} = 0.0129066784. \tag{4.6-3}$$

Substituting (4.4-5) in (4.3-21) and using $G_j(z) = \frac{D_j(z)}{D(z)}$ (see (4.3-16) and (4.3-18)) we get $G_0(1) = 0.6019$, $G_1(1) = 0.3040$, $G_2(1) = 0.0800$, $G_3(1) = 0.0130$, $G_4(1) = 0.0011$.

$$\text{Clearly, } \sum_{j=0}^{4} G_j(1) = 1. \tag{4.6-4}$$

The above yields the expected number of satellite customers

$$E[J] = \sum_{j=0}^{4} jG_j(1) = 0.5064. \tag{4.6-5}$$

Let $I$ denote the number of customers in the intermediate buffer. Then (Figure 2),

$$E(I) = 1(P_{0,0,3} + G_1(1) - P_{0,1,0} - P_{0,1,1} - P_{0,1,2}) + 2(G_0(1) - P_{0,0,0} - P_{0,0,1} - P_{0,0,2} - P_{0,0,3}).$$

This follows since, when $X = 3$ ($X = 4$) only one (two) customer(s) stay(s) in the intermediate buffer.

Thus, using (4.4-5) and (4.4-6) we get

$$E(I) = 0.1981509. \tag{4.6-6}$$

The expected number of jobs in the controller's buffer is given by

$$E(N) = \sum_{j=0}^{4} E[N|J = j]P(J = j).$$

$$\text{Since } E[N|J = j] = \frac{\frac{d}{dz}(G_j(z))|_{z=1}}{G_j(1)}, \quad \text{we have}$$

$$E[N] = \sum_{j=0}^{4} \frac{d}{dz}(G_j(z))|_{z=1} = 0.143.$$

Thus, the total number of waiting jobs, either in the controller's buffer or in the intermediate buffer, is $E[L_q] = 0.143 + 0.198 = 0.341$.

This numerical example demonstrates the *reduction* in the mean queue size accomplished by our 'intermediate buffer' model, as compared to the system analyzed in Litvak and Yechiali [3], where no intermediate buffer is used. Using the same values for the parameters $\lambda, \mu$ and $\gamma$, and applying the equation for the mean queue size $E[L_q]$ given there (page 156), we get $E[L_q = 1.49$. (Litvak and Yechiali [3]) Clearly, this last value is much larger than 0.341.

## 5. THE *c*-SERVER MODEL

In this section we analyze the $c$–server $M/M/c$-type queue with intermediate buffer of size $= c$. The model is depicted in Figure 5.

Again, the state space is $\{N, J, X\}$ with the following interpretation:

$x = 0$ denotes that all servers are idle; $1 \le x \le c$ indicates that exactly $x$ servers are busy.

When $x = c + k$, $c$ customers are being served and another $k$ customers are waiting, $k = 1, 2, \ldots, c$. Note, that the controller only knows the sum $X + J \le 2c$ without having a precise knowledge of the *specific* values of $X$ or $J$.

Transition-rate diagrams for $N = 0$ and for $N = n$ $(n = 1, 2, 3, \ldots)$ are depicted in Figures 6 and 7, respectively.

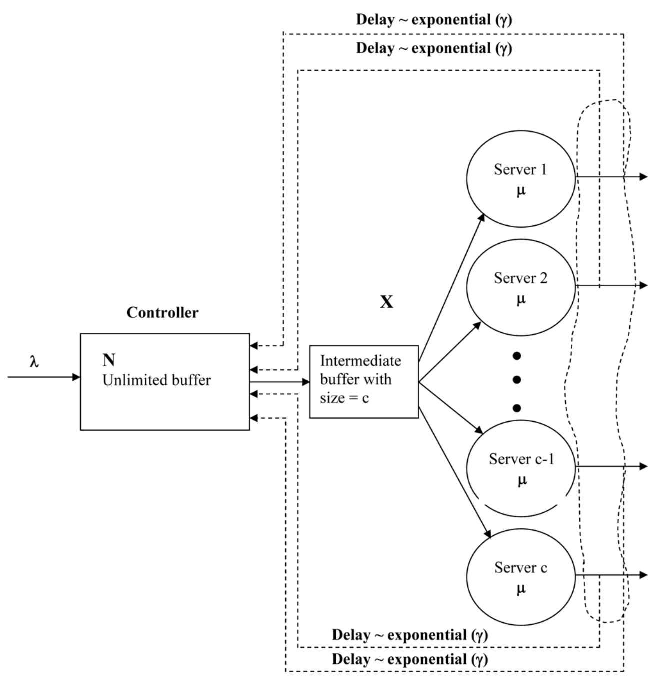


**FIGURE 5** A $c$-server model with intermediate buffer of size $= c$.

## 5.1. Balance Equations

Define $P_{n,j,x} = \text{Probability } (N = n, J = j, X = x)$, where $n = 0, \ldots \infty$, $j = 0, \ldots 2c$, $x = 0, \ldots, 2c$.

For each value of $N$ $(N = 0, 1, 2, \ldots, n, \ldots)$ we write, for $J+X \le 2c-1$, the set of corresponding balance equations, as follows. Consider Figure 6, then

$$\begin{cases} \lambda P_{0,0,0} = \gamma P_{0,1,0}, \\ (\lambda+\mu)P_{0,0,1} = \lambda P_{0,0,0} + \gamma P_{0,1,1}, \\ (\lambda+2\mu)P_{0,0,2} = \lambda P_{0,0,1} + \gamma P_{0,1,2}, \\ (\lambda+c\mu)P_{0,0,c} = \lambda P_{0,0,c-1} + \gamma P_{0,1,c}, \\ (\lambda+c\mu)P_{0,0,c+1} = \lambda P_{0,0,c} + \gamma P_{0,1,c+1}, & J=0, N=0 \\ (\lambda+c\mu)P_{0,0,2c-1} = \lambda P_{0,0,2c-2} + \gamma P_{0,1,2c-1}. & 0 \le X \le 2c-1 \end{cases} \quad (5.1\text{-}1)$$

$$\begin{cases} (\lambda+\gamma)P_{0,1,0} = \mu P_{0,0,1} + 2\gamma P_{0,2,0}, \\ (\lambda+\mu+\gamma)P_{0,1,1} = \lambda P_{0,1,0} + 2\mu P_{0,0,2} + 2\gamma P_{0,2,1}, \\ (\lambda+2\mu+\gamma)P_{0,1,2} = \lambda P_{0,1,1} + 3\mu P_{0,0,3} + 2\gamma P_{0,2,2}, \\ (\lambda+c\mu+\gamma)P_{0,1,c} = \lambda P_{0,1,c-1} + c\mu P_{0,0,c+1} + 2\gamma P_{0,2,c}, & J=1, N=0 \\ (\lambda+c\mu+\gamma)P_{0,1,c+1} = \lambda P_{0,1,c} + c\mu P_{0,0,c+2} + 2\gamma P_{0,2,c+1}, & 0 \le X \le 2c-2 \\ (\lambda+c\mu+\gamma)P_{0,1,2c-2} = \lambda P_{0,1,2c-3} + c\mu P_{0,0,2c-1} + 2\gamma P_{0,2,2c-2}. \end{cases} \quad (5.1\text{-}2)$$

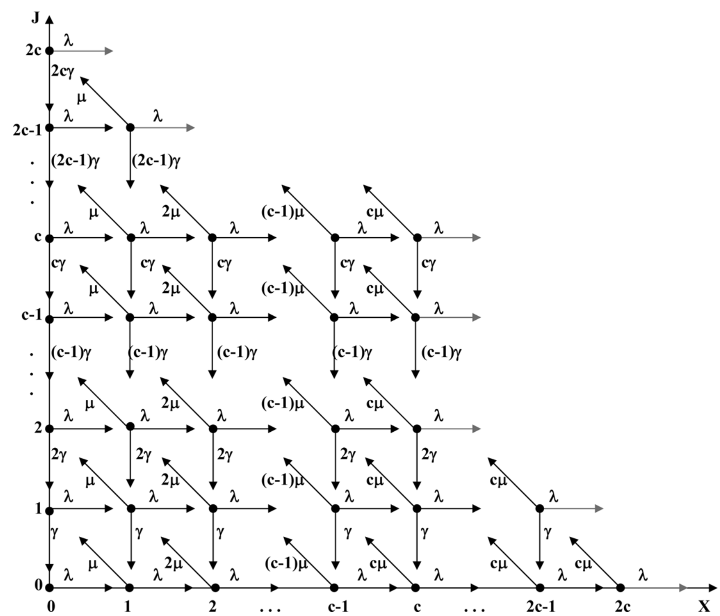


**FIGURE 6** Transition-rate diagram for the $c$-server case, $N = 0$.

$$\begin{cases}(\lambda + c\gamma)P_{0,c,0} = \mu P_{0,c-1,1} + (c+1)\gamma P_{0,c+1,0}, \\ (\lambda + \mu + c\gamma)P_{0,c,1} = \lambda P_{0,c,0} + 2\mu P_{0,c-1,2} + (c+1)\gamma P_{0,c+1,1}, \\ (\lambda + 2\mu + c\gamma)P_{0,j,2} = \lambda P_{0,c,1} + 3\mu P_{0,c-1,3} \\ \quad + (c+1)\gamma P_{0,c+1,2}, & J = c, N = 0 \\ (\lambda + (c-1)\mu + c\gamma)P_{0,c,c-1} = \lambda P_{0,c,c-2} + c\mu P_{0,c-1,c} \\ \quad + (c+1)\gamma P_{0,c+1,c-1}. & 0 \le X \le c-1 \end{cases} \tag{5.1-3}$$

$$\begin{cases}(\lambda + (2c-2)\gamma)P_{0,2c-2,0} = \mu P_{0,2c-3,1} \\ \quad + (2c-1)\gamma P_{0,2c-1,0}, & J = 2c-2, N = 0 \\ (\lambda + \mu + (2c-2)\gamma)P_{0,2c-2,1} = \lambda P_{0,2c-2,0} \\ \quad + 2\mu P_{0,2c-3,2} + (2c-1)\gamma P_{0,2c-1,1}. & 0 \le X \le 1 \end{cases} \tag{5.1-4}$$

$$(\lambda + (2c-1)\gamma)P_{0,2c-1,0} = \mu P_{0,2c-2,1} + 2c\gamma P_{0,2c,0}. \quad J = 2c-1, N = 0, X = 0 \tag{5.1-5}$$

The points on the 'diagonal' where $N = 0, J + X = 2c$ yield the following:

$$\begin{cases}(\lambda + c\mu)P_{0,0,2c} = \lambda P_{0,0,2c-1} + \gamma P_{1,1,2c-1}, & J + X = 2c, \\ (\lambda + c\mu + \gamma)P_{0,1,2c-1} = \lambda P_{0,1,2c-2} + c\mu P_{0,0,2c} + 2\gamma P_{1,2,2c-2}, & N = 0 \\ (\lambda + 2c\gamma)P_{0,2c,0} = \mu P_{0,2c-1,1}. \end{cases} \tag{5.1-6}$$

We define the set (5.1-1)–(5.1-5) together with the last equation of (5.1-6) as the "boundary equations".

Consider now Figure 7. For $N = n \ge 1$ we have

$$\begin{cases}(\lambda + c\mu)P_{n,0,2c} = \lambda P_{n-1,0,2c} + \gamma P_{n+1,1,2c-1}, \\ (\lambda + c\mu + \gamma)P_{n,1,2c-1} = \lambda P_{n-1,1,2c-1} + c\mu P_{n,0,2c} + 2\gamma P_{n+1,2,2c-2}, \\ (\lambda + c\mu + 2\gamma)P_{n,2,2c-2} = \lambda P_{n-1,2,2c-2} + c\mu P_{n,1,2c-1} \\ \quad + 3\gamma P_{n+1,3,2c-3}, \quad J + X = 2c, \\ (\lambda + c\mu + c\gamma)P_{n,c,c} = \lambda P_{n-1,c,c} + c\mu P_{n,c-1,c+1} \\ \quad + (c+1)\gamma P_{n+1,c+1,c-1}, \quad N = n, \\ (\lambda + (c-1)\mu + (c+1)\gamma)P_{n,c+1,c-1} = \lambda P_{n-1,c+1,c-1} + c\mu P_{n,c,c} \\ \quad + (c+2)\gamma P_{n+1,c+2,c-2}, \quad n = 1, 2, 3, \ldots \\ (\lambda + 2\mu + (2c-2)\gamma)P_{n,2c-2,2} = \lambda P_{n-1,2c-2,2} + 3\mu P_{n,2c-3,3} + (2c-1)\gamma P_{n+1,2c-1,1}, \\ (\lambda + \mu + (2c-1)\gamma)P_{n,2c-1,1} = \lambda P_{n-1,2c-1,1} + 2\mu P_{n,2c-2,2} + 2c\gamma P_{n+1,2c,0}, \\ (\lambda + 2c\gamma)P_{n,2c,0} = \lambda P_{n-1,2c,0} + \mu P_{n,2c-1,1}. \end{cases} \tag{5.1-7}$$

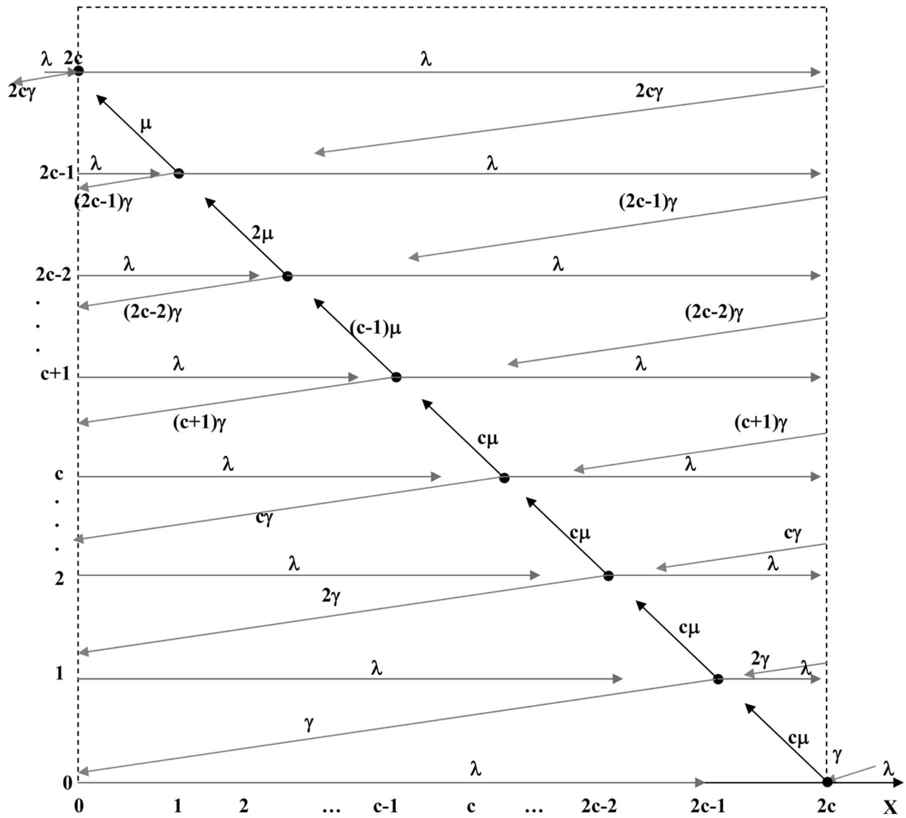


**FIGURE 7** Transition-rate diagram for the $c$-server case, $N = n > 0$.

## 5.2. Partial Generating Functions

For each level of the satellite customers $J = 0, 1, 2, \ldots, 2c$, we define the corresponding (partial) generating function (PGF) as follows:

$$
\begin{aligned}
G_0(z) &= \sum_{x=0}^{2c-1} P_{0,0,x} + \sum_{n=0}^{\infty} P_{n,0,2c} z^n, \\
G_1(z) &= \sum_{x=0}^{2c-2} P_{0,1,x} + \sum_{n=0}^{\infty} P_{n,1,2c-1} z^n, \\
G_c(z) &= \sum_{x=0}^{c-1} P_{0,c,x} + \sum_{n=0}^{\infty} P_{n,c,c} z^n, \\
G_{2c}(z) &= \sum_{n=0}^{\infty} P_{n,2c,0} z^n.
\end{aligned}
\qquad (5.2\text{-}1)
$$

To illustrate how one obtains the set of equations relating the $G_j(z)$ to each other we take $j = c$. We write

$$\begin{cases} (\lambda + c\mu + c\gamma)P_{0,c,c} = \lambda P_{0,c,c-1} + c\mu P_{0,c-1,c+1} & \\ \quad + (c+1)\gamma P_{1,c+1,c-1} & \text{from (5.1-6)}, \\ (\lambda + c\mu + c\gamma)P_{1,c,c}z = \lambda P_{0,c,c}z + c\mu P_{1,c-1,c+1}z & \\ \quad + (c+1)\gamma P_{2,c+1,c-1}z & \text{from (5.1-7)}, n = 1, \\ (\lambda + c\mu + c\gamma)P_{2,c,c}z^2 = \lambda P_{1,c,c}z^2 + c\mu P_{2,c-1,c+1}z^2 & \\ \quad + (c+1)\gamma P_{3,c+1,c-1}z^2 & \text{from (5.1-7)}, n = 2, \\ (\lambda + c\mu + c\gamma)P_{n,c,c}z^n = \lambda P_{n-1,c,c}z^n + c\mu P_{n,c-1,c+1}z^n & \\ \quad + (c+1)\gamma P_{n+1,c+1,c-1}z^n & \text{from (5.1-7)}, \end{cases}$$

Summing the above equations over $n$ we obtain

$$z(\lambda(1-z) + c\mu + c\gamma)\sum_{n=0}^{\infty} P_{n,c,c}z^n - c\mu z\sum_{n=0}^{\infty} P_{n,c-1,c+1}z^n$$
$$- (c+1)\gamma\sum_{n=0}^{\infty} P_{n,c+1,c-1}z^n = \lambda z P_{0,c,c-1} - (c+1)\gamma P_{0,c+1,c-1}.$$

This equation may be written in terms of the generating functions as

$$z(\lambda(1-z) + c\mu + c\gamma)(G_c(z) - (P_{0,c,0} + P_{0,c,1} + \cdots + P_{0,c,c-1}))$$
$$-c\mu z(G_{c-1}(z) - (P_{0,c-1,0} + P_{0,c-1,1} + \cdots + P_{0,c-1,c}))$$
$$-(c+1)\gamma(G_{c+1}(z) - (P_{0,c+1,0} + \cdots + P_{0,c+1,c-2})) = \lambda z P_{0,c,c-1} - (c+1)\gamma P_{0,c+1,c-1}.$$

That is,

$$\begin{aligned} &z(\lambda(1-z) + c\mu + c\gamma)G_c(z) - c\mu z G_{c-1}(z) - (c+1)\gamma G_{c+1}(z) \\ &\quad = \lambda z P_{0,c,c-1} - (c+1)\gamma P_{0,c+1,c-1} + z(\lambda(1-z) + c\mu + c\gamma) \\ &\qquad \times (P_{0,c,0} + P_{0,c,1} + \cdots + P_{0,c,c-1}) - (c+1)\gamma(P_{0,c+1,0} + \cdots + P_{0,c+1,c-2}) \\ &\qquad - c\mu z(P_{0,c-1,0} + P_{0,c-1,1} + \cdots + P_{0,c-1,c}). \end{aligned} \tag{5.2-2}$$

Define the right hand side of (5.2-1) as $b_c(z)$. That is,

$$\begin{aligned} &\lambda z P_{0,c,c-1} - (c+1)\gamma P_{0,c+1,c-1} + z(\lambda(1-z) + c\mu + c\gamma) \\ &\quad \times (P_{0,c,0} + P_{0,c,1} + \cdots + P_{0,c,c-1}) - (c+1)\gamma(P_{0,c+1,0} + \cdots + P_{0,c+1,c-2}) \\ &\quad - c\mu z(P_{0,c-1,0} + P_{0,c-1,1} + \cdots + P_{0,c-1,c}) = b_c(z). \end{aligned} \tag{5.2-3}$$

Also, let $a_c(z) = \lambda(1-z) + c\mu + c\gamma$. Then, (5.2-3) can be written as

$$z a_c(z) G_c(z) - c\mu z G_{c-1}(z) - (c+1)\gamma G_{c+1}(z) = b_c(z). \tag{5.2-4}$$

Similarly, define

$$
\begin{aligned}
a_0(z) &= \lambda(1-z) + c\mu,\\
a_1(z) &= \lambda(1-z) + c\mu + \gamma,\\
a_2(z) &= \lambda(1-z) + c\mu + 2\gamma,\\
a_c(z) &= \lambda(1-z) + c\mu + c\gamma,\\
a_{c+1}(z) &= \lambda(1-z) + (c-1)\mu + (c+1)\gamma,\\
a_{2c-1}(z) &= \lambda(1-z) + \mu + (2c-1)\gamma,\\
a_{2c}(z) &= \lambda(1-z) + 2c\gamma.
\end{aligned}
$$

Proceeding in a similar manner the sets of equations (5.1-6) and (5.1-7) are transformed into the matrix representation $A(z)G(z) = b(z)$, where $A(z)$ is given in Figure 8, $G(z)$ is the vector of PGFs and $b(z)$ is the $(2c+1)$ vector of right-hand side values $b_j(z)$.

We note that the determinant of $A(z)$ can be calculated recursively as follows:

Defining $B_1(z) \equiv a_{2c}(z)$,

$$B_2(z) \equiv \begin{Bmatrix} za_{2c-1}(z) & -2c\gamma \\ -\mu & a_{2c}(z) \end{Bmatrix} \text{ Then }, |B_2(z)| = za_{2c-1}(z)|B_1(z)| - 2c\mu\gamma.$$

$$B_3(z) \equiv \begin{Bmatrix} za_{2c-2}(z) & -(2c-1)\gamma & 0 \\ -c\mu z & za_{2c-1} & -2c\gamma \\ 0 & -\mu & a_{2c}(z) \end{Bmatrix} \text{ Then,}$$

$$
\begin{array}{c}
\begin{array}{ccccccccccccc}
0 & 1 & 2 & 3 & \dots & c-2 & c-1 & c & c+1 & \dots & 2c-2 & 2c-1 & 2c
\end{array}\\
A_1(z) = \left\{\begin{array}{ccccccccccccc}
za_0(z) & -\gamma & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
-c\mu z & za_1(z) & -2\gamma & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -c\mu z & za_2(z) & -3\gamma & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & -c\mu z & za_{c-1}(z) & -c\gamma & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & -c\mu z & za_c(z) & -(c+1)\gamma & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -c\mu z & za_{2c-1}(z) & -2c\gamma \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -\mu & a_{2c}(z)
\end{array}\right\}
\end{array}
$$

**FIGURE 8** $A(z)$ for the $c$-server case.

$$|B_3(z)| = z\{a_{2c-2}(z)|B_2(z)| - c(2c-1)\mu\gamma|B_1(z)|\}.$$

$B_{2c+1} = A(z)$. Now $|A(z)|$ is calculated recursively:

$$|A(z)| = |B_{2c+1}(z)| = z[a_0(z)|B_{2c}(z)| - c\mu\gamma|B_{2c-1}(z)|]. \qquad (5.2\text{-}5)$$

### 5.3. Procedure for Solving the General *c*-Server Model

For the general $c$-server case, proceeding in a similar method as for the special cases $c = 1$ and $c = 2$, presented in previous sections, we can write $\frac{(2c+1)(2c+2)}{2} - 2c = 2c^2 + c + 1$ equations for the corresponding "boundary probabilities" (Figure 6). The boundary probabilities are those for which $n = 0$ and their balance equations involve only probabilities for which $n = 0$. Indeed, for single-server case we have 4 boundary probabilities, and for the case $c = 2$ there were 11 such probabilities. On the other hand, $2c^2 + c + 1$ boundary equations contain $\frac{(2c+1)(2c+2)}{2} - 1 = 2c^2 + 3c$ different unknown probabilities. This implies that we need additional $2c - 1$ equations in order to solve the model. For $c = 1$ the extra equation was the 'total probability' equation, while for $c = 2$ we generated 3 additional equations, one of which is the 'total probability' equation. (see last 3 equations in (4.4-1)). For the general case we utilize the properties of the determinant of the matrix $A(z)$ which is a polynomial of degree $4c + 1$ ($a_{2c}(z)$ has degree 1 and $za_j(z)$ has degree 2, for $j = 0, 1, \ldots, 2c - 1$). Indeed, for $c = 1$, the degree of $|A(z)|$ is 5, while for $c = 2$ it is 9. We have the following:

**Theorem 5.3.1.** *The polynomial $|A(z)|$ has a root of multiplicity $c$ at $z = 0$; $c - 1$ distinct roots in $(0, 1)$; a single root at $z = 1$; and $2c + 1$ roots in $(1, \infty)$.*

*Proof.* From the recursive equation (5.2-5) it is easy to see that each of the polynomials $|B_{2j+1}(z)|$ and $|B_{2j+2}(z)|$, for $j = 1, 2, \ldots, c - 1$ has a root of multiplicity $j$ at $z = 0$. Also, $|B_{2c+1}(z)| = |A(z)|$ has a root of multiplicity $c$ at $z = 0$. The rest of the proof is similar to that of Theorem 3.1 in Litvak and Yechiali [3] since the matrix $A(z)$ here has exactly the same structure as the matrix $A(z)$ there, where the only real difference is that the size of $A(z)$ here is $2c + 1$ compared to $c + 1$ there (another insignificant change is that $\mu$ and $\gamma$ switch their positions in the matrix). □

The solution procedure is now concluded by deriving $c - 1$ independent equations from the $c - 1$ distinct roots in $(0, 1)$, additional $c - 1$ equations from the $c$ roots at $z = 0$ (note that for $c = 1$ there was a single root at $z = 0$, but no additional equation could be derived from this fact, and for $c = 2$ there where two roots at $z = 0$, but only one additional equation could be derived). To summarize, together with the 'total probability' equation, we obtain the required $2c-1$ additional equations. Adding the $2c^2+c+1$ boundary equations we get a set of $2c^2+3c$ equations in the $2c^2+3c$ unknown probabilities.

*Note:* When $\mu \to \infty$, the system converges to a $M/M/2c$-type queue with arrival rate $\lambda$ and 'service' rate $\gamma$ for each individual server.

## 5.4. Matrix Geometric Approach

For the general case of $c \geq 2$ the block-diagonal transition matrix $Q$ looks the same as in Section 3.6, but $A_0$, $A_1$ and $A_2$ are of order $2c+1$. Specifically, with $I$ being the unit diagonal matrix,

$$A_2 = \begin{bmatrix} 0 & 2c\gamma & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & (2c-1)\gamma & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (c+1)\gamma & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & c\gamma & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & (c-1)\gamma & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 2\gamma & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \gamma \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

$A_0 = \lambda I.$

$$A_1 = \begin{bmatrix} -(\lambda+2c\gamma) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \mu & -(\lambda+\mu+(2c-1)\gamma) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -(\lambda+(c-1)\mu+(c+1)\gamma) & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & c\mu & -(\lambda+c\mu+c\gamma) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & -(\lambda+c\mu+\gamma) & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & c\mu & -(\lambda+c\mu) \end{bmatrix}$$

The dimension of the vector $\pi$ is $2c+1$ and the solution of

$$\begin{cases} \pi A = 0, \\ \pi_0 + \pi_1 + \pi_2 + \cdots + \pi_{2c-1} + \pi_{2c} = 1 \end{cases}$$

is given by

$$\begin{aligned} \pi_1 &= 2c\theta\pi_0, \\ \pi_k &= \binom{2c}{k}\theta^k\pi_0 \quad \text{for } 0 \le k \le c, \\ \pi_{c+1} &= \theta\pi_c, \\ \pi_{c+m} &= \frac{c!}{(c-m)!}\left(\frac{\theta}{c}\right)^m \pi_c = \frac{2c!}{c!(c-m)!}\theta^c\left(\frac{\theta}{c}\right)^m, \end{aligned}$$

for $0 \le m \le c$.

The stability condition $\pi A_2 e > \pi A_0 e$ is translated into

$$\lambda < \gamma(2c\pi_0 + (2c-1)\pi_1 + \cdots + (2c-k)\pi_k + 2\pi_{2c-2} + \pi_{2c-1}). \qquad (5.4\text{-}1)$$

It is interesting to note that the probabilities $\{\pi_j : 0 \le j \le 2c\}$ are the steady state solution of a machine-repair model with $2c$ identical machines, and $c$ servers. The time until breakdown of a machine and the repair time are exponentially distributed with parameters $\gamma$ and $\mu$, respectively. $\pi_j$ denotes the probability that $j$ machines undergo repair.

## 6. CONCLUSION

In this work we have analyzed multi-server queues with intermediate buffer and delayed information on service completions. We used two methods of analysis: (1) via probability generation functions, which leads to calculating roots of a certain polynomial and using the roots in order to find the values of what we call 'boundary probabilities'. (2) via matrix–geometric approach, which enabled us to specifically calculate the stationary condition of the system for any number of servers $c \ge 1$. It is shown that by using an intermediate buffer in front of the servers, queue sizes (and waiting times) are reduced.

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