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Generalized control-limit preventive repair policies for deteriorating cold and warm standby Markovian systems

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ABSTRACT

Consider a deteriorating repairable Markovian system with N stochastically independent identical units. The lifetime of each unit follows a discrete phase-type distribution. There is one online unit and the others are in standby status. In addition, there is a single repair facility and the repair time of a failed unit has a geometric distribution. The system is inspected at equally spaced points in time. After each inspection, either repair or a full replacement is possible. We consider state-dependent operating costs, repair costs that are dependent on the extent of the repair, and failure penalty costs. Applying dynamic programming, we show that under reasonable conditions on the system's law of evolution and on the state-dependent costs, a generalized control-limit policy is optimal for the expected total discounted criterion for both cold standby and warm standby systems. Illustrative numerical examples are presented and insights are provided.

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1. Introduction

System maintenance is necessary in order to improve overall reliability, prevent system failures, and reduce costs. A common industrial and military activity is the periodic inspection of a system or of one of its units to keep it operative. After each inspection, a decision has to be made as to whether or not to alter the system's state at that time. The main objective of this article is to shed light on optimality issues regarding maintenance strategies for multi-unit repairable systems.

In this article, a repairable system with N stochastically independent and identical units (components) is analyzed. There is one operating unit online while the others are on standby, in repair, or waiting to be repaired. Upon an online unit failure, it goes to a repair queue and is immediately replaced by a standby unit (if available) that becomes the operating unit online. When the number of good units reduces to zero, the system fails.

A multi-unit k-out-of-N standby system (which is a generalization of the above 1-out-of-N system) is defined as a system with N units that is functioning when at least k of the units work; otherwise, the system fails. Note that both parallel ($k = N$) and series ($k = 1$) systems are special cases of k-out-of-N system. Examples abound in real-word systems: multiple pumps in a hydroelectric plant; computer networks with multiple servers; a four-engine aircraft that can continue to operate as long as at least two engines are working; systems in remote locations that are often designed as k-out-of-N systems, etc. A great deal of research has been devoted to k-out-of-N systems; papers by Smith and Dekker (1997), Tian *et al.* (2008), Levitin *et al.* (2014), Xie *et al.* (2014), Barron (2015), Fernández (2015), Wu *et al.* (2015), Babishin and Taghipour (2016), and Chalabi *et al.* (2016) are just a sample. Even though the reliability and availability of k-out-of-N systems have been studied in the literature, not many models have been proposed for inspection and maintenance optimization of such systems.

In a repairable system, units are replaced upon failure while failed units are sent to be repaired. A preventive maintenance policy does not wait for units to fail; rather, it replaces or repairs them in accordance with a predetermined maintenance schedule. Such preventive maintenance policies reduce unexpected failures of units and of the entire system. Condition-based maintenance decisions are scheduled based on the condition of single or multiple units and have been extensively studied and widely applied. Known results on replacement and maintenance policies were summarized in Barlow and Proschan (1965). They derived the optimal sequential policy for a replacement, using dynamic programming. The finite-time inspection model with discounted costs was discussed by Hariga and AI-Fawzan (2000). For the discrete case, the basic models introduced by Derman (1963) and Kolesar (1966) consider a system deteriorating based on a Markov process, where different states correspond to different levels of deterioration. In these models, the only possible action is the replacement of the system with a new one. They established conditions on the transition probabilities and the cost functions under which the optimal average cost criterion policy is a Control-Limit Rule (CLR). Ross (1969) generalized the above models and extended the results to the continuous state-space case. Other models, such as Kijima *et al.* (1988) and Douer and Yechiali (1994), allow partial repairs, which are cheaper than a complete replacement. For a deterioration process that changes its characteristics as the system ages, we cite Kao (1973), Stadje and Zuckerman (1991), So (1992), Lam and Yeh (1994), Benyamini and Yechiali (1999),

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and, more recently, De Smidt-Destombes *et al.* (2006), Nakagawa and Mizutani (2009), Huynh *et al.* (2012), and Bajestani and Banjevic (2016), among others. Laggounea *et al.* (2010) present a preventive maintenance model based on a partial periodic renewal policy and non-negligible replacement times. Zhong and Jin (2014) include preventive maintenance in a cold standby two-unit system using a semi-Markov process. An optimal replacement policy is developed by Zhang and Wang (2011). Ahmadi (2014) presents a generalized approach to maintenance scheduling of repairable systems whose resulting output is subject to a system's state. Zhang, Wu, Li, and Lee (2015) study maintenance policies for a multi-unit system by decomposing such a system into mutually influential single-unit systems; each single unit system is formulated as a Markov decision process, with the objective of minimizing its long-run average maintenance cost. Olde Keizer *et al.* (2016) develop a dynamic programming model to optimize the condition-based maintenance strategy for a k-out-of-N system with both economic dependencies and redundancy. Shi and Zeng (2016) present a dynamic opportunistic condition-based maintenance strategy based on real-time predictions of the remaining useful life under the simultaneous consideration of economic and stochastic dependence. Integrated models based on statistical process control (for example, control charts) and maintenance decisions for manufacturing and supply chain systems can be found in Yin *et al.* (2015), Zhang, Deng, Zhu, and Yin (2015), Gunay and Kula (2016), and Zhong *et al.* (2016). Practical examples of systems using CLRs are manufacturing machines whose components' wear arises from continual use over time. To produce items of acceptable quality, such systems must be maintained by routine inspections for detecting the system state and performing an appropriate action. As an example, consider the operation of a gas turbine. In order to maintain the system's efficiency, it is required to inspect the components' condition from time to time. Another example comes from air conditioner maintenance; due to electrostatic effects and air circulation, some units are covered with dirt and dust, which, in turn, results in the frequent overload of electronic components, a significant increase in power consumption, and, even worse, the burning out of electronic components. Hence, proper maintenance can extend an air conditioner's service life and prevent fatal damage. Another notable example is the periodic inspection of a private car, after which various degrees of maintenance are possible.

In this article, we focus on modeling and analyzing the effect of preventive maintenance policies where the failure times of the units follow a discrete phase-type distribution. Although reliability systems are usually studied in the continuous time space (Barron *et al.*, 2004; Fernández, 2015; Shue *et al.*, 2015; Wu *et al.*, 2015; Shu *et al.*, 2016), not all systems can be continuously monitored, and some are observed at specified times. Given that all discrete distributions with finite support can be represented by discrete phase-type distributions (Neuts, 1981; Alfa, 2004; Moghaddass and Zuo, 2014), phase distributions are useful for representing evolutionary processes, such as degradation of a unit, and, hence, are appropriate for the present investigation. Reliability systems that evolve in discrete time have been used to analyze the behavior of devices in fields such as civil and aeronautical engineering (Ruiz-Castro, 2016).

We assume a single repair facility where repair times are independent and identically distributed (i.i.d.) random variables and follow a geometric distribution (which is the counterpart of the continuous exponential distribution) with probability p for repairing a unit to "as good as new" during a time period. Note that when p is very close to one, a good item is almost always available and, thus, the repairman's contribution is negligible. When p is close to zero, the mean repair time is huge and the repair facility is hardly used. Hence, we consider values of p that are neither close to one nor to zero. We discuss both a "Replacement-Only" model, in which the operating unit is replaced by a new one, and a "Repair–Replacement" model, in which we allow for a general-degree repair action from any state to any better state at any time of inspection. We consider state-dependent operating costs, failure costs, and repair costs that are dependent on the degree of repair. Applying dynamic programming, we focus on the discounted cost criterion, using a discount factor $0 < \alpha < 1$. Two (classic) models are investigated: (i) the cold standby system, in which a standby element is unpowered and does not operate until needed to replace a faulty unit, and (ii) the warm standby system, where an element, while in standby mode, is partially powered. Therefore, the failure rate of a warm standby element is typically less than its full operational failure rate. The cold standby technique is commonly used in applications where energy consumption is critical, while examples of a warm standby system are redundant hard disks used to replace failed disks in a computer storage system. Another example of a warm system is a power plant in which extra generating units are waiting in standby mode. (Clearly, the cold standby system is a special case of the warm one, for a zero failure rate of a standby unit.) For more examples, we refer the reader to Amari *et al.* (2012), Levitin *et al.* (2014), and Ruiz-Castro (2016).

The contribution of this article, motivated by the models of Douer and Yechiali (1994), who consider a single-unit deteriorating system in discrete time, is fourfold:

(i) we study multi-unit systems;
(ii) we analyze both cold and warm standby systems;
(iii) the lifetime of an operative unit has a phase-type distribution; and
(iv) we include a repair facility.

To the best of our knowledge, although the related literature is large, our model is new and has not previously been investigated in the reliability literature. Following Douer and Yechiali (1994), we introduce a generalized CLR defined as follows: Repair to a better state (or replace) if and only if the state of the system exceeds some control-limit state. We show that, under reasonable conditions on the system's transition laws and cost values, the optimal policy has the structure of a generalized CLR. In general, it is known that in a finite state space there exists a non-randomized optimal policy; however, a generalized CLR serves in two aspects: computability, by reducing the number of possible optimal policies, and practically, by providing a useful and easy-to-implement tool for the controller; hence, CL rules are the most practical to implement maintenance rules.

This article is organized as follows. In Section 2 we introduce the model, the associated expected discounted costs, and the concept of a generalized CLR. Section 3 discusses the Replacement-Only model and the optimality of CLRs. Section 4 offers an extension by studying the Repair–Replacement model. More restrictive conditions are suggested and discussed. In both Sections 3 and 4, the cold and warm

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standby models are treated. In Section 5, numerical examples are presented and insights are provided.

2. Mathematical description

Consider a system of N stochastically independent and identical units (components). There is one operating unit online and the others are either in a standby position, in a repair position, or waiting to be repaired. Upon failure, the unit goes to a repair queue and is immediately replaced by a standby unit (if there is any) that becomes the operating unit online. When all N units are in a repair position, the system fails. Let B_i be the lifetime of an operating unit i $(i = 1, \dots, N)$; we assume that $B_i \sim B$ are i.i.d. random variables having a discrete phase-type distribution with representation $PH_d(\beta, \mathbb{S})$ of order m.

A discrete phase-type distribution $B \sim PH_d(\beta, \mathbb{S})$ is the distribution of the absorption time into state m in a discrete-time Markov chain with an initial probability vector (β, β_m) and a transition probability $((m+1) \times (m+1))$ matrix $\widetilde{\mathbb{S}}$:

$$\widetilde{\mathbb{S}} = \begin{bmatrix} \mathbb{S} & \mathbb{S}^o \\ \mathbf{0} & 1 \end{bmatrix},$$

where $\mathbb{S}^o + \mathbb{S}e = 1$, e is the unit vector, and the $(1 \times m)$ vector $\mathbf{0} = (0, \dots, 0)$. We denote the (ij)th element of the transition matrix $\widetilde{\mathbb{S}}$ by S_{ij} and $S_{ij} \geq 0$ for $0 \leq i, j \leq m$. Note that $E(B) = \beta(I - \mathbb{S})^{-1}e$. The working unit is classified into one of the $m+1$ states: $0, 1, \dots, m$. Without loss of generality, we assume that 0 denotes the state of a new unit or "as good as new" and m denotes the state of a failed unit; hence, the $(1 \times m)$ vector $\beta = (1, 0, \dots, 0)$ and $\beta_m = 0$. State i is better than state j if $i < j$. Throughout this article, we assume that S_{ij} $(0 \leq i, j \leq m)$ satisfies the following condition.

Condition 1. An Increasing Failure Rate condition: For each $k = 0, 1, 2, \dots, m$, the function

$$D_k(i) = \sum_{j=k}^{m} S_{ij}, \quad i = 0, 1, \dots, m-1, \tag{1}$$

is a nondecreasing function of i. It may be shown (see a similar result in Derman (1963)) that Condition 1 is equivalent to the following: for any nondecreasing function $h(j)$, $j = 0, 1, \dots, m$, the function

$$K(i) = \sum_{j=0}^{m} S_{ij} h(j), \quad i = 0, 1, \dots, m-1, \tag{2}$$

is also nondecreasing in i.

The system is inspected at equally spaced points in time. Let the instants of inspection be $t = 1, 2, \dots$. At time t, denote by X_t the number of good standby units plus the online unit ($X_t \in \{0, 1, \dots, N\}$) and by I_t the observed state of the working unit ($I_t \in \{0, 1, \dots, m\}$). Note that a component in one of the states $\{0, \dots, m-1\}$ is considered to be a good functioning component and state m indicates a bad component that should be replaced. Thus, for $X_t = i$, if the state of the online unit is $I_t \in \{0, \dots, m-1\}$, then there are i good units, of which $(i-1)$ are in standby. If $I_t = m$, then there are just $(i-1)$ good standby units. The two-dimensional stochastic process $\{(X_t, I_t) \mid t = 1, 2, \dots\}$ is a finite-state discrete Markov chain. Furthermore, we assume that a failed unit is replaced immediately by a new one from the standby units (if one is available) and the replacement is instantaneous. In addition, there is a single repair facility with a random variable repair time Y of a failed unit; Y has a geometric distribution with probability p for repairing a unit to "as good as new" during one time period. The repair time is independent of the state of the repaired item or the number of failed units; however, if there are no failed units (i.e., $X_t = N$), the repair facility is idle during interval $[t, t+1]$. After each inspection, the working unit can be replaced by a new one (if one is available) and the replacement takes no time. Furthermore, we allow partial repair—i.e., maintenance—such that a repair from state i to state $j < i$ is immediate and costs some fee $(2 \leq i \leq m-1,\ j = 1, \dots i-1)$. We assume that maintenance or full replacement can be performed upon any state $I_t = 1, 2, \dots, m-1$ of the operating unit. To emphasize, when a unit (failed or not) arrives at the repair facility, a major overall is being done, which takes considerable time, while replacement or improvement of an online unit takes negligible time. Clearly, the motivation for performing the maintenance action of a full replacement is to prevent the severe consequences of a system failure or of letting the unit operate under "bad" conditions. We consider three costs, as follows.

(i) *Replacement/repair cost*. The expected cost of replacing an online unit in state i $(1 \leq i \leq m)$ by a new unit is denoted by c_i; this cost includes adjustments and setup. Alternatively, the cost c_i can be interpreted as the repair cost to a "as good as new" unit. The cost of repairing an online unit in state i $(1 \leq i \leq m)$ to state $j < i$ is denoted by c_{ij}. Note that c_{i0} is the cost of replacing the unit by a new one and hence $c_{i0} = c_i$; we assume that $c_{ii} = 0$. Recall that in state m a replacement is mandatory, costing c_{m0}.

(ii) *Operating cost*. The expected operating cost of an active unit in state i $(0 \leq i \leq m-1)$ during the period $[t, t+1]$ is denoted by r_i.

(iii) *Penalty cost*. Whenever the number of good units is reduced to zero, the system fails until the number of good units increases to one (which does not occur earlier than the next inspection time). Each down time of the system costs c_d per time period ($c_d >> r_i, c_i, c_{ij}\ \forall i, j$).

The goal of this article is to derive and characterize optimal maintenance and repair rules under the total expected discounted cost for an unbounded horizon.

We focus our attention on the class of non-randomized stationary maintenance rules. As mentioned before, for the Replacement-Only model, it is known (Derman, 1963; Kolesar, 1966; Ross, 1969) that for a finite state and a finite action space and under specific conditions, there exists an optimal threshold policy that depends only on the state of the system at decision epochs. We direct our attention to the class of non-randomized stationary rules of the form, "For $X_t = x$, replace the operative unit, at time t, if and only if $I_t \geq i^*(x)$" where $i^*(x)$ is the control limit $(0 < i^*(x) \leq m)$.

3. Replacement-only model

We now consider the case where only replacement is allowed; i.e., after each inspection time $t = 1, 2, \dots,$ a decision must be made as to whether or not to replace the online unit by a standby one. We impose the following conditions on the costs.

Condition 2. The cost of a replacement is an increasing function of the state; i.e., $c_1 \le c_2 \le \cdots \le c_m$.

Condition 3. The operating cost is an increasing function of the state; i.e., $r_0 \le r_1 \le \cdots \le r_{m-1}$.

Note that Conditions 2 and 3 both imply that as the state of the system deteriorates, the costs increase.

Condition 4. For each i, $i < m$, $c_i + r_0 \ge r_i$, Condition 4 implies that for a one time period, it does not pay to perform any replacement; it is valid for systems in which the cost of replacement is large relative to the operating cost.

3.1. Cold standby system

We start with the cold standby model; although the cold standby model is a special case of the warm, analyzing it first makes the presentation more transparent. Assume a 1-out-of-N cold standby system where only the online unit can fail. Denote by $g_R(X_t, I_t)$ the one-step expected cost when the system is in state (X_t, I_t) at time t, under some maintenance rule R. Denote by $\varphi_R(x, i)$ the total expected discounted cost for an unbounded horizon if the process starts at state (x, i) and some maintenance rule R is applied. Then,

$$\varphi_R(x, i) = \left\{ \sum_{t=1}^{\infty} \alpha^{t-1} g_R(X_t, I_t) \mid (X_1, I_1) = (x, i) \right\}. \tag{3}$$

Here, x denotes the number of good standby units plus the online one ($x \in \{0, 1, \ldots, N\}$) and i denotes the state of the online unit ($i \in \{0, \ldots, m\}$).

For a given discount factor α denote the optimal maintenance policy for Criterion (3) by R^*_α and denote the total discounted minimal cost by $\varphi(x, i)$

$$\varphi(x, i) = \min_R \varphi_R(x, i) = \varphi_{R^*_\alpha}(x, i).$$

Denote by $\varphi_T(x, i)$ the minimal total expected discounted cost for a finite horizon T and for an initial state (x, i):

$$\varphi_T(x, i) = \min_R \left\{ \sum_{t=1}^{T} \alpha^{t-1} g_R(X_t, I_t) \mid (X_1, I_1) = (x, i) \right\}.$$

Using a standard argument of dynamic programming, the function $\varphi_T(x, i)$ satisfies the following functional set of equations (Equations (4a)–(4f)):

$$\varphi_T(N, 0) = r_0 + \alpha \sum_{j=0}^{m} S_{0j} \varphi_{T-1}(N, j). \tag{4a}$$

Equation (4a) refers to the case of N good units and the online unit is new. Clearly, in this case no decision has to be made. Regarding p, the probability for repairing a unit to "as good as new" during one time period, and setting $q = 1 - p$, we write

$$\underset{\substack{x=1,\ldots,N-1,i=0 \\ \text{or } x=1, i\neq m}}{\varphi_T(x, i)} = r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)]. \tag{4b}$$

Equation (4b) refers to the case of a new online unit (in state 0) or the case where only one good unit is left. Since the penalty cost is assumed to be huge, both cases imply no replacement:

$$\underset{x=2,\ldots,N}{\varphi_T(x, m)} = c_m + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]. \tag{4c}$$

Equation (4c) refers to the case of a failed unit, where a replacement is mandatory:

$$\varphi_T(1, m) \equiv \varphi_T(0) = c_d + \alpha[p\varphi_{T-1}(1, 0) + q\varphi_{T-1}(0)]. \tag{4d}$$

The cost $\varphi_T(1, m) \equiv \varphi_T(0)$ refers to the penalty cost in the case of a system failure; i.e., when all of the units have failed and the system does not function until one of the units is repaired. Clearly, we assume that the cost of that situation is very high.

Next, we obtain $\varphi_T(x, i)$ for the cases in which a decision has to be made. We start with the case where all of the units are good and therefore the repair facility is idle (Equation (4e)). Note that if a replacement is performed, the repair facility starts to repair the replaced unit; with probability p the unit is fixed until the next inspection time, and with probability $q = 1 - p$, the unit continues to be inoperative. Then, for $x = N$ we have Equation (4e) and for $1 < x < N$ we write Equation (4f):

$$\underset{i=1,\ldots,m-1}{\varphi_T(N, i)} = \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}\varphi_{T-1}(N, j), \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) \\ \quad + q\varphi_{T-1}(N-1, j)]. \end{array} \right\} \tag{4e}$$

$$\underset{\substack{x\neq 1,N \\ i=1,\ldots,m-1}}{\varphi_T(x, i)} = \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}\left[p\varphi_{T-1}(x+1, j) \right. \\ \quad \left. + q\varphi_{T-1}(x, j)\right], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ \quad + q\varphi_{T-1}(x-1, j)]. \end{array} \right\} \tag{4f}$$

The following initial conditions are derived from the use of Conditions 2 to 4 and the fact that for a one-time period, it does not pay to perform any replacement or repair (note that in state m the unit must be replaced, if one is available):

$$\begin{aligned} \varphi_1(1, m) &= c_d, \\ \varphi_1(x, m) &= c_m + r_0 \quad x > 1, \\ \varphi_1(x, i) &= r_i \quad x \ge 1, \quad i \neq m. \end{aligned} \tag{5}$$

Lemma 1. *For fixed α, T, and i, $\varphi_T(x, i)$ is a nonincreasing function of x ($x = 1, \ldots, N$). That is, the overall cost is (weakly) smaller when the number of good units is larger.*

Proof. We prove the lemma by using a double induction on T and on i.

Step 1. We start with $T = 1$ and prove that $\varphi_1(x, i) \ge \varphi_1(x+1, i)$ for all x. We consider the following four cases: (i) For $i \neq m$, we obtain $\varphi_1(x, i) = r_i = \varphi_1(x+1, i) = r_i$.

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(ii) For $i = m, x > 1$, we obtain $\varphi_1(x, i) = c_m + r_0 = \varphi_1(x+1, i) = c_m + r_0$. (iii) For $i = m, x = 1$, we obtain $\varphi_1(1, m) = c_d \geq \varphi_1(2, m) = c_m + r_0$ since $c_d >> r_0$. (iv) For a failed system, $\varphi_1(0) = c_d = \varphi_1(1, m)$.

Induction Step. Assume that Lemma 1 holds for $T-1$; we prove it for T. We start with $i = 0$, proceed with $0 < i < m$, and finally show for $i = m$.

Step 2. For $i = 0$.

(i) For $1 \leq x < N-1, i = 0$. We use Equation (4b) to get

$$\begin{aligned}\varphi_T(x, 0) &= r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) \\ &\quad + q\varphi_{T-1}(x, j)] \\ &\geq r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+2, j) \\ &\quad + q\varphi_{T-1}(x+1, j)] \\ &= \varphi_T(x+1, 0).\end{aligned}$$

(ii) For $x = N-1, i = 0$. Equations (4a) and (4b) yield

$$\begin{aligned}\varphi_T(N-1, 0) &= r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) \\ &\quad + q\varphi_{T-1}(N-1, j)] \\ &\geq r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) \\ &\quad + q\varphi_{T-1}(N, j)] \\ &\geq r_0 + \alpha \sum_{j=0}^{m} S_{0j}\varphi_{T-1}(N, j) = \varphi_T(N, 0).\end{aligned}$$

Assume that Lemma 1 holds for $i-1$; we prove that it holds for $0 < i < m$. Then, the special case $i = m$ is considered.

(i) For $0 < i < m, x = 1$. By Equations (4b) and (4f) we have

$$\begin{aligned}\varphi_T(1, i) &= r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(2, j) + q\varphi_{T-1}(1, j)] \\ &\geq r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(3, j) + q\varphi_{T-1}(2, j)] \\ &\geq \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(3, j) \\ \quad + q\varphi_{T-1}(2, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(2, j) \\ \quad + q\varphi_{T-1}(1, j)] \end{array} \right\} \\ &= \varphi_T(2, i).\end{aligned}$$

(ii) For $0 < i < m, 1 < x < N-1$. Equation (4f) leads to

$$\begin{aligned}\varphi_T(x, i) &= \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ \quad + q\varphi_{T-1}(x-1, j)] \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+2, j) \\ \quad + q\varphi_{T-1}(x+1, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)] \end{array} \right\} \\ &= \varphi_T(x+1, i).\end{aligned}$$

(iii) For $0 < i < m, x = N-1$. By Equations (4e) and (4f) we obtain

$$\begin{aligned}\varphi_T(N-1, i) &= \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(N, j) \\ \quad + q\varphi_{T-1}(N-1, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N-1, j) \\ \quad + q\varphi_{T-1}(N-2, j)] \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(N, j) \\ \quad + q\varphi_{T-1}(N, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) \\ \quad + q\varphi_{T-1}(N-1, j)] \end{array} \right\} \\ &= \varphi_T(N, i).\end{aligned}$$

Finally, for $i = m, x \neq 1$, applying Equation (4c) results in

$$\begin{aligned}\varphi_T(x, m) &= c_m + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ &\quad + q\varphi_{T-1}(x-1, j)] \\ &\geq c_m + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) \\ &\quad + q\varphi_{T-1}(x, j)] \\ &= \varphi_T(x+1, m).\end{aligned}$$

In the case of $x = 1, i = m$ clearly supports $\varphi_T(1, m) \geq \varphi_T(2, m)$, since we assume that a system failure incurs a huge penalty cost. □

Lemma 2. *For fixed α, T, and $x > 0$, $\varphi_T(x, i)$ is a nondecreasing function of i ($i = 0, \ldots, m$). That is, as the state of the online unit deteriorates, the overall cost (weakly) increases.*

Proof. We use a double induction on T and on x.

Step 1. We start with $T = 1$ and prove that $\varphi_1(x, i+1) \geq \varphi_1(x, i)$ for $i \in \{0, \ldots, m\}$. We consider three cases as follows. (i) For $i \neq m-1$ we obtain $\varphi_1(x, i+1) = r_{i+1} \geq \varphi_1(x, i) = r_i$. (ii) For $i = m-1, x > 1$, we obtain $\varphi_1(x, m) = c_m + r_0 \geq c_{m-1} + r_0 \geq r_{m-1} = \varphi_1(x, m-1)$. (iii) For $i = m-1, x = 1$, we obtain $\varphi_1(1, m) = c_d >> r_{m-1} = \varphi_1(1, m-1)$.

Induction Step. Assume that Lemma 2 holds for $T-1$; we prove it for T by induction on x and by applying Condition 1. The proofs for $x = 1$ and $x = N$ are provided in Appendix A1.

Step 2. Assume that Lemma 2 holds for $x-1$ good units; we prove that it holds for x $(1 < x < N)$ units.

(i) For $1 < x < N, i = 0$. Equations (4b) and (4f) lead to

$$\varphi_T(x, 1) = \min \left\{ \begin{array}{l} r_1 + \alpha \sum_{j=0}^{m} S_{1j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)], \\ c_1 + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ \quad + q\varphi_{T-1}(x-1, j)] \end{array} \right\}$$

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$$\geq \min \left\{ \begin{array}{l} r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)], \\ r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)] (\text{Lemma 1}) \end{array} \right\}$$

$$= r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)] = \varphi_T(x, 0).$$

(ii) For $1 < x < N, 0 < i < m-1$. By applying Equation (4f) we get

$$\varphi_T(x, i+1) = \min \left\{ \begin{array}{l} r_{i+1} + \alpha \sum_{j=0}^{m} S_{i+1,j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)], \\ c_{i+1} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ \quad + q\varphi_{T-1}(x-1, j)] \end{array} \right\}$$

$$\geq \min \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{i,j}[p\varphi_{T-1}(x+1, j) \\ \quad + q\varphi_{T-1}(x, j)], \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) \\ \quad + q\varphi_{T-1}(x-1, j)] \end{array} \right\}$$

$$= \varphi_T(x, i).$$

(iii) For $1 < x < N, i = m-1$. By Equation (4c) we obtain

$$\varphi_T(x, m) = c_m + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]$$

$$\geq c_{m-1} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]$$

$$\geq \varphi_T(x, m-1).$$

□

For the following proposition, the following condition is required.

Proposition 1. *Suppose that for a fixed x, and for all states i, υ such that $i < \upsilon < m$, the condition*

$$c_\upsilon - c_i \leq r_\upsilon - r_i \tag{6}$$

holds; then if in state i $(0 < i < m)$ the minimum of Equations (4e) and (4f) is achieved by a replacement, then for every state $\upsilon > i$ the minimum of Equations (4e) and (4f) is also achieved by a replacement. Condition 1 implies that the higher the state, the higher the incentive to replace (see also theorem 2.1 of Douer and Yechiali (1994)).

Proof. We prove it by using an induction on T. For $T = 1$, m is the only state in which a replacement is beneficial. Suppose that the proposition holds for $T-1$. We will show that it also holds for T. For $1 < x < N$, suppose that at state i $(0 < i \leq m-1)$ the minimum of Equation (4f) is achieved by replacing the unit; that is,

$$\varphi_T(x, i) = c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]$$
$$\leq r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)]. \tag{7}$$

Now, when $i = m-1$, it is clear that the proposition is true, as the only larger state than $i = m-1$ is m, for which a replacement is mandatory. If $i < m-1$, let υ be such that $i < \upsilon < m$ and $c_\upsilon - c_i \leq r_\upsilon - r_i$. Adding the last inequality to both sides of Equation (7) results in

$$c_\upsilon + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]$$
$$\leq r_\upsilon + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)] \quad i < \upsilon < m. \tag{8}$$

The case of $x = N$ (Equation (4e)) is similar. This concludes the proof. □

As $\varphi_T(x, i)$ is nondecreasing in $i = 0, \ldots, m$ (Lemma 2), it follows from Condition 1 that

$$\sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)]$$
$$\leq \sum_{j=0}^{m} S_{\upsilon j}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)]. \tag{9}$$

Substituting the above in the right-hand side of Equation (8) we get

$$c_\upsilon + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x, j) + q\varphi_{T-1}(x-1, j)]$$
$$\leq r_\upsilon + \alpha \sum_{j=0}^{m} S_{\upsilon j}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)],$$

which implies that in state υ the optimal policy is achieved by replacing the unit, rather than doing nothing.

Proposition 1 simply states that under the above cost conditions and Conditions 2 to 4, if it pays to replace in state (x, i), it also pays to replace in state (x, υ) for $\upsilon > i$. Let $0 < i(x, T) < m$ be the smallest state in which there are x good units and a replacement is beneficial when T periods are left for the operating horizon (such a state always exists because at state 0 no maintenance is needed, and in state m replacement is mandatory). Therefore, it readily follows from Proposition 1 that $\varphi_T(x, i)$ has the form

$$\varphi_T(N, i) = r_i + \alpha \sum_{j=0}^{m} S_{ij}\varphi_{T-1}(N, j), \quad 0 \leq i < i(x, T)$$

$$\varphi_{T}(x, i)_{x \neq N} = r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(x+1, j) + q\varphi_{T-1}(x, j)] \quad 0 \leq i < i(x, T)$$

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$$\varphi_T(x,i)_{x\neq 1} = c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(x,j) + q\varphi_{T-1}(x-1,j)] \quad i(x,T) < i < m. \tag{10}$$

Note that Equation (10) lists only the costs in the cases where a decision has to be made; the other costs are given in Equations (4a) to (4d).

Since

$$\varphi(x,i) = \lim_{T\longrightarrow\infty} \varphi_T(x,i)$$

and, for each T, $\varphi_T(x,i)$ is nonincreasing in x ($x = 1,\ldots,N$) (Lemma 1) and is nondecreasing in i ($i = 0,\ldots,m$) (Lemma 2), it follows that $\varphi(x,i)$ is also nonincreasing in x and nondecreasing in i. Using Conditions 2 to 4 and the conditions of Proposition 1, it can be shown (in a similar way as before) that there exists $i(\alpha)$, $0 < i(\alpha) < m$, such that $\varphi(x,i)$ has the form of a CLR.

3.2. Warm standby system

A natural extension to the cold standby system is the warm standby system, in which each good standby unit can fail. Motivated by Ruiz-Castro (2016), we assume that any warm standby unit can fail at any time unit with probability γ, where we assume that the expected time of a failure of a warm standby unit is much higher than that of an online unit; i.e., $\gamma^{-1} > E(B) = \beta(I - \mathbb{S})^{-1}e$. The warm standby concept considers two features: fast recovery (i.e., low restoration cost) and energy conservation (i.e., low operational cost). While in a standby mode, an element is partially powered and partially exposed to operational stresses. Therefore, the failure rate of a warm standby component is typically less than its full operational failure rate. Thus, it is reasonable to assume, for warm standby systems, that standby components have time-dependent failure behavior. An example of warm standby systems are redundant hard disks used to replace the failed disks in a storage system. The spare disks are spinning and, thus, can be exposed to operation stresses. On the other hand, the warm standby disks do not provide access to information and, therefore, their positioning mechanisms are idle, which makes the disks in a standby mode less failure-prone than those in the operation mode. Another example is a power plant in which extra generating units are waiting in a standby mode. The standby units can fail, but their failure rates, as well as operational costs, are less than those for the primary unit working under full load. Wireless sensor networks also use warm standby redundancy to keep a balance between energy consumption and recovery time needed for switching the backup sleeping sensor to operational mode. Another example is a gas company that holds tanks with redundant pumps, to ensure continuous operation of the system. A sudden disaster, bad weather, or fire, can cause a standby tank to be useless (we refer the reader to Levitin *et al.* (2014) and Ruiz-Castro (2016) for additional examples).

For a given discount factor α, the number of good standby units plus the online unit x ($x \in \{0,1,\ldots,N\}$) and the state i of the online unit ($i \in \{0,\ldots,m\}$) denote the total discounted minimal cost for an unbounded horizon by $\omega(x,i)$ and for the T periods left by $\omega_T(x,i)$. As indicated, the cold standby model is a special case where $\gamma = 0$. Let

$$C(k,n) = \binom{n}{k}\gamma^k(1-\gamma)^{n-k}$$

denote the probability of exactly k failures among n good warm standby units. Using a standard argument of dynamic programming, the function $\omega_T(x,i)$ satisfies the following functional set of equations:

$$\omega_T(N,0) = r_0 + \alpha \sum_{k=0}^{N-1}\sum_{j=0}^{m} C(k,N-1)S_{0j}\omega_{T-1}(N-k,j). \tag{11a}$$

Equation (11a) refers to the case of N good units, where the online unit is new; thus, $N-1$ units are in warm standby status; each of them can fail with probability γ:

$$\omega_T(x,i)_{\substack{x=1,\ldots,N-1,i=0\\ \text{or } x=1,i\neq m}} = r_i + \alpha \sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k,x-1)S_{ij} \times [p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x-k,j)] \tag{11b}$$

$$\omega_T(x,m)_{x=2,\ldots,N} = c_m + r_0 + \alpha \sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k,x-2)S_{0j} [p\omega_{T-1}(x-k,j) + q\omega_{T-1}(x-1-k,j)]. \tag{11c}$$

Equation (11c) refers to the cases where $(x-1)$ good units are in a warm standby status and the online unit fails. Thus, the failed unit is replaced by one of the standby units, which leaves $x-2$ standby units, a new online unit, and a failed one under repair:

$$\omega_T(1,m) \equiv \omega_{T-1}(0) = c_d + \alpha[p\omega_{T-1}(1,0) + q\omega_{T-1}(0)]. \tag{11d}$$

Similar to Section 3.1, Equations (11e) and (11f) refer to the case where a decision is called for:

$$\omega_T(N,i)_{i=1,\ldots,m-1} = \min\left\{\begin{array}{l} r_i + \alpha\sum_{k=0}^{N-1}\sum_{j=0}^{m} C(k,N-1) \\ \quad S_{ij}\omega_{T-1}(N-k,j), \\ c_i + r_0 + \alpha\sum_{k=0}^{N-2}\sum_{j=0}^{m} C(k,N-2) \\ \quad S_{0j}[p\omega_{T-1}(N-k,j) \\ \quad +q\omega_{T-1}(N-k-1,j)]. \end{array}\right\} \tag{11e}$$

$$\omega_T(x,i)_{\substack{x\neq 1,N-1\\ i=1,\ldots,m-1}} = \min\left\{\begin{array}{l} r_i + \alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k,x-1)S_{ij} \\ \quad [p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x-k,j)], \\ c_i + r_0 + \alpha\sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k,x-2)S_{0j} \\ \quad [p\omega_{T-1}(x-k,j) + q\omega_{T-1}(x-k-1,j)]. \end{array}\right\} \tag{11f}$$

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The initial conditions are

$$\begin{aligned}\omega_1(1,m) &= c_d,\\ \omega_1(x,m) &= c_m + r_0 \quad x>1,\\ \omega_1(x,i) &= r_i \quad x\geq 1, i\neq m.\end{aligned} \tag{12}$$

Condition 5. Assume that $\gamma \leq \min(q,p)$.

Condition 5 is a peculiar one. A partial explanation is as follows. Regarding the equality $\gamma \leq q$, which becomes essential in Lemma 3, if the system is left to operate on its own without the replacement option and the repair queue is not empty, the repair facility is working on the broken units and is loaded by failed warm standby units. It is reasonable to assume that, during one unit of time, the failure probability of a standby unit is smaller than the probability that a unit under repair will not be fixed. In addition, the condition $\gamma \leq p$ is obtained by examining the departure and arrival processes of failed units. The number of failed units decreases with p and increases with γ. Thus, in order to keep the system working and stable, we also need the latter condition. Combining both conditions, we obtain $\gamma \leq \min(q,p)$.

Lemma 3. *For a fixed α, T, and i, $\omega_T(x,i)$ is a nonincreasing function of x ($x = 1,\ldots,N$) under Conditions 2 to 4.*

Proof. We use a double induction on T and on i.
Step 1. For $T=1$ the proof coincides with that of Lemma 1.
Induction Step. Assume that Lemma 3 holds for $T-1$ and we prove it for T. We start with $i=0$ and $1\leq x<N-1$. The proofs for $x=N-1$ and for $i>0$ are similar and are provided in Appendix A2.

Step 2. For $i=0$.

(i) For $i=0$, $1\leq x<N-1$:

$$\omega_T(x+1,0) = r_0 + \alpha\sum_{k=0}^{x}\sum_{j=0}^{m} C(k,x)S_{0j}$$
$$[p\omega_{T-1}(x-k+2,j) + q\omega_{T-1}(x-k+1,j)]. \tag{13}$$

Recall that

$$\binom{x}{k} = \binom{x-1}{k-1} + \binom{x-1}{k}. \tag{14}$$

By substituting Equation (14) in Equation (13) we get

$$\begin{aligned}\omega_T(x+1,0) = r_0 &+ \alpha\sum_{j=0}^{m}\tbinom{x-1}{0}\gamma^0(1-\gamma)^x S_{0j}\\
&\times[p\omega_{T-1}(x+2,j)+q\omega_{T-1}(7+1,j)]\\
&+\alpha\sum_{j=0}^{m}\left[\tbinom{x-1}{1}+\tbinom{x-1}{0}\right]\gamma^1(1-\gamma)^{x-1}S_{0j}\\
&\times[p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j)]+\\
&\vdots\\
&+\alpha\sum_{j=0}^{m}\left[\tbinom{x-1}{x-1}+\tbinom{x-1}{x-2}\right]\gamma^{x-1}(1-\gamma)^1 S_{0j}\\
&\times[p\omega_{T-1}(3,j)+q\omega_{T-1}(2,j)]\\
&+\alpha\sum_{j=0}^{m}\tbinom{x-1}{x-1}\gamma^x(1-\gamma)^0 S_{0j}\\
&\times[p\omega_{T-1}(2,j)+q\omega_{T-1}(1,j)].\end{aligned} \tag{15}$$

Arranging Equation (15) yields

$$\begin{aligned}\omega_T(x+1,0) = r_0 &+ \alpha\sum_{j=0}^{m}\tbinom{x-1}{0}\gamma^0(1-\gamma)^{x-1}S_{0j}\\
&\times[\gamma p\omega_{T-1}(x+1,j)+\gamma q\omega_{T-1}(x,j)\\
&+(1-\gamma)p\omega_{T-1}(x+2,j)\\
&+(1-\gamma)q\omega_{T-1}(x+1,j)]+\\
&\vdots\\
&+\alpha\sum_{j=0}^{m}\tbinom{x-1}{x-1}\gamma^{x-1}(1-\gamma)^0 S_{0j}\\
&\times[\gamma p\omega_{T-1}(2,j)+\gamma q\omega_{T-1}(1,j)\\
&+(1-\gamma)p\omega_{T-1}(3,j)+(1-\gamma)q\omega_{T-1}(2,j)].\end{aligned}$$

By the induction assumption, note that for every x ($1\leq x<N-1$),

$$\begin{aligned}&\gamma[p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j)]\\
&\quad+(1-\gamma)[p\omega_{T-1}(x+2,j)+q\omega_{T-1}(x+1,j)]\\
&\leq\gamma[p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j)]\\
&\quad+(1-\gamma)[p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j)]\\
&=p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j).\end{aligned} \tag{16}$$

Substituting Equation (16) in Equation (15) leads to

$$\begin{aligned}\omega_T(x+1,0) \leq r_0 &+ \alpha\sum_{j=0}^{m}\tbinom{x-1}{0}\gamma^0(1-\gamma)^{x-1}S_{0j}\\
&\times[p\omega_{T-1}(x+1,j)+q\omega_{T-1}(x,j)]\\
&\vdots\\
&+\alpha\sum_{j=0}^{m}\tbinom{x-1}{x-1}\gamma^{x-1}(1-\gamma)^0 S_{0j}\\
&\times[p\omega_{T-1}(2,j)+q\omega_{T-1}(1,j)]\\
&=r_0+\alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m}C(k,x-1)S_{0j}\\
&\times[p\omega_{T-1}(x+1-k,j)+q\omega_{T-1}(x-k,j)]\\
&\leq r_i+\alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m}C(k,x-1)S_{ij}\\
&\times[p\omega_{T-1}(x+1-k,j)+q\omega_{T-1}(x-k,j)]\\
&=\omega_T(x,0).\end{aligned}$$

□

Lemma 4. *For fixed α, T, and x, $\omega_T(x,i)$ is a nondecreasing function of i ($i=0,\ldots,m$).*

Proof. The proofs are similar to those of Lemma 2 and are provided in Appendix A3. □

Proposition 2. *Suppose that for a fixed number of good units x, and for all states i, υ such that $i<\upsilon<m$, Condition (6) holds;*

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then if in state i $(0 < i < m)$ the minimum of Equations (11e) and (11f) is achieved by a replacement, then for every state $\upsilon > i$ the minimum of Equations (11e) and (11f) is also achieved by a replacement.

Proof. By using an induction on T. For $T = 1$, the state m is the only state in which a replacement is beneficial. Suppose the proposition holds for $T - 1$; we prove it for T. Assume that there are x good units and suppose that for some state i $(0 < i < m)$, the minimum is achieved by a replacement. Following Equation (11f) we obtain

$$\begin{aligned}\omega_T(x, i) &= c_i + r_0 + \alpha \sum_{k=0}^{x-2} \sum_{j=0}^{m} C(k, x-2) S_{0j} \\ &\quad \times [p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)] \\ &\leq r_i + \alpha \sum_{k=0}^{x-1} \sum_{j=0}^{m} C(k, x-1) S_{ij} \\ &\quad \times [p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)].\end{aligned} \tag{17}$$

Consider a state $\upsilon, i < \upsilon < m$, such that $c_\upsilon - c_i \leq r_\upsilon - r_i$. Adding the last inequality to both sides of Equation (17) leads to

$$\begin{aligned}c_\upsilon &+ r_0 + \alpha \sum_{k=0}^{x-2} \sum_{j=0}^{m} C(k, x-2) S_{0j} \\ &\times [p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)] \\ &\geq r_\upsilon + \alpha \sum_{k=0}^{x-1} \sum_{j=0}^{m} C(k, x-1) S_{ij} \\ &\times [p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)] \\ &\geq r_\upsilon + \alpha \sum_{k=0}^{x-1} \sum_{j=0}^{m} C(k, x-1) S_{\upsilon j} \\ &\times [p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)],\end{aligned}$$

which concludes that the minimum is achieved by a replacement also for state υ. The fact that in states $i = 0$ and $i = m$ we must replace completes the proof. □

In a nutshell, Proposition 2 states that under the above cost conditions and Conditions 2 to 4, if it pays to replace in state (x, i), it also pays to replace in state (x, υ) for $\upsilon > i$. Let $0 < i_{(x,T)} < m$ be the smallest state in which a replacement is beneficial when T periods are left for the operating horizon. Since

$$\omega(x, i) = \lim_{T \longrightarrow \infty} \omega_T(x, i)$$

and, for each T, $\omega_T(x, i)$ is nonincreasing in x $(x = 1, \ldots, N)$ (Lemma 3) and is nondecreasing in i $(i = 0, \ldots, m)$ (Lemma 4), it follows that $\omega(x, i)$ is also nonincreasing in x and nondecreasing in i. Using Conditions 2 to 4 and the conditions of Proposition 2, it is easy to show (in a similar way as before) that there exists $i(\alpha), 0 < i(\alpha) < m$, such that $\omega(x, i)$ has the form of a generalized CLR.

Claim 1. For fixed α, γ, x, and i , it can be shown that for each T, $\omega_T(x, i) \geq \varphi_T(x, i)$, which implies that $\omega(x, i) \geq \varphi(x, i)$.

4. Repair–replacement model

In this section, we generalize the above models to include a repair facility; namely, after each inspection, in addition to a full replacement, a repair to a better state is possible. We show that under reasonable conditions on the system's law of evolution and on the operating, repair, and replacement costs, the generalized CLR is optimal for the expected total discounted cost.

Specifically, in this section we assume that after each inspection and in each state $0 < i < m$ the system can be replaced by a new one or be repaired to a better state than state i such that its state after the repair is $0 < k < i$. We assume that a repair (as well as an initiated replacement) takes no time. Denote by A_i the collection of maintenance actions possible when the system is in state i $(1 \leq i \leq m-1)$. We assume that for each i, $A_i \in \{0, \ldots, i\}$. An action $k \in A_i$ means a repair of the working unit to state k ($k = 0$ means a replacement of the unit by a new one, and $k = i$ means no repair at all). Note that when the system is observed in state i and action $0 \leq k \leq i$ is taken, an expected operating cost of $r_k \geq 0$ is incurred until the next inspection. The repair action itself costs $c_{ik} \geq 0$ (c_{i0} is the cost of replacement and $c_{ii} = 0$). In addition, each time period in which the system is in a failed status costs c_d ($c_d >> r_i, c_{ik}, \forall i, k$). Furthermore, note that when a full replacement is performed (where $k = 0$), the unit is sent to the repair queue, whereas in the case of repair only (so that $0 < k \leq i$), the unit continues to be online. In addition to Condition 3, and similar to Conditions 2 and 4, we assume the following.

Condition 6. For each i, j, k such that $0 \leq k \leq i \leq j \leq m, c_{jk} \geq c_{ik}$. That is, the cost of a repair to a certain state, $k > 0$, as well as the cost of initiated replacement is an increasing function of the state from which the repair is performed. Note that the condition $c_{m0} \geq c_{i0}$ for $m > i$ implies that an initiated replacement costs less than the mandatory replacement in the case of failure.

Condition 7. For each $i, i < m$, and $k \leq i$, the condition $c_{ik} + r_k \geq r_i$ holds. Condition 7 implies that for a one-time period, it does not pay to perform any replacement or repair.

Here, too, we consider two systems: the cold and the warm standby systems.

4.1. Cold standby system

For initial state (x, i), denote by $\varphi^R(x, i)$ and $\varphi^R_T(x, i)$ the total discounted minimal cost for an unbounded horizon and for T periods left, respectively. The function $\varphi^R_T(x, i)$ satisfies the functional set of equations

$$\varphi^R_T(N, 0) = r_0 + \alpha \sum_{j=0}^{m} S_{0j} \varphi^R_{T-1}(N, j) \tag{18a}$$

$$\underset{x=1,\ldots,N-1, i=0}{\varphi^R_T(x, i)} = r_i + \alpha \sum_{j=0}^{m} S_{ij} \left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right] \tag{18b}$$

$$\begin{aligned}\underset{x=2,\ldots,N}{\varphi^R_T(x, m)} &= c_{m0} + r_0 + \alpha \sum_{j=0}^{m} S_{0j} \\ &\quad \times \left[p\varphi^R_{T-1}(x, j) + q\varphi^R_{T-1}(x-1, j)\right]\end{aligned} \tag{18c}$$

$$\varphi^R_T(1, m) \equiv \varphi^R_T(0) = c_d + \alpha \left[p\varphi^R_{T-1}(1, 0) + q\varphi^R_{T-1}(0)\right]. \tag{18d}$$

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Note that Equations (18a) to (18d) refer to the cases where the actions are uniquely dictated by the system's states.

Next, we obtain $\varphi_T^R(x,i)$ for the cases in which a decision has to be made. We start with N good units (Equation (18e)); then the equation is extended to $x<N$ (Equation (18f)). In both equations, (18e) and (18f), the first line refers to the case where no action is taken, the second line refers to a replacement, and the third line to a repair:

$$\varphi_T^R(N,i) \underset{i=1,\dots,m-1}{=} \min \times \left\{ \begin{array}{ll} r_i + \alpha \sum_{j=0}^m S_{ij}\varphi_{T-1}^R(N,j) & \text{(no action)} \\ c_{i0} + r_0 + \alpha \sum_{j=0}^m S_{0j} & \\ \quad \left[p\varphi_{T-1}^R(N,j) + q\varphi_{T-1}^R(N-1,j)\right] & \text{(replacement)} \\ \min\limits_{1\le k\le i-1} \left[c_{ik} + r_k + \alpha \sum_{j=0}^m S_{kj}\varphi_{T-1}^R(N,j)\right] & \text{(repair)} \end{array} \right\} \tag{18e}$$

$$\varphi_T^R(x,i) \underset{\substack{x\neq N\\ i=1,\dots,m-1}}{=} \min \times \left\{ \begin{array}{l} r_i + \alpha \sum_{j=0}^m S_{ij}\left[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)\right], \\ c_{i0} + r_0 + \alpha \sum_{j=0}^m S_{0j}\left[p\varphi_{T-1}^R(x,j) + q\varphi_{T-1}^R(x-1,j)\right], \\ \min\limits_{1\le k\le i-1} \\ \times \left[c_{ik} + r_k + \alpha \sum_{j=0}^m S_{kj}\left[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)\right]\right]. \end{array} \right\} \tag{18f}$$

Note that although in Equation (4f) the condition $x\neq 1$ is necessary, since replacing the last unit causes a system failure, here, in Equation (18f), the option of repairing the unit to a better state should be considered.

The following initial conditions are derived from Conditions 7 and the fact that for a one time period, it does not pay to perform any replacement or repair:

$$\begin{aligned} \varphi_1^R(1,m) &\equiv \varphi_1^R(0) = c_d, \\ \varphi_1^R(x,m) &= c_{m0} + r_0 \quad x>1, \\ \varphi_1^R(x,i) &= r_i \quad x\ge 1, i\neq m. \end{aligned} \tag{19}$$

Lemma 5. *For a fixed α, T, and i, $\varphi_T^R(x,i)$ is a nonincreasing function of x ($x=1,\dots,N$).*

Lemma 6. *For a fixed α, T, and $x>0$, $\varphi_T^R(x,i)$ is a nondecreasing function of i ($i=0,\dots,m$).*

Proof. Applying Conditions 2, 7, and 6, the proofs of Lemmas 5 and 6 are similar to those of Lemmas 1 and 2, respectively, and thus are omitted. □

Proposition 3. *Suppose that for a fixed x, and for all states i,k,υ such that $0\le k<i<\upsilon<m$, the condition*

$$c_{\upsilon k} - c_{ik} \le r_\upsilon - r_i, \tag{20}$$

holds; then if in state i ($0<i<m$) the minimum of Equations (18e) and (18f) is achieved by a maintenance action (either by a repair to some state $1\le k<i$ or by a replacement), then for every state $j>i$, the minimum of Equations (18e) and (18f) is also achieved by a maintenance action.

Remark 1. Consider the specific case where for all states i,k,υ such that $0\le k<i<\upsilon<m$, the equality $c_{\upsilon k}-c_{ik}=r_\upsilon-r_i$ holds. Then, Proposition 3 can be stated as follows: If in state i ($0<i<m$) the minimum of Equations (18e) and (18f) is achieved by a repair (respectively, a replacement) to some state $0<k<i$ (a replacement, respectively), then for every state $j>i$, the minimum of Equations (18e) and (18f) is also achieved by a repair (respectively, a replacement).

Proof. By using an induction on T. For $T=1$, the state (x,m) is the only one for which a replacement is beneficial. Suppose that the proposition holds for $T-1$. We show that it also holds for T. Assume initial state (x,i), $1<x<N$, $0<i<m$ and suppose that in state i the minimum of Equation (18f) is achieved by a maintenance action. We distinguish between two maintenance policies: a replacement or a repair. For each policy, we prove Proposition 3 and extend it to include Remark 1.

(i) *Replacement*. If a replacement by a new unit is preferred, then from the first and second lines of Equation (18f) we obtain

$$\begin{aligned} \varphi_T^R(x,i) &= c_{i0} + r_0 + \alpha\sum_{j=0}^m S_{0j}[p\varphi_{T-1}^R(x,j) + q\varphi_{T-1}^R(x-1,j)] \\ &\le r_i + \alpha\sum_{j=0}^m S_{ij}\left[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)\right]. \end{aligned} \tag{21}$$

Thus, for $i<m$, let υ be such that $i<\upsilon<m$. Adding Equation (20) for $k=0$—i.e., $c_{\upsilon 0}-c_{i0}\le r_\upsilon - r_i$—to both sides of Equation (21) leads to

$$\begin{aligned} & c_{\upsilon 0} + r_0 + \alpha\sum_{j=0}^m S_{0j}[p\varphi_{T-1}^R(x,j) + q\varphi_{T-1}^R(x-1,j)] \\ &\le r_\upsilon + \alpha\sum_{j=0}^m S_{ij}[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)] \\ &\le r_\upsilon + \alpha\sum_{j=0}^m S_{\upsilon j}[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)], \end{aligned} \tag{22}$$

which implies that in state υ a maintenance action is also preferred.

Furthermore, by assuming that in state i ($0<i<m$), the minimum of Equations (18e) and (18f) is achieved by a replacement yields

$$\begin{aligned} \varphi_T^R(x,i) &= c_{i0} + r_0 + \alpha\sum_{j=0}^m S_{0j}[p\varphi_{T-1}^R(x,j) + q\varphi_{T-1}^R(x-1,j)] \\ &\le \min_{1\le k\le i-1}\left[c_{ik} + r_k + \alpha\sum_{j=0}^m S_{kj}[p\varphi_{T-1}^R(x+1,j) + q\varphi_{T-1}^R(x,j)]\right]. \end{aligned} \tag{23}$$

Now, assume that Remark 1 holds. That is, for each k:

$$c_{\upsilon k} - c_{ik} = r_\upsilon - r_i,$$

and, specifically,

$$c_{\upsilon 0} - c_{i0} = c_{\upsilon k} - c_{ik}. \tag{24}$$

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Adding Equation (24) to both sides of Equation (23) yields

$$c_{\upsilon 0} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi'_{T-1}(x, j) + q\varphi'_{T-1}(x-1, j)]$$

$$\leq \min_{1\leq k\leq i-1}\left[c_{\upsilon k} + r_k + \alpha \sum_{j=0}^{m} S_{kj}\left[p\varphi'_{T-1}(x+1, j) + q\varphi'_{T-1}(x, j)\right]\right],$$

which implies that in state υ a replacement is preferred. The case of $i = N$ is similar.

(ii) *Repair*. If a repair to a better state is preferred, then for some $k > 0$, the first and third lines of Equation (18f) yield

$$\varphi^R_T(x, i) = c_{ik} + r_k + \alpha \sum_{j=0}^{m} S_{kj}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right]$$

$$t \leq r_i + \alpha \sum_{j=0}^{m} S_{ij}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right]. \quad (25)$$

Adding Equation (20)—i.e., $c_{\upsilon k} - c_{ik} \leq r_\upsilon - r_i$—to both sides of Equation (25) leads to

$$c_{\upsilon k} + r_k + \alpha \sum_{j=0}^{m} S_{kj}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right]$$

$$\leq r_\upsilon + \alpha \sum_{j=0}^{m} S_{ij}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right]$$

$$\leq r_\upsilon + \alpha \sum_{j=0}^{m} S_{\upsilon j}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right], \quad (26)$$

which implies that in state υ a maintenance action is also preferred. Next, assume that Remark 1 holds. Since a repair action is preferred, then for some $k > 0$ we have

$$c_{ik} + r_k + \alpha \sum_{j=0}^{m} S_{kj}\left[p\varphi^R_{T-1}(x+1, j) + q\varphi^R_{T-1}(x, j)\right]$$

$$\leq c_{i0} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}\left[p\varphi^R_{T-1}(x, j) + q\varphi^R_{T-1}(x-1, j)\right]. \quad (27)$$

Applying the same method as in case (i) yields the preferability of the repair. □

We have thus proved the following theorem.

Theorem 1. *Under the condition of Proposition 3, the optimal policy has the form of a generalized CLR.*

Remark 2. Assume that the costs of a replacement are equal in the Replacement-Only model and the Repair–Replacement model; that is, $c_i = c_{i0}$ for $0 < i \leq m$; then, in a similar manner as above, it can be shown that $\varphi^R_T(x, i) \leq \varphi_T(x, i)$, $(0 \leq x \leq N, 0 \leq i \leq m)$.

4.2. Warm standby system

Assume that the system starts at state (x, i). As before, we assume that any warm standby unit can fail at any time with probability γ. Denote the total discounted minimal cost of the warm standby model for an unbounded horizon by $w^R(x, i)$ and for T periods left by $w^R_T(x, i)$. The function $w^R_T(x, i)$ satisfies the functional set of equations

$$w^R_T(N, 0) = r_0 + \alpha \sum_{u=0}^{N-1}\sum_{j=0}^{m} C(u, N-1) S_{0j} w^R_{T-1}(N-u, j) \quad (28a)$$

$$\underset{x=1,\ldots,N-1, i=0}{w^R_T(x, i)} = r_i + \alpha \sum_{u=0}^{x-1}\sum_{j=0}^{m} C(u, x-1) S_{ij} \times \left[p w^R_{T-1}(x+1-u, j) + q w^R_{T-1}(x-u, j)\right] \quad (28b)$$

$$\underset{x=2,\ldots,N}{w^R_T(x, m)} = c_{m0} + r_0 + \alpha \sum_{u=0}^{x-2}\sum_{j=0}^{m} C(u, x-2) S_{0j} \times \left[p w^R_{T-1}(x-u, j) + q w^R_{T-1}(x-1-u, j)\right] \quad (28c)$$

$$w^R_T(1, m) \equiv w^R_T(0) = c_d + \alpha\left[p w^R_{T-1}(1, 0) + q w^R_{T-1}(0)\right]. \quad (28d)$$

Note that Equations (28a) to (28d) refer to the cases where the actions are uniquely determined by the system's states.

Next, we obtain $w^R_T(x, i)$ for the cases in which a decision has to be made. We start with N good units (Equation (28e)); then the model is generalized to $x < N$ good units (Equation (28f)). In both equations, (28e) and (28f), the first line refers to the case where no action is taken, the second line refers to a replacement, and the third line to a repair:

$$\underset{i=1,\ldots,m-1}{w^R_T(N, i)} = \min \begin{Bmatrix} r_i + \alpha \sum_{u=0}^{N-1}\sum_{j=0}^{m} C(u, N-1) S_{ij} w^R_{T-1}(N-u, j), \\ c_{i0} + r_0 + \alpha \sum_{u=0}^{N-2}\sum_{j=0}^{m} C(u, N-2) S_{0j} \\ \times \left[p w^R_{T-1}(N-u, j) + q w^R_{T-1}(N-u-1, j)\right] \min_{1\leq k\leq i-1} \\ \times \left[c_{ik} + r_k + \alpha \sum_{u=0}^{N-1} \right. \\ \left. \sum_{j=0}^{m} C(u, N-1) S_{kj} w^R_{T-1}(N-u, j)\right] \end{Bmatrix} \quad (28e)$$

$$\underset{\substack{x\neq N \\ i=1,\ldots,m-1}}{w^R_T(x, i)} = \min \begin{Bmatrix} r_i + \alpha \sum_{u=0}^{x-1}\sum_{j=0}^{m} C(u, x-1) S_{ij} \\ \times \left[p w^R_{T-1}(x+1-u, j) + q w^R_{T-1}(x-u, j)\right], \\ c_{i0} + r_0 + \alpha \sum_{u=0}^{x-2}\sum_{j=0}^{m} C(u, x-2) S_{0j} \\ \times \left[p w^R_{T-1}(x-u, j) + q w^R_{T-1}(x-u-1, j)\right] \\ \min_{1\leq k\leq i-1} \\ \times \left[c_{ik} + r_k + \alpha \sum_{u=0}^{x-1}\sum_{j=0}^{m} C(u, x-1) S_{kj} \right. \\ \left. \times \left[p w^R_{T-1}(x+1-u, j) + q w^R_{T-1}(x-u, j)\right]\right], \end{Bmatrix} \quad (28f)$$

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with the initial conditions

$$\begin{aligned} w_1^R(1,m) &\equiv w_1^R(0) = c_d, \\ w_1^R(x,m) &= c_{m0} + r_0 \quad x > 1, \\ w_1^R(x,i) &= r_i \quad x \geq 1, i \neq m. \end{aligned} \tag{29}$$

The warm standby system has a similar behavior as the associated cold system; thus, we cite the following lemmas and proposition without proofs.

Lemma 7. *For fixed α, T, and i, $w_T^R(x,i)$ is a nonincreasing function of x ($x = 1, \ldots, N$).*

Lemma 8. *For fixed α, T, and $x > 0$, $w_T^R(x,i)$ is a nondecreasing function of i ($i = 0, \ldots, m$).*

Proposition 4. *Suppose that Equation (20) holds, then if in state i ($0 < i < m$) the minimum of Equations (28e) and (28f) is achieved by a maintenance action (either by a repair to some state $1 \leq k < i$ or by a replacement), then for every state $j > i$, the minimum of Equations (28e) and (28f) is also achieved by a maintenance action. Furthermore, assume that Equation (20) holds; then if in state i ($0 < i < m$) the minimum of Equations (28e) and (28f) is achieved by a repair (respectively, a replacement) to some state $0 < k < i$, then for every state $j > i$, the minimum of Equations (28e) and (28f) is also achieved by a repair (respectively, a replacement).*

Theorem 2. *Under the condition of Proposition 4, the optimal policy has the form of a generalized CLR.*

Remark 3. Assume that $c_i = c_{i0}$ for $0 < i \leq m$; then, in a similar manner as above, it can be shown that $w_T^R(x,i) \leq w_T(x,i)$, ($0 \leq x \leq N, 0 \leq i \leq m$).

Claim 2. For fixed α, γ, x, and i, it can be shown that for each T, $\omega_T^R(x,i) \geq \varphi_T^R(x,i)$, which implies $\omega^R(x,i) \geq \varphi^R(x,i)$.

5. Numerical study and insights

Consider a discrete phase-type distribution with $m = 4$ states having the transition probability matrix

$$\mathbb{S} = \begin{pmatrix} 0.7 & 0.1 & 0.1 & 0.05 \\ 0 & 0.1 & 0.05 & 0.8 \\ 0 & 0 & 0.1 & 0.7 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}.$$

Let the initial probability vector $\beta = (1, 0, 0, 0)$, the discount factor $\alpha = 0.95$, the penalty cost $c_d = 1000$, and the operating cost vector $r = [1, 1, 4, 6]$. We begin our examples with the Replacement-Only model. The first example is associated with the cold standby system, the second example generalizes it to include the warm standby one, and the last example is devoted to the Repair–Replacement model.

Example 1: Consider the Replacement-Only cold standby system. Assume $N = 10$ units and let $p = 0.7$ and $q = 0.3$. Let C denote the replacement cost vector—i.e., $C = (c_1, c_2, c_3, c_4)$—and let

$$C_1 = (1, 1, 3, 5), C_2 = (5, 6, 7, 8), C_3 = (10, 12, 14, 16),$$
$$C_4 = (18, 19, 20, 21), C_5 = (25, 27, 30, 32).$$

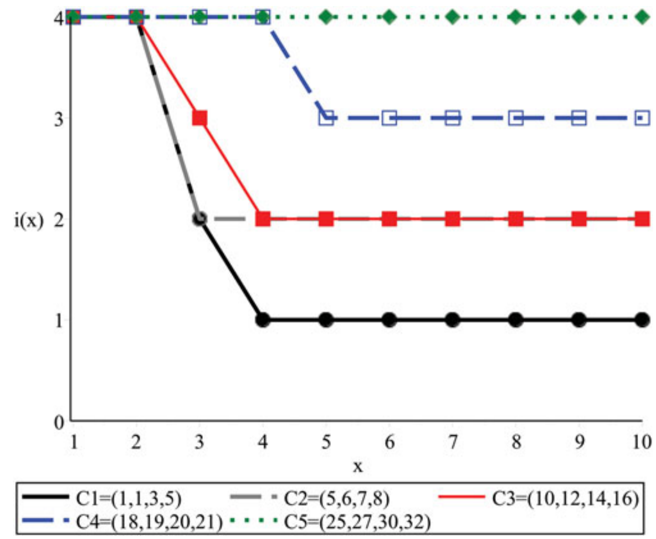

Figure 1. Control-limit level, $i(x)$, as a function of x for several vectors C.

Figure 1 presents the control-limit level $i(x)$ ($i(x) \in \{1, .., 4\}$) as a function of $x = 1, 2, \ldots, 10$, for $\{C_1, C_2, C_3, C_4, C_5\}$. Note that $i(x) = 4$ means that a replacement is preferred only when a units fails, whereas $i(x) = k \neq 4$ means that a replacement is performed when the online unit is in state k. As expected, we see that $i(x)$ is a nonincreasing function of x and a nondecreasing function of C; clearly, for a high replacement cost (C_5) it is never worthwhile to perform maintenance (unless the unit fails).

Evidently, the control-limit level is influenced by all of the parameters and costs. In order to examine numerically the impact of the repair probability p, we fix $x = 4$. Figure 2 depicts the control-limit level as a function of p for the same cost vectors C. Figure 2 shows that $i(x)$ is a nonincreasing function of p, since a higher probability p results in a shorter repair time.

Next, we study the behavior of the warm standby system and compare it to its cold standby counterpart.

Example 2: Consider a Replacement-Only warm standby system with the same data as in Example 1, $p = 0.7, C_1 = (1, 1, 3, 5)$, and $x \in \{1, .., 10\}$. Figure 3 shows the control-limit level $i(x)$ for several values of the probability γ. It is seen that

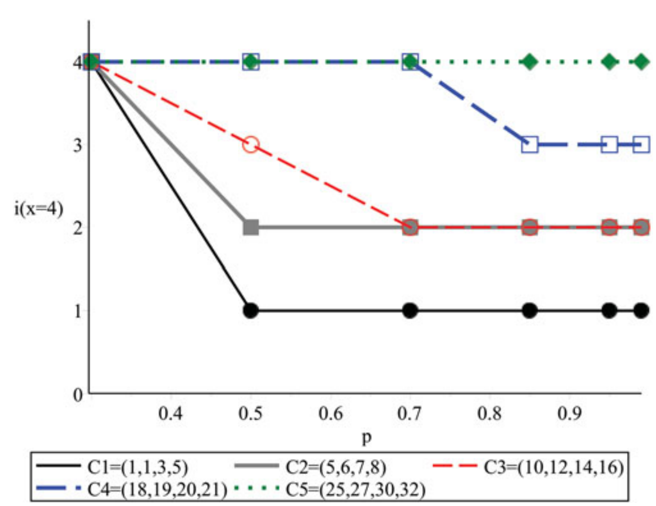

Figure 2. Control-limit level as a function of p for several vectors C.

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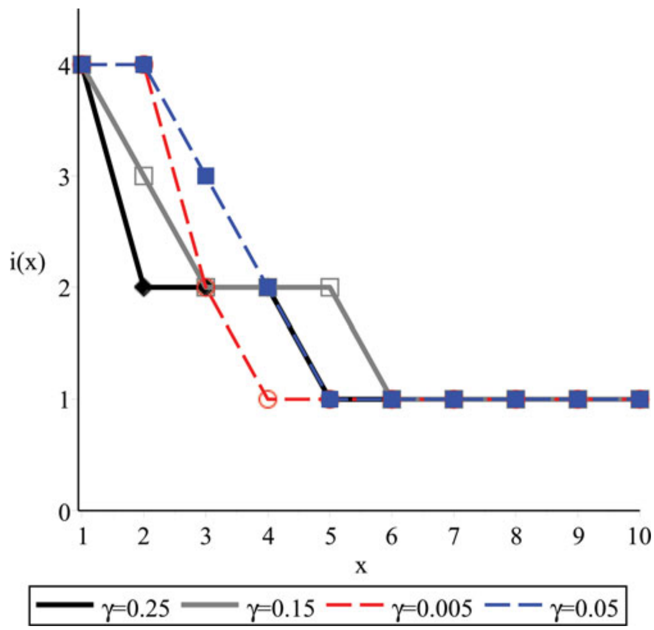

Figure 3. Control-limit level, $i(x)$, for several values of γ.

$i(x)$ is a nonincreasing function of x, as is the case in the associated cold standby model. In general, Figure 3 shows that as γ increases, $i(x)$ decreases. This behavior can be partially explained as follows. When γ is high, it is worthwhile for the decision maker to use a good standby unit before it fails; hence, $i(x)$ decreases.

In addition, it is interesting to compare the behavior of the cold and warm standby systems. For $\gamma = 0.15$, Figure 4 presents the discounted total costs, $\varphi(x, 0)$ (the black surface) and $w(x, 0)$ (the gray surface) as functions of x and C, respectively. As expected, Figure 4 shows that $w(x, 0) > \varphi(x, 0)$ (see Claim 1) and both costs decrease in x and increase in C. Furthermore, Table 1 presents the control-limit level $i(x)$ in both systems. Regarding the warm system, although $i(x)$ is influenced by many factors, we see that it is a nonincreasing function of x and

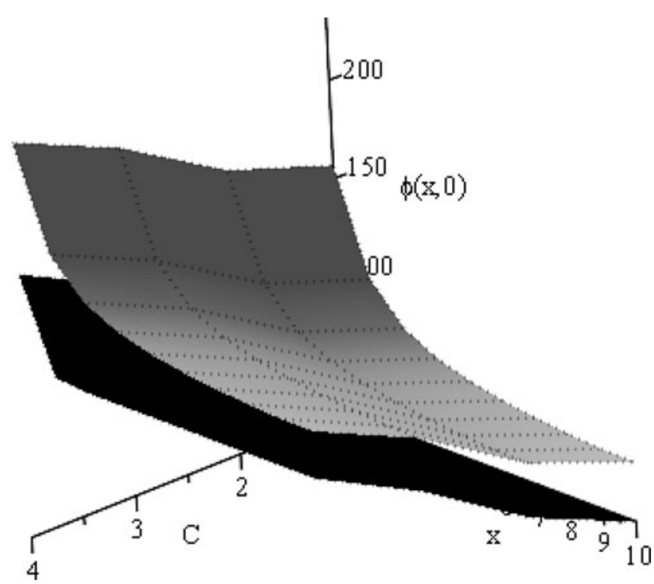

Figure 4. Costs $\varphi(x, 0)$ (black surface) vs. $w(x, 0)$ (gray surface) as functions of x and c.

Table 1. Control-limit level for the cold and warm standby systems for several vectors C.

	C_1		C_2		C_3		C_4	
x	*Cold*	*Warm*	*Cold*	*Warm*	*Cold*	*Warm*	*Cold*	*Warm*
1	4	4	4	4	4	4	4	4
2	4	3	4	3	4	3	4	4
3	2	2	2	3	3	3	4	4
4	1	2	2	2	2	3	4	4
5	1	2	2	2	2	3	3	4
6	1	1	2	2	2	2	3	4
7	1	1	2	2	2	2	3	4
8	1	1	2	2	2	2	3	4
9	1	1	2	2	2	2	3	4
10	1	1	2	1	2	1	3	3

a nondecreasing function of C (as in the cold model). However, knowing $i(x)$ for one of the models doesnot provide a specific knowledge of its value for the other.

Example 3: Consider a Repair–Replacement cold standby system with the same data as in Example 1 and with the following repair and replacement costs:

$$C_I : c_{10} = 10, c_{20} = 11, c_{30} = 13, c_{40} = 14, c_{21} = 7,$$
$$c_{31} = 9, c_{32} = 7,$$
$$C_{II} : c_{10} = 17, c_{20} = 18, c_{30} = 20, c_{40} = 21, c_{21} = 7,$$
$$c_{31} = 9, c_{32} = 7,$$
$$C_{III} : c_{10} = 25, c_{20} = 26, c_{30} = 28, c_{40} = 29, c_{21} = 7,$$
$$c_{31} = 9, c_{32} = 7.$$

Let p vary in $\{0.3, 0.7, 0.9\}$ and the vector C vary in $\{C_I, C_{II}, C_{III}\}$. Tables 2 and 3 present the optimal decisions for state (x, i) for different values of p and for C_I and C_{II}, respectively; each entry of the tables presents a matrix with indices $(x, i)x = (1, .., 10)$ and $i = (0, 1, 2, 3, 4)$. Note that for $x = 0$, or when i equals either $m = 4$ or $i = 0$, no decision has to be made. For each state (x, i), we mark the optimal policy as follows: "do nothing" by -1, replace to a new unit by 0, and repair to a better state by k $(1 \le k < i)$. Table 4 displays the optimal policies for the Replacement-Only cold standby model with the same parameters as in Table 3. For the replacement cost, we take $c_i = c_{i0}$ $i = 1, \ldots, m$. Here, too, -1 is marked as "do nothing" and 0 for a replacement by a new unit. Regarding Tables 2 and 3, our observation leads to some important insights:

1. The preferred state to be repaired to (in the case of repair) seems to be state 1. Examining the transition probabilities (see Example 1), it is seen that when the system is new, it moves with high probability to state 1. State 1 is more stable, and the system stays there with high probability. When out of state 1, the probability of failure increases. In addition, the operating costs of states 0 and 1 are equal, whereas the cost of replacing to state 0 is higher than the cost of repairing; thus, state 1 is preferred.
2. Increasing the probability p causes the repair policy to be less common; instead, the replacement policy or even doing nothing (which leads to replacement, eventually) becomes widespread. Clearly, when the repair probability is high, new units are preferred.

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Table 2. Optimal policies for the Repair–Replacement model, cost vector C_I.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	1	1	1	-1	-1	-1	-1	-1	0
3	1	1	1	1	1	-1	-1	-1	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 2(a). p=0.1, cost vector C_I.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	1	-1	-1	-1	0	0	0	0	0
3	1	1	1	-1	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 2(b). p=0.3, cost vector C_I.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	-1	0	0	0	0	0	0	0	0
3	1	-1	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 2(c). p=0.9, cost vector C_I.

Table 3. Optimal policies for the Repair–Replacement model, cost vector C_{II}.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	1	1	1	-1	-1	-1	-1	-1	-1
3	1	1	1	1	1	-1	-1	-1	-1	-1
4	0	0	0	0	0	0	0	0	0	0

Table 3(a). p=0.1, cost vector C_{II}.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	1	-1	-1	-1	-1	-1	-1	-1	-1
3	1	1	1	-1	-1	-1	-1	-1	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 3(b). p=0.3, cost vector C_{II}.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	1	-1	-1	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 3(c). p=0.9, cost vector C_{II}.

3. For high replacement costs and large values of p, the do nothing policy becomes common; however, for low p, (e.g., $p = 0.1$), a replacement or doing nothing may prove too risky and hence the repair policy has a significant impact.

Furthermore, it is interesting to compare the optimal costs and the optimal policies of the Repair–Replacement model versus those of the Replacement-Only model. Figure 5 plots $\varphi^R(x, i = 0)$ (solid lines) and $\varphi(x, i = 0)$ (dashed lines) as a function of x, for $C \in \{C_I, C_{II}, C_{III}\}$ (the blue lines represent C_I, the gray-C_{II}, and the black-C_{III}). Table 4 presents the optimal decisions under the Replacement-Only policy for state (x, i) for different values of p and for cost vector C_I. Observing Table 2, Table 4, and Figure 5 leads to the following insights:

1. For small values of p, replacing a unit to a new one is rare, while repairing to a better state is common.
2. For high values of p, the decision to replace the unit as a function of (x, i) is similar in both models since the repair option is rare.
3. Figure 5 shows that $\varphi^R(x, 0) \leq \varphi(x, 0)$ (see Remark 2); however, as x increases, the difference between these two cost functions decreases and the costs eventually become equal. Furthermore, although the costs $\varphi^R(x, 0)$ and $\varphi(x, 0)$ increase as C increases, the difference $\varphi(x, 0) - \varphi^R(x, 0)$ is mainly determined by x and is not affected by C.

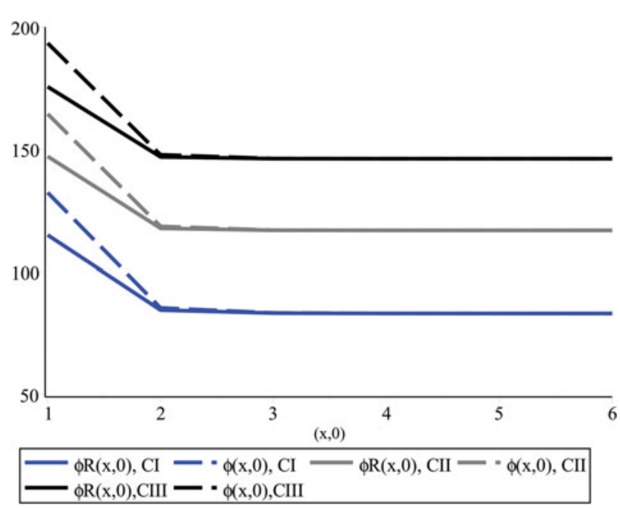

Figure 5. $\varphi^R(x, i = 0)$ (solid lines) vs. $\varphi(x, i = 0)$ (dashed lines) for $C = (C_I, C_{II}, C_{III})$.

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Table 4. Optimal policies for the Replacement-Only model, cost vector C_I, for different values of p.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
4	0	0	0	0	0	0	0	0	0	0

Table 4(a). p=0.1, cost vector C_I.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	-1	-1	-1	-1	-1	-1	0	0	0
3	-1	-1	-1	-1	-1	-1	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 4(b). p=0.3, cost vector C_I.

i \ x	1	2	3	4	5	6	7	8	9	10
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	-1	-1	0	0	0	0	0	0	0	0
3	-1	-1	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0

Table 4(c). p=0.9, cost vector C_I.

6. Concluding remarks and further research

This article studies the effect of preventive maintenance policies for a 1-out-of-N repairable system, using the dynamic programming tool. The unit failure time follows a discrete phase-type distribution and there is a single repair facility. Repair times are i.i.d. geometric random variables. For both cold and warm standby systems, we study the expected discounted cost under two policies, a Replacement-Only policy and a Repair–Replacement policy. Motivated by Douer and Yechiali (1994), we introduce a generalized CLR and show that, under reasonable conditions on the system's transition laws and cost values, the optimal policy has the structure of a generalized CLR.

Regarding the Replacement-Only policy, numerical examples of both the cold and the warm standby systems show a similar behavior: the control-limit level is a nonincreasing function of the number of good units x and the repair probability p and a nondecreasing function of the cost vector C. Under the Repair–Replacement policy, our observations show that increasing p causes the repair policy to be less common; instead, the replace policy or even doing nothing becomes widespread. It is proved that the expected discounted costs for all models (i.e., $\varphi(x, i)$, $w(x, i)$, $\varphi^R(x, i)$, and $w^R(x, i)$) are a nonincreasing function of x and a nondecreasing function of the state i. Furthermore, although the costs $\varphi^R(x, 0)$ and $\varphi(x, 0)$ decrease in x and increase in C, the difference $\varphi(x, 0) - \varphi^R(x, 0)$ decreases in x (and eventually becomes zero) and is not affected by C.

There are several possible extensions of the above models. Motivated by Ross (1969) and Benyamini and Yechiali (1999), the case of a continuous state-space (for both failure times and repair times) is probably tractable; in this case, a generalization to exponential (or even phase-type) distribution seems possible. In addition, it may be worthwhile to consider $S \geq 2$ parallel identical repair facilities. Another interesting and practical extension is to consider a phase-type distribution for the warm standby units.

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References

Ahmadi, R. (2014) Maintenance scheduling for a complex manufacturing system subject to deterioration. *Annals of Operation Research*, **217**, 1–29.

Alfa, A.S. (2004) Markov chain representations of discrete distributions applied to queueing models. *Computers & Operations Research*, **31**(14), 2365–2385.

Amari, S.V., Pham, H. and Misra, R.B. (2012) Reliability characteristics of-out-of-warm standby systems. *IEEE Transactions on Reliability*, **61**(4), 1007–1018.

Babishin, V. and Taghipour, S. (2016) Joint maintenance and inspection optimization of a k-out-of-n system, in *Proceedings of the 2016 Annual Reliability and Maintainability Symposium*, IEEE Press, Piscataway, NJ, pp. 1–6.

Bajestani, M.A. and Banjevic, D. (2016) Calendar-based age replacement policy with dependent renewal cycles. *IIE Transactions*, **48**, 1016–1026.

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Barlow, R.W. and Proschan, F. (1965) *Mathematical Theory of Reliability*, Wiley, New York, NY.

Barron, Y. (2015) Group replacement policies for a repairable cold standby system with fixed lead times. *IIE Transactions*, **47**, 1139–1151.

Barron, Y., Frostig, E. and Levikson, B. (2004) Analysis of R-out-of-N repairable systems: The case of phase-type distribution. *Advances in Applied Probability*, **36**, 116–138.

Benyamini, Z. and Yechiali, U. (1999) Optimality of control limit maintenance policies under nonstationary deterioration. *Probability in the Engineering and Informational Sciences*, **13**, 55–70.

Chalabi, N., Dahane, M., Beldjilali, B. and Neki, A. (2016) Optimization of preventive maintenance grouping strategy for multi-component series systems: Particle swarm based approach. *Computers & Industrial Engineering*, **102**, 440–451.

Derman, C. (1963) On optimal replacement rules when changes of states are Markovian, in R Bellman, (ed), *Mathematical Optimization Techniques, The Rand Corporation*, pp. 201–210. University of California Press, Berkeley and Los Angeles, CA.

De Smidt-Destombes, K.S., van der Heijden, M.C. and van Harten, A. (2006) On the interaction between maintenance, spare part inventories and repair capacity for a k-out-of-N system with wear-out. *European Journal of Operational Research*, **174**(1), 182–200.

Douer, N. and Yechiali, U. (1994) Optimal repair and replacement in Markovian systems. *Stochastic Models*, **10**, 253–270.

Fernández, A.J. (2015) Optimum attributes component test plans for k-out-of-n: F Weibull systems using prior information. *European Journal of Operational Research*, **240**(3), 688–696.

Gunay, E.E. and Kula, U. (2016) Integration of production quantity and control chart design in automotive manufacturing. *Computers & Industrial Engineering*, **102**, 374–382.

Hariga, M. and AI-Fawzan, M.A. (2000) Discounted models for the single machine inspection problem, in *Maintenance, Modeling and Optimization*, Kluwer Academic Publisher, Boston, MA,pp. 215–243.

Huynh, K.T., Castro, I.T., Barros, A. and Berenguer, C. (2012) Modeling age-based maintenance strategies with minimal repairs for systems subject to competing failure modes due to degradation and shocks. *European Journal of Operational Research*, **218**(1), 140–151.

Kao, E.P.C. (1973) Optimal replacement rules when changes of state are semi-Markovian. *Operations Research*, **21**, 1231–1249.

Kijima, M., Morimura, H. and Suzuki, Y. (1988) Periodical replacement problem without assuming minimal repair. *European Journal of Operational Research*, **37**, 194–2003.

Kolesar, P. (1966) Minimum cost replacement under Markovian deterioration. *Management Science*, **12**, 694–705.

Laggounea, R., Chateauneuf, A. and Aissania, D. (2010) Preventive maintenance scheduling for a multi-component system with non-negligible replacement time. *International Journal of Systems Science*, **41**(7), 747–761.

Lam, C.T. and Yeh, R.H. (1994) Optimal replacement policies for multi state deteriorating systems. *Naval Research Logistics*, **41**, 303–315.

Levitin, G., Xing, L. and Dai, Y. (2014) Optimal component loading in 1-out-of-N cold standby systems. *Reliability Engineering & System Safety*, **127**, 58–64.

Moghaddass, R. and Zuo, M.J. (2014) Multistate degradation and supervised estimation methods for a condition-monitored device. *IIE Transactions*, **46**, 131–148.

Nakagawa, T. and Mizutani, S. (2009) A summary of maintenance policies for a finite interval. *Reliability Engineering & System Safety*, **94**(1), 89–96.

Neuts, M.F. (1981) *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, Johns Hopkins University Press, Baltimore.

Olde Keizer, M.C., Teunter, R.H. and Veldman, J. (2016) Clustering condition-based maintenance for systems with redundancy and economic dependencies. *European Journal of Operational Research*, **251**(2), 531–540.

Ross, S.M. (1969) A Markovian replacement model with generalization to include stocking. *Management Science*, **15**, 702–715.

Ruiz-Castro, J.E. (2016) Complex multi-state systems modelled through marked Markovian arrival processes. *European Journal of Operational Research*, **252**, 852–865.

Sheu, S.H., Chang, C.C., Chen, Y.L. and Zhang, Z.G. (2015) Optimal preventive maintenance and repair policies for multi-state systems. *Reliability Engineering & System Safety*, **140**, 78–87.

Shi, A.H. and Zeng, B.J.C. (2016) Real-time prediction of remaining useful life and preventive opportunistic maintenance strategy for multi-component systems considering stochastic dependence. *Computers & Industrial Engineering*, **93**, 192–204.

Shu, Y., Feng, Q., Kao, E.P. and Liu, H. (2016) Lévy driven non-Gaussian Ornstein-Uhlenbeck processes for degradation-based reliability analysis. *IIE Transactions*, **48**(11), 993–1003.

Smith, M.A.J. and Dekker, R. (1997) Preventive maintenance in a 1-out-of-n system: The uptime, downtime and costs. *European Journal of Operational Research*, **99**, 565–583.

So, K.C. (1992) Optimality of control limit policies in replacement models. *Naval Research Logistics*, **39**, 685–697.

Stadje, W. and Zuckerman, D. (1992) Optimal maintenance strategies for repairable systems with general degree of repair. *Journal of Applied Probability*, **28**, 384–396.

Tian, Z., Zuo, M.J. and Yam, R.C. (2008) Multi-state k-out-of-n systems and their performance evaluation. *IIE Transactions*, **41**(1), 32–44.

Wu, W., Tang, Y., Yu, M. and Jiang, Y. (2015) Computation and transient analysis of a k-out-of-n: G repairable system with general repair times. *Operational Research*, **15**(2), 307–324.

Xie, W., Liao, H. and Jin, T. (2014) Maximizing system availability through joint decision on component redundancy and spares inventory. *European Journal of Operational Research*, **237**(1), 164–176.

Yin, H., Zhang, G., Zhu, H., Deng, Y. and He, F. (2015) An integrated model of statistical process control and maintenance based on the delayed monitoring. *Reliability Engineering & System Safety*, **133**, 323–333.

Zhang, G., Deng, Y., Zhu, H. and Yin, H. (2015) Delayed maintenance policy optimisation based on control chart. *International Journal of Production Research*, **53**(2), 341–353.

Zhang, Y.L. and Wang, G.J. (2011) An extended replacement policy for a deteriorating system with multi-failure modes. *Applied Mathematics and Computation*, **218**, 1820–1830.

Zhang, Z., Wu, S., Li, B. and Lee, S. (2015) (n, N) type maintenance policy for multi-component systems with failure interactions. *International Journal of Systems Science*, **46**(6), 1051–1064.

Zhong, C.Q. and Jin, H. (2014) A novel optimal preventive maintenance policy for a cold standby system based on semi-Markov theory. *European Journal of Operational Research*, **232**, 405–411.

Zhong, J., Ma, Y. and Tu, Y.L. (2016) Integration of SPC and performance maintenance for supply chain system. *International Journal of Production Research*, **54**(19), 5932–5945.

A Appendix: Proofs.

A1. Proof of Lemma 2.

Proof. For $x = 1$.

(i) For $x = 1$, $i < m - 1$. Following Equation (4b) we have

$$\begin{aligned}\varphi_T(1, i+1) &= r_{i+1} + \alpha \sum_{j=0}^{m} S_{i+1,j}[p\varphi_{T-1}(2, j) + q\varphi_{T-1}(1, j)] \\ &\geq r_i + \alpha \sum_{j=0}^{m} S_{ij}[p\varphi_{T-1}(2, j) + q\varphi_{T-1}(1, j)] \\ &= \varphi_T(1, i).\end{aligned}$$

(ii) For $x = 1$, $i = m - 1$, clearly the inequality $\varphi_T(1, m) \geq \varphi_T(1, m-1)$ holds, since we assume that system failure incurs a huge cost.

For $x = N$.

(i) For $x = N$, $i = 0$ the proof is similar to the case of $1 < x < N$, $i = 0$.

(ii) For $x = N$, $0 < i < m - 1$:

$$\begin{aligned}\varphi_T(N, i+1) &= \min\left\{\begin{array}{l} r_{i+1} + \alpha \sum_{j=0}^{m} S_{i+1,j}\varphi_{T-1}(N, j), \\ c_{i+1} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) + q\varphi_{T-1}(N-1, j)] \end{array}\right\} \\ &\geq \min\left\{\begin{array}{l} r_i + \alpha \sum_{j=0}^{m} S_{i,j}\varphi_{T-1}(N, j), \\ c_i + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) + q\varphi_{T-1}(N-1, j)] \end{array}\right\} \\ &= \varphi_T(N, i).\end{aligned}$$

(iii) For $x = N$, $i = m - 1$:

$$\begin{aligned}\varphi_T(N, m) &= c_m + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) + q\varphi_{T-1}(N-1, j)] \\ &\geq c_{m-1} + r_0 + \alpha \sum_{j=0}^{m} S_{0j}[p\varphi_{T-1}(N, j) + q\varphi_{T-1}(N-1, j)] \\ &\geq \varphi_T(N, m-1).\end{aligned}$$

□

A2. Proof of Lemma 3.

Assume that Lemma 3 holds for $T - 1$ and we prove for T. The case of $i = 0$ and $1 \leq x \leq N - 1$ is given in Lemma 3; we continue with $i = 0$, $x = N - 1$, proceed with $0 < i < m$, and finally show for $i = m$.

Proof. For $i = 0$, $x = N - 1$. Following Equation (11a) we obtain

$$\omega_T(N, 0) = r_0 + \alpha \sum_{k=0}^{N-1} \sum_{j=0}^{m} C(k, N-1) S_{0j} p \omega_{T-1}(N-k, j).$$

Applying Equation (14) and following similar steps as in Equation (15) yields

$$\begin{aligned}\omega_T(N, 0) = r_0 &+ \alpha \sum_{j=0}^{m} \binom{N-2}{0}\gamma^0(1-\gamma)^{N-2} S_{0j}[\gamma\omega_{T-1}(N-1, j) + (1-\gamma)\omega_{T-1}(N, j)] \\ &+ \alpha \sum_{j=0}^{m} \binom{N-1}{1}\gamma^1(1-\gamma)^{N-3} S_{0j}[\gamma\omega_{T-1}(N-2, j) + (1-\gamma)\omega_{T-1}(N-1, j)] \\ &\cdots \\ &+ \alpha \sum_{j=0}^{m} \binom{N-2}{N-2}\gamma^{N-2}(1-\gamma)^0 S_{0j}[\gamma\omega_{T-1}(1, j) + (1-\gamma)\omega_{T-1}(2, j)].\end{aligned} \tag{A1}$$

Using induction and Condition 5, for all x we obtain

$$(1-\gamma-p)\omega_{T-1}(x-1, j) \geq (1-\gamma-p)\omega_{T-1}(x, j)$$

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or

$$p\omega_{T-1}(x, j) + (1-p)\omega_{T-1}(x-1, j) \geq \gamma\omega_{T-1}(x-1, j) + (1-\gamma)\omega_{T-1}(x, j). \tag{A2}$$

Substituting Equation (A2) in Equation (A1) and applying Equation (11b) leads to

$$\begin{aligned}\omega_T(N,0) \leq r_0 &+ \alpha\sum_{j=0}^{m}\tbinom{N-2}{0}\gamma^0(1-\gamma)^{N-2}S_{0j}[p\omega_{T-1}(N,j) + q\omega_{T-1}(N-1,j)] \\ &+ \alpha\sum_{j=0}^{m}\tbinom{N-2}{1}\gamma^1(1-\gamma)^{N-3}S_{0j}[p\omega_{T-1}(N-1,j) + q\omega_{T-1}(N-2,j)] \\ &\ldots \\ &+ \alpha\sum_{j=0}^{m}\tbinom{N-2}{N-2}\gamma^{N-2}(1-\gamma)^0S_{0j}[p\omega_{T-1}(2,j) + q\omega_{T-1}(1,j)] \\ = \omega_T(N-1,0).&\end{aligned} \tag{A3}$$

Assuming that Lemma 3 holds for $i-1$, we now prove that it holds for $0 < i < m$.

(i) For $0 < i < m$, $x \neq 1, N-1$. By Equation (11f) we get

$$\omega_T(x+1,i) = \min\left\{\begin{array}{l} r_i + \alpha\sum_{k=0}^{x}\sum_{j=0}^{m}C(k,x)S_{ij}[p\omega_{T-1}(x+2-k,j) + q\omega_{T-1}(x+1-k,j)], \\ c_i + r_0 + \alpha\sum_{k=0}^{x}\sum_{j=0}^{m}C(k,x-1)S_{0j}[p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x+1-k-1,j)]. \end{array}\right\}$$

Consider the first option, in which no maintenance is taken. Then, by applying the same methods as in Equations (14) to (16) we arrive at

$$\begin{aligned} r_i &+ \alpha\sum_{k=0}^{x}\sum_{j=0}^{m}C(k,x)S_{ij}[p\omega_{T-1}(x+2-k,j) + q\omega_{T-1}(x+1-k,j)] \\ &\leq r_i + \alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m}C(k,x-1)S_{ij}[p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x-k,j)]. \end{aligned} \tag{A4}$$

Similarly, for the second option, in which a replacement is performed, we get

$$\begin{aligned} c_i + r_0 &+ \alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m}C(k,x-1)S_{0j}[p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x-k,j)] \\ &\leq c_i + r_0 + \alpha\sum_{k=0}^{x-2}\sum_{j=0}^{m}C(k,x-2)S_{0j}[p\omega_{T-1}(x-k,j) + q\omega_{T-1}(x-k-1,j)]. \end{aligned} \tag{A5}$$

Thus, from Equations (A4) and (A5) we obtain

$$\begin{aligned}\omega_T(x+1,i) &\leq \min\left\{\begin{array}{l} r_i + \alpha\sum_{k=0}^{x-1}\sum_{j=0}^{m}C(k,x-1)S_{ij}[p\omega_{T-1}(x+1-k,j) + q\omega_{T-1}(x-k,j)] \\ c_i + r_0 + \alpha\sum_{k=0}^{x-2}\sum_{j=0}^{m}C(k,x-2)S_{0j}[p\omega_{T-1}(x-k,j) + q\omega_{T-1}(x-k-1,j)] \end{array}\right\} \\ &= \omega_T(x,i). \end{aligned} \tag{A6}$$

The proof for the case of $x = 1$ is similar and hence is omitted.

(ii) For $0 < i < m$, $x = N-1$. Applying Equations (A1) to (A3) proves the case of $x = N-1$.
Finally, to end the proof, we consider Equation (11c), the case of $i = m$, $x \neq 1$ (note that the case of $x = 1$ incurs a huge cost of c_d):

$$\begin{aligned}\underset{x\neq 1}{\omega_T(x,m)} &= c_m + r_0 + \alpha\sum_{k=0}^{x-2}C(k,x-2)S_{0j}[p\omega_{T-1}(x-k,j) + q\omega_{T-1}(x-1-k,j)] \\ &= c_m + \omega_T(x-1,0) \\ &\geq c_m + \omega_T(x,0) \text{ (by induction)} \\ &= \omega_T(x+1,m). \end{aligned} \tag{A7}$$

□

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A3. Proof of Lemma 4.

Proof. We use a double induction on T and x.

(1) **Step 1.** The case of $T = 1$ is similar to that of Lemma 2.

(2) **Induction Step.** Assume that Lemma 4 holds for $T - 1$; we prove it for the case of T periods left.

(3) **Step 2.** For $x = 1$, $i < m - 1$. Applying Condition 1, it is easy to show that $\omega_T(1, i) \le \omega_T(x, i+1)$ holds. For $x = 1$ and $i = m - 1$, clearly $\omega_T(i, m)$ represents a system failure with a high cost.

Assume that Lemma 4 holds for $x - 1$; we show that it holds for $1 < x < N$.

(i) For $1 < x < N$, $i = 0$. Applying Condition 1 and Condition 3 leads immediately to the proof.

(ii) For $1 < x < N$, $0 < i < m - 1$. Equation (11f) leads to

$$\omega_T(x, i+1) = \min\left\{\begin{array}{l} r_{i+1} + \alpha \sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k, x-1)S_{i+1,j}[p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)], \\ c_{i+1} + r_0 + \alpha \sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k, x-2)S_{0j}[p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)]. \end{array}\right\} \tag{A8}$$

Consider the first option, in which no maintenance is performed. Then

$$\begin{aligned} \omega_T(x, i+1) &= r_{i+1} + \alpha \sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k, x-1)S_{i+1,j}[p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)] \\ &\ge r_i + \alpha \sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k, x-1)S_{i,j}[p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)] \\ &= \omega_T(x, i), \end{aligned} \tag{A9}$$

which follows due to Condition 1, Condition 3, and the induction hypothesis. For the case of a replacement, applying Condition 2 yields a similar proof. Note that following Equation (11c) for $i = m - 1$ we have

$$\begin{aligned} \omega_T(x, m-1) &= \min\left\{\begin{array}{l} r_{m-1} + \alpha \sum_{k=0}^{x-1}\sum_{j=0}^{m} C(k, x-1)S_{m-1,j}[p\omega_{T-1}(x+1-k, j) + q\omega_{T-1}(x-k, j)], \\ c_{m-1} + r_0 + \alpha \sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k, x-2)S_{0j}[p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)] \end{array}\right\} \\ &\le c_{m-1} + r_0 + \alpha \sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k, x-2)S_{0j}[p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)] \\ &\le c_m + r_0 + \alpha \sum_{k=0}^{x-2}\sum_{j=0}^{m} C(k, x-2)S_{0j}[p\omega_{T-1}(x-k, j) + q\omega_{T-1}(x-k-1, j)] \\ &= \omega_T(x, m). \end{aligned} \tag{A10}$$

Finally, the case of $x = N$ is similar and is omitted. □

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