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The Israeli Queue with a general group-joining policy

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Abstract We consider a single-server multi-queue system with unlimited-size batch service where the next queue to be served is the one with the most senior customer (the so called 'Israeli Queue'). We study a Markovian system with state-dependent group-joining policy and derive results for various performance measures, such as steady-state distribution of the number of groups in the system, sojourn times, group sizes, and lengths of busy periods. Closed-form expressions are obtained for both the Uniform and the Geometric joining policies. Numerical results are presented.

Keywords Queueing · Unlimited-size batch service · Israeli Queue · Sojourn times · Group sizes · Busy periods

Mathematics Subject Classification 60K25 · 90B22

1 Introduction

The so called 'Israeli Queue' is a queue of groups, instead of individuals. Each arriving customer joins a group already waiting in line, or creates a new group and becomes its leader. When reaching the server, the entire group is being served, where service time is independent

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Uri Yechiali dedicates this paper to Benny Avi-Itzhak, his first lecturer in Probability Theory, and to Matt Sobel, a long time colleague.

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of the group's size. The order in which groups are served is determined by the order of arrival of its leaders. The term 'Israeli Queue' originated from a real-life situation when considering a physical waiting line for buying tickets to a movie or a show. A line of groups is formed, headed by a 'leader', the first customer that originates the group. New arrival that knows a leader already standing in line joins his group. When the leader reaches the cashier he buys tickets for the entire group. It is assumed that the buying process is (almost) not affected by the number of tickets purchased.

This system resembles a polling system with batch service, in which a single server circles between the different queues, where the next queue to be served is the one with the most senior customer (i.e., the leader that has been waiting for the longest time). Unlimited-size batch service in an N-queue polling system was first studied by van der Wal and Yechiali (2003) when analyzing a computer tape-reading problem in a system where large amounts of information are stored on tapes, and requests for retrieving information from the various tapes arrive randomly. Optimal visiting rules of the server were derived for various objective functions without requiring the steady-state distribution function of the system's state. Probabilistic properties of such a system were analyzed in Boxma et al. (2007, 2008).

Unlimited-size batch service models were also considered in the literature as application to videotex, telex and TDMA (Time Division Multiple Access) systems (Dykeman et al. 1986; Ammar and Wong 1987; Liu and Nain 1992). In addition, an Automated Guided Vehicle system was formulated as a polling model with an infinite capacity batch service (Van Oyen and Teneketzis 1996).

Subsequently, in Perel and Yechiali (2013, 2014a, b), systems with unlimited-size batch service were studied, where the individual customers' group joining policy is Geometric(p). That is, if n groups are present in the system, then a newly arriving customer joins group k ($k \leq 1 \leq n$) with probability $(1-p)^{k-1}p$, or creates a new group with probability $(1-p)^n$. Single-server and multi-server queues (2014a), priority queues (2013) and retrial queues (2014b) were analyzed. In this paper we consider the Israeli Queue under *general* group-joining policy. That is, we assume that when n groups are present in the system, the probability that a new arrival joins the kth group ($1 \leq k \leq n$) is $p_{n,k}$ and the probability for a new group to be formed (last in the line of groups) is $p_{n,n+1}$, where $\sum_{k=1}^{n+1} p_{n,k} = 1$. The overall arrival process is Poisson with rate λ, and the service is given in unlimited-size batches. That is, it takes one (random) service duration to serve a group, independent of its size. We assume that a service duration of each group is exponentially distributed with parameter μ. We further assume that an arriving customer can join the group which is being served.

In Section 2 we present the general model and derive: (i) the steady-state distribution of the number of groups in the system; (ii) the Laplace-Stieltjes Transforms (LST's), as well as the means, of the sojourn time, both of a group leader and of an arbitrary customer; (iii)the mean groups' sizes right after a service completion or an arrival; and (iv) the mean length of a busy period starting with $n \geq 1$ groups. In Section 3 we assume a Uniform group-joining Policy. That is, when the number of groups in the system is n, for $n \geq 0$, a newly arriving customer joins any of the existing groups with probability $\frac{1}{n+1}$, or creates a new group, the $(n+1)$-st, with the same probability. We analyze this system both for finite, or possibly infinite, number of groups. In Section 4 we assume that the number of groups present in the system is at most N, and consider Geometric group-joining policy. That is, if there are $1 \leq n \leq N-1$ groups in the system, then a new arrival joins the kth group with probability $(1-p)^{k-1}p$, for $1 \leq k \leq n$, or creates a new group (the $(n+1)$-st) with probability $(1-p)^n$. Also, if N groups are present and a new arrival does not join any of the first $N-1$ groups,

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he/she will necessarily join the last group (in the Nth position). The arrival process and group service times are exponential, as described above. The contribution in this section is a vast extension and elaborate treatment of the Geometric model, including issues that were not studied in Perel and Yechiali (2014a). Finally, in Section 5 we present numerical results for all models considered, and discuss the parameters' effects on the various performance measures.

2 General joining probabilities

In this section we consider a single-sever queueing system where the arrival process of individual customers is Poisson with rate λ and the queue is comprised of groups. Service to a group is given simultaneously to all its members (batch service) and the service time of a batch is exponentially distributed with parameter μ. We assume that when there are $n \geq 1$ groups in the system, an arriving customer joins the kth group with probability p_{nk}, for $k = 1, 2, \ldots, n$, or creates a new group (the last in the line of groups) with probability $p_{n,n+1}$. When the system is empty, an arriving customer creates the first group in line with probability 1, that is $p_{01} = 1$. Clearly, for all $n \geq 0$, $\sum_{k=1}^{n+1} p_{nk} = 1$. We study the case where the number of groups is unbounded, and derive various performance measures. Throughout the paper, we use the following notation: X= number of groups in the system in steady-state; $\pi_n = \mathbb{P}(X = n)$; W= sojourn time of a group leader; W^a= sojourn time of an arbitrary customer; L_k= size of the group in the kth position after an arrival or service completion; and Θ_n= busy period starting with n groups.

2.1 Steady-state probabilities

We assume that X, the number of possible groups, is unlimited. For stability, we assume that there exists an M such that for all $n > M$, $\lambda p_{n,n+1} < \mu$. The balance equations determining the probability distribution of the number of groups in the system are

$$\lambda \pi_n p_{n,n+1} = \mu \pi_{n+1}, \quad n \geq 0. \tag{2.1}$$

Iteration of (2.1) yields

$$\pi_n = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \prod_{i=0}^{n-1} p_{i,i+1}, \tag{2.2}$$

where $\pi_0 = \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \prod_{i=0}^{n-1} p_{i,i+1}\right)^{-1}$, with $\prod_{i=0}^{-1} (\cdot) \triangleq 1$.
The mean number of groups in the system is $\mathbb{E}[X] = \sum_{n=0}^{\infty} n\pi_n$.

2.2 Sojourn times

We wish to derive the LST and mean of the sojourn time in the system of a group leader, and of an arbitrary customer. We first calculate P_{new}, the probability that an arriving customer creates a new group. We have,

$$P_{new} = \sum_{n=0}^{\infty} \pi_n p_{n,n+1} = \sum_{n=0}^{\infty} \frac{\mu}{\lambda} \pi_{n+1} = \frac{\mu}{\lambda}(1 - \pi_0). \tag{2.3}$$

Therefore,

$$\lambda P_{new} = \mu(1 - \pi_0). \tag{2.4}$$

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Equation (2.4) simply states that the rate of group generation equals the rate of group departures. Let W denote the total sojourn time of a group *leader* in the system and let $\widetilde{W}(\cdot)$ denote its LST. Then, using (2.1),

$$\widetilde{W}(s) = \frac{1}{P_{new}} \sum_{n=0}^{\infty} \pi_n p_{n,n+1} \left(\frac{\mu}{\mu+s}\right)^{n+1} = \frac{1}{1-\pi_0} \sum_{n=0}^{\infty} \pi_{n+1} \left(\frac{\mu}{\mu+s}\right)^{n+1}, \tag{2.5}$$

and

$$\mathbb{E}[W] = -\widetilde{W}'(s)|_{s=0} = \frac{1}{1-\pi_0} \sum_{n=0}^{\infty} (n+1)\pi_{n+1} \frac{1}{\mu} = \frac{\mathbb{E}[X]}{\mu(1-\pi_0)} = \frac{\mathbb{E}[X]}{\lambda P_{new}}. \tag{2.6}$$

Define Z as the *position* in which a new group is formed. Then,

$$\mathbb{P}(Z=n) = \frac{1}{P_{new}} \pi_{n-1} p_{n-1,n} = \frac{1}{P_{new}} \frac{\mu}{\lambda} \pi_n = \frac{\pi_n}{1-\pi_0} = \mathbb{P}(X=n|X>0), \quad n=1,2,\ldots$$

which implies that

$$\mathbb{E}[Z] = \frac{\mathbb{E}[X]}{1-\pi_0}.$$

That is, $\mathbb{E}[W] = \frac{1}{\mu}\mathbb{E}[Z]$, which is the mean service time, $\frac{1}{\mu}$, multiplied by $\mathbb{E}[Z]$, the mean position in which a new group is formed.

To calculate the LST and mean of W^a, the sojourn time of an *arbitrary* customer, we condition on the position of the group that the customer joins. Since the LST of a group's service time is $\frac{\mu}{\mu+s}$, we have,

$$\widetilde{W}^a(s) = \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n+1} p_{n,k} \left(\frac{\mu}{\mu+s}\right)^k,$$

and

$$\mathbb{E}\left[W^a\right] = \frac{1}{\mu} \sum_{n=0}^{\infty} \pi_n \sum_{k=1}^{n+1} k p_{n,k}.$$

Define Z^a as the position of the group that an arbitrary customer joins. Then,

$$\mathbb{P}(Z^a = n) = \sum_{k=n-1}^{\infty} \pi_k p_{k,n},$$

and

$$\mathbb{E}[Z^a] = \sum_{n=1}^{\infty} n \mathbb{P}(Z^a = n) = \sum_{n=1}^{\infty} n \sum_{k=n-1}^{\infty} \pi_k p_{k,n} = \sum_{k=0}^{\infty} \pi_k \sum_{n=1}^{k+1} n p_{k,n}$$

As expected, $\mathbb{E}[W^a] = \frac{1}{\mu}\mathbb{E}[Z^a]$.

2.3 Number of customers in the kth group

Define a Poissonian event as either an arrival of a new customer or a group service completion. Let L_k^m denote the number of *customers* present in the kth group ($k = 1, 2, \ldots$) immediately after the mth Poissonian event occurs, for $m \geq 1$, and let $\vec{L}^m = \left(L_1^m, L_2^m, \ldots\right)$. We now

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observe the system at two successive Poissonian events, m and $m+1$. Note that, if the system is not empty, the time elapsing until the next Poissonian event is exponentially distributed with mean $\frac{1}{\lambda+\mu}$, whereas, if the system is empty, the time elapsing until the next Poissonian event is exponentially distributed with mean $\frac{1}{\lambda}$.

Let $\{Y_m, m \geq 1\}$ be the number of *groups* in the system a moment before the mth Poissonian event occurs. $\{Y_m, m \geq 1\}$ defines an infinite (semi) Markov chain with one-step transition probabilities $\nu_{ij} = \mathbb{P}(Y_{m+1} = j | Y_m = i)$, for $i, j = 0, 1, 2 \ldots$. Let $Q = \left[\nu_{ij}\right]$ be the one step transition probability matrix of the process $\{Y_m, m \geq 1\}$. Then, Q is given by

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-p_{1,2})}{\lambda+\mu} & \frac{\lambda p_{1,2}}{\lambda+\mu} & 0 & 0 & \cdots & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-p_{2,3})}{\lambda+\mu} & \frac{\lambda p_{2,3}}{\lambda+\mu} & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-p_{n,n+1})}{\lambda+\mu} & \frac{\lambda p_{n,n+1}}{\lambda+\mu} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots)$ denote the steady-state distribution of $Y = \lim_{m\to\infty} Y_m$, where $\sigma_k = \mathbb{P}(Y = k)$, $\vec{\sigma} Q = \vec{\sigma}$, and $\sum_{k=0}^{\infty} \sigma_k = 1$. By performing standard calculations we get, for $k \geq 1$,

$$\sigma_k = \sigma_0(\lambda+\mu)\frac{\lambda^{k-1}}{\mu^k}\prod_{i=1}^{k-1} p_{i,i+1}, \tag{2.7}$$

where σ_0 is obtained from the normalization equation, $\sum_{k=0}^{\infty} \sigma_k = 1$. We thus have

$$\sigma_0 = \left(1 + (\lambda+\mu)\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{\mu^k}\prod_{i=1}^{k-1} p_{i,i+1}\right)^{-1}. \tag{2.8}$$

In fact, σ_k is the long-run fraction of visits of the process Y at state k. Then, the proportion of time that there are k groups in the system is given by Ross (1997)

$$\pi_0 = \frac{\frac{\sigma_0}{\lambda}}{\frac{\sigma_0}{\lambda} + \frac{1}{\lambda+\mu}\sum_{j=1}^{\infty}\sigma_j},$$

$$\pi_k = \frac{\frac{\sigma_k}{\lambda+\mu}}{\frac{\sigma_0}{\lambda} + \frac{1}{\lambda+\mu}\sum_{j=1}^{\infty}\sigma_j}, \quad k \geq 1. \tag{2.9}$$

Indeed, substituting in Eq. (2.9) the expressions for σ_k given in equations (2.7) and (2.8), results in Eq. (2.2).

Consider the process $\left(\vec{L}^m\right)_{m=1}^{\infty}$ in steady state, so that $L_k^m \to L_k$ when $m \to \infty$. If the system is empty a moment before a Poissonian event (with probability σ_0), the next Poissonian event will be an arrival, so that the first group will contain a single customer. Next, assume that only a single group is in the system (with probability σ_1). Then, if the next event is an arrival (with probability $\frac{\lambda}{\lambda+\mu}$), then the new customer will join the single group with probability $p_{1,1}$ or will create a new (second) group (with probability $p_{1,2}$)). However, if a service completion occurs before an arrival, the system will become empty. This occurs with probability $\frac{\mu}{\lambda+\mu}$. In this manner, we consider all possible vectors of group sizes and all possible events. Thus, if k groups are present (with probability σ_k), an arriving customer may either join one of these groups, or create a new group (with the corresponding probabilities). In all cases, when the system is not empty, a service completion before an arrival causes each group to move one position forward towards the server. We then have

$$(L_1, L_2, L_3, \ldots) \stackrel{d}{=} \begin{cases} (1, 0, 0, 0, \ldots,) & \text{w.p.} \quad \sigma_0 \\ (L_1 + 1, 0, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{1,1}}{\lambda+\mu}\sigma_1 \\ (L_1, 1, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{1,2}}{\lambda+\mu}\sigma_1 \\ (0, 0, 0, 0, \ldots) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_1 \\ (L_1 + 1, L_2, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{2,1}}{\lambda+\mu}\sigma_2 \\ (L_1, L_2 + 1, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{2,2}}{\lambda+\mu}\sigma_2 \\ (L_1, L_2, 1, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{2,3}}{\lambda+\mu}\sigma_2 \\ (L_2, 0, 0, 0, \ldots) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_2 \\ \vdots & \vdots \\ (L_1 + 1, L_2, \ldots, L_k, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{k,1}}{\lambda+\mu}\sigma_k \\ \vdots & \vdots \\ (L_1, L_2, \ldots, L_k + 1, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{k,k}}{\lambda+\mu}\sigma_k \\ (L_1, L_2, \ldots, L_k, 1, 0, 0, \ldots) & \text{w.p.} \quad \frac{\lambda p_{k,k+1}}{\lambda+\mu}\sigma_k \\ (L_2, L_3, \ldots, L_{k-1}, 0, 0, \ldots) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_k \\ \vdots & \vdots \end{cases} \tag{2.10}$$

From relation (2.10) we have, for all $k \geq 1$,

$$\mathbb{E}[L_k] = \mathbb{E}_Y\left[\mathbb{E}[L_k|Y]\right] = \sum_{j=0}^{\infty} \mathbb{P}(Y = j)\mathbb{E}[L_k|Y = j] = \sum_{j=0}^{\infty} \sigma_j \mathbb{E}[L_k|Y = j]. \tag{2.11}$$

Specifically,

$$\mathbb{E}[L_1] = \sigma_0 + \sum_{j=1}^{\infty} \frac{\lambda p_{j,1}\sigma_j}{\lambda+\mu} + \mathbb{E}[L_1]\sum_{j=1}^{\infty} \frac{\lambda\sigma_j}{\lambda+\mu} + \mathbb{E}[L_2]\sum_{j=2}^{\infty} \frac{\mu\sigma_j}{\lambda+\mu}, \tag{2.12}$$

$$\mathbb{E}[L_k] = \frac{\lambda p_{k-1,k}\sigma_{k-1}}{\lambda+\mu} + \sum_{j=k}^{\infty} \frac{\lambda p_{j,k}\sigma_j}{\lambda+\mu} + \mathbb{E}[L_k]\sum_{j=k}^{\infty} \frac{\lambda\sigma_j}{\lambda+\mu} + \mathbb{E}[L_{k+1}]\sum_{j=k+1}^{\infty} \frac{\mu\sigma_j}{\lambda+\mu}, \quad k \geq 2. \tag{2.13}$$

Define:

$$q_1 = \sigma_0 + \frac{\lambda}{\lambda+\mu}\sum_{j=1}^{\infty} p_{j,1}\sigma_j,$$

$$q_k = \frac{\lambda p_{k-1,k}}{\lambda+\mu}\sigma_{k-1} + \frac{\lambda}{\lambda+\mu}\sum_{j=k}^{\infty} p_{j,k}\sigma_j, \quad k \geq 2,$$

$$\alpha_k = \frac{\lambda}{\lambda+\mu}\sum_{j=k}^{\infty}\sigma_j = \frac{\sigma_0}{\pi_0}\sum_{j=k}^{\infty}\pi_j, \quad k \geq 1,$$

$$\beta_k = \frac{\mu}{\lambda+\mu}\sum_{j=k+1}^{\infty}\sigma_j = \frac{\mu}{\lambda}\alpha_{k+1} \ \ k \geq 1.$$

Then, Eqs. (2.12) and (2.13) can be written as

$$\mathbb{E}[L_k] = q_k + \mathbb{E}[L_k]\alpha_k + \mathbb{E}[L_{k+1}]\beta_k, \quad k \geq 1,$$

or

$$\mathbb{E}[L_k] = \frac{q_k}{1-\alpha_k} + \frac{\beta_k}{1-\alpha_k}\mathbb{E}[L_{k+1}]. \tag{2.14}$$

Iterating equation (2.14) n times gives

$$\mathbb{E}[L_k] = \sum_{j=0}^{n-1}\frac{q_{k+j}}{1-\alpha_{k+j}}\prod_{i=0}^{j-1}\frac{\beta_{k+i}}{1-\alpha_{k+i}} + \mathbb{E}[L_{k+n}]\prod_{j=0}^{n-1}\frac{\beta_{k+j}}{1-\alpha_{k+j}}. \tag{2.15}$$

Since both α_k and β_k tend to zero when k becomes large, the expression $\prod_{j=0}^{n-1}\frac{\beta_{k+j}}{1-\alpha_{k+j}}$ tends to 0 as $n \longrightarrow \infty$, so that $\mathbb{E}[L_k]$ may be well approximated by considering only the first term in Eq. (2.15) for n sufficiently large.

2.4 The busy period

Let Θ_n ($n = 1, 2, \ldots$) denote the time from a moment when there are n groups in the system until the first moment thereafter when no groups are present. Define for $n \geq 0$, $\lambda_n = \lambda p_{n,n+1}$. Let $Exp(\lambda)$ denote an exponential distribution with parameter λ. Then, for $n \geq 1$, the following relation holds,

$$\Theta_n \stackrel{d}{=} Exp\left(\lambda p_{n,n+1} + \mu\right) + \begin{cases} \Theta_{n-1} & w.p. \ \frac{\mu}{\lambda_n+\mu} \\ \Theta_{n+1} & w.p. \ \frac{\lambda_n}{\lambda_n+\mu} \end{cases}, \tag{2.16}$$

where $\Theta_0 = 0$. This gives,

$$\mathbb{E}[\Theta_n] = \frac{1}{\lambda_n+\mu} + \frac{\mu}{\lambda_n+\mu}\mathbb{E}[\Theta_{n-1}] + \frac{\lambda_n}{\lambda_n+\mu}\mathbb{E}[\Theta_{n+1}],$$

or

$$\mathbb{E}[\Theta_{n+1}] = \frac{\lambda_n+\mu}{\lambda_n}\mathbb{E}[\Theta_n] - \frac{\mu}{\lambda_n}\mathbb{E}[\Theta_{n-1}] - \frac{1}{\lambda_n}. \tag{2.17}$$

To derive $\mathbb{E}[\Theta_1]$, the mean period of time during which the server is working continuously, starting from the first arrival to an empty system, we note that the idle time of the server is $\text{Exp}(\lambda)$. Thus, we get

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$$\frac{\mathbb{E}[\Theta_1]}{\frac{1}{\lambda}+\mathbb{E}[\Theta_1]} = 1-\pi_0,$$

resulting in

$$\mathbb{E}[\Theta_1] = \frac{1-\pi_0}{\lambda\pi_0}. \tag{2.18}$$

To solve the recurrence relation (2.17), we rewrite it as follows:

$$\mathbb{E}[\Theta_n]-\mathbb{E}[\Theta_{n-1}] = \frac{\mu}{\lambda_{n-1}}\Big(\mathbb{E}[\Theta_{n-1}]-\mathbb{E}[\Theta_{n-2}]\Big)-\frac{1}{\lambda_{n-1}}.$$

Iterating the above equation leads to

$$\mathbb{E}[\Theta_n]-\mathbb{E}[\Theta_{n-1}] = \mu^{n-1}\prod_{i=1}^{n-1}\frac{1}{\lambda_{n-i}}\mathbb{E}[\Theta_1]-\sum_{j=0}^{n-2}\mu^j\prod_{i=0}^{j}\frac{1}{\lambda_{n-i-1}}. \tag{2.19}$$

Finally, moving $\mathbb{E}[\Theta_{n-1}]$ to the RHS of (2.19) and iterating again leads to

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_1]\sum_{k=1}^{n}\mu^{n-k}\prod_{i=1}^{n-k}\frac{1}{\lambda_{n-k-i+1}}-\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}\mu^j\prod_{i=0}^{j}\frac{1}{\lambda_{n-i-k}}, \tag{2.20}$$

where $\mathbb{E}[\Theta_1]$ is given in (2.18).

In the next sections we consider both Uniform (Section 3) and Geometric (Section 4) group-joining policies. In these models we also consider the case where the number of groups present in the system is finite and can be at most N.

3 Model 1: Uniform joining probability

3.1 Unbounded number of groups

3.1.1 Steady-state probabilities

We assume that X, the number of possible groups, is unbounded. If n groups are present, $n \geq 0$, an arriving customer can join any of the existing groups with probability $p_{n,k} = \frac{1}{n+1}$, $k = 1, 2, \ldots, n$; or creates a new group (the last in the line of groups) with probability $p_{n,n+1} = \frac{1}{n+1}$. A customer arriving to an empty queue initiates the first group in the system. Equation (2.1) now results in

$$\pi_n = \pi_0\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, \tag{3.1}$$

where $\pi_0 = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = e^{-\frac{\lambda}{\mu}}$.

That is, X is a Poisson random variable with parameter $\left(\frac{\lambda}{\mu}\right)$, which, interestingly, is the same as the distribution of the number of customers in an $M/M/\infty$ queue with Poisson arrival rate λ and exponentially distributed service time with parameter μ. This follows since in the Uniform-joining Israeli Queue $\lambda_n = \frac{\lambda}{n+1}$ and $\mu_{n+1} = \mu$, while in the $M/M/\infty$ queue, $\lambda_n = \lambda$ and $\mu_{n+1} = (n+1)\mu$. This leads to the same ratio $\frac{\lambda_n}{\mu_{n+1}} = \frac{\lambda}{\mu(n+1)}$ in both models.

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3.1.2 Sojourn times

Under the Uniform group-joining policy equation (2.3) results in

$$P_{new} = \frac{\mu}{\lambda}(1-\pi_0) = \frac{\mu}{\lambda}\left(1-e^{-\frac{\lambda}{\mu}}\right).$$

Equation (2.5) becomes

$$\widetilde{W}(s) = \frac{1}{P_{new}}\sum_{n=0}^{\infty}\pi_n\frac{1}{n+1}\left(\frac{\mu}{\mu+s}\right)^{n+1} = \frac{e^{\frac{\lambda}{\mu+s}}-1}{e^{\frac{\lambda}{\mu}}-1},$$

and

$$\mathbb{E}[W] = -\widetilde{W}'(s)|_{s=0} = \frac{\lambda e^{\frac{\lambda}{\mu}}}{\mu^2(e^{\frac{\lambda}{\mu}}-1)} = \frac{\lambda}{\mu^2(1-e^{-\frac{\lambda}{\mu}})}. \tag{3.2}$$

The distribution of Z, the position in which a new group is formed, is given by

$$\mathbb{P}(Z=n) = \frac{1}{P_{new}}\pi_{n-1}\frac{1}{n} = \frac{\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}/n!}{1-e^{-\frac{\lambda}{\mu}}} = \mathbb{P}(X=n|X>0),\quad n=1,2,\ldots$$

and

$$\mathbb{E}[Z] = \frac{1}{P_{new}}\sum_{n=1}^{\infty}n\pi_{n-1}\frac{1}{n} = \frac{\lambda}{\mu(1-e^{-\frac{\lambda}{\mu}})} = \mu\mathbb{E}[W].$$

The calculations of the mean and LST of W^a, the sojourn time of an *arbitrary* customer, yield

$$\widetilde{W}^a(s) = \sum_{n=0}^{\infty}\pi_n\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{\mu}{\mu+s}\right)^k,$$

which after some algebra results in

$$\widetilde{W}^a(s) = \frac{\mu^2}{\lambda s}\left(1-e^{\frac{\lambda s}{\mu(\mu+s)}}\right).$$

Differentiation gives

$$\mathbb{E}\left[W^a\right] = \frac{1}{\mu}+\frac{\lambda}{2\mu^2}. \tag{3.3}$$

Note that $\mathbb{E}[W^a]$ is linear in λ. Furthermore,

$$\mathbb{P}(Z^a=n) = \sum_{k=n-1}^{\infty}\pi_k\frac{1}{k+1},$$

which leads to

$$\mathbb{E}[Z^a] = \sum_{n=1}^{\infty}n\mathbb{P}(Z^a=n) = \frac{\lambda}{2\mu}+1 = \mu\mathbb{E}[W^a]. \tag{3.4}$$

Intuitively, the sojourn time of an arbitrary customer should not exceed the sojourn time of a group leader. In the "Appendix" we prove the following:

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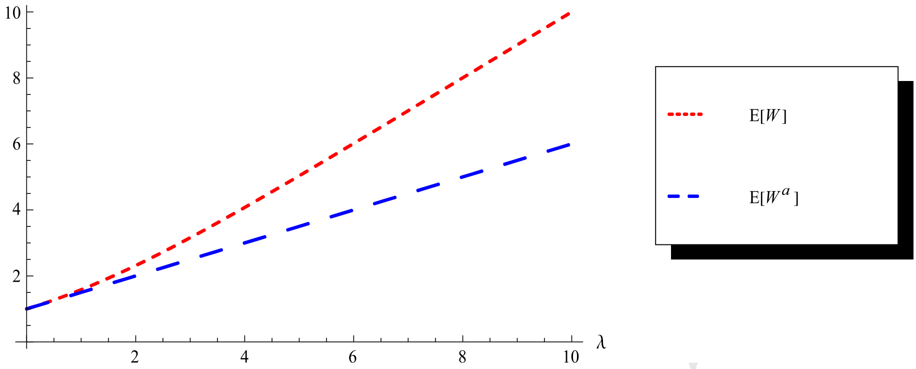

Fig. 1 $\mathbb{E}[W]$ and $\mathbb{E}[W^a]$ as a function of λ for $\mu = 1$

Proposition 3.1 *For any* $\lambda, \mu \geq 0$, $\mathbb{E}[W^a] \leq \mathbb{E}[W]$.

Furthermore, for large values of λ, we have

$$\lim_{\lambda\to\infty} \frac{\mathbb{E}[W^a]}{\mathbb{E}[W]} = \lim_{\lambda\to\infty} \frac{(2\mu+\lambda)(1-e^{-\frac{\lambda}{\mu}})}{2\lambda} = \frac{1}{2}.$$

Indeed, an arbitrary customer joins, on the average, the middle group, while a group leader forms a new group, last in the line of groups. $\mathbb{E}[W]$ and $\mathbb{E}[W^a]$ are depicted in Fig. 1 below.

3.1.3 Number of customers in the kth group

Following the general results of Section 2.3, in the case of Uniform group-joining policy, the matrix Q and the vector $\vec{\sigma}$ are given by:

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{2(\lambda+\mu)} & \frac{\lambda}{2(\lambda+\mu)} & 0 & 0 & \cdots & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{2\lambda}{3(\lambda+\mu)} & \frac{\lambda}{3(\lambda+\mu)} & 0 & \cdots & \cdots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{3\lambda}{4(\lambda+\mu)} & \frac{\lambda}{4(\lambda+\mu)} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

$$\sigma_k = \sigma_0(\lambda+\mu)\frac{\lambda^{k-1}}{k!\mu^k}, \quad k = 1, 2, \ldots \tag{3.5}$$

and

$$\sigma_0 = \frac{\lambda}{(\lambda+\mu)e^{\frac{\lambda}{\mu}} - \mu}. \tag{3.6}$$

Consider now the group sizes at Poissonian events. Using equations (3.5) and $p_{n,k} = \frac{1}{n+1}$ for $k = 1, 2, \ldots, n+1$, Eqs. (2.12) and (2.13) become, for $k \geq 1$,

$$\mathbb{E}[L_k] = \sigma_0 \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{k!} + \sigma_0 \sum_{j=k}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j} \frac{1}{(j+1)!} + \mathbb{E}[L_k]\sigma_0 \sum_{j=k}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j} \frac{1}{j!}$$
$$+ \mathbb{E}[L_{k+1}]\sigma_0 \sum_{j=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j-1} \frac{1}{j!}.$$

Finally, $\mathbb{E}[L_k]$ is given by Eq. (2.15) with

$$\alpha_k = \sigma_0 \sum_{j=k}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j} \frac{1}{j!},$$
$$\beta_k = \sigma_0 \sum_{j=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j-1} \frac{1}{j!},$$
$$q_k = \sigma_0 \left(\frac{\lambda}{\mu}\right)^{k-1} \frac{1}{k!} + \sigma_0 \sum_{j=k}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j} \frac{1}{(j+1)!}.$$

Figure 2 depicts $\mathbb{E}[L_k]$ for $k = 1, 5, 10$. Evidently, the mean group size decreases with k.

3.1.4 The busy period

Equation (2.16) becomes

$$\Theta_n \stackrel{d}{=} Exp\left(\frac{\lambda}{n+1} + \mu\right) + \begin{cases} \Theta_{n-1} & w.p. \quad \frac{\mu}{\frac{\lambda}{n+1}+\mu} \\ \Theta_{n+1} & w.p. \quad \frac{\frac{\lambda}{n+1}}{\frac{\lambda}{n+1}+\mu} \end{cases}. \tag{3.7}$$

Now, (2.18) results in

$$\mathbb{E}[\Theta_1] = \frac{e^{\frac{\lambda}{\mu}} - 1}{\lambda}, \tag{3.8}$$

and (2.20) is given by

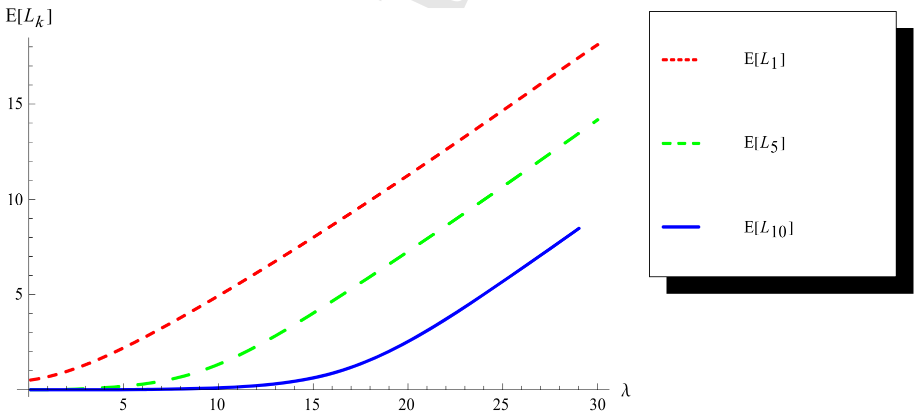

Fig. 2 $\mathbb{E}[L_k]$ as a function of λ for $k = 1, 5, 10$

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_1] \sum_{k=1}^{n} k! \left(\frac{\mu}{\lambda}\right)^{k-1} - \sum_{k=0}^{n-2} \sum_{j=1}^{n-1-k} \frac{(n-k)!}{(n-k-j)!} \frac{\mu^{j-1}}{\lambda^j}. \tag{3.9}$$

3.2 Finite number of groups

In this section we assume that the number of groups in the system is at most N. If $0 \le n \le N-1$ groups are present, then an arriving customer can join any of the existing groups with probability $\frac{1}{n+1}$, or create a new group (the last in the line of groups) with probability $\frac{1}{n+1}$. However, if N groups are in the system, an arriving customer can only join any of the existing N groups (with probability $\frac{1}{N}$), but can not create a new group. The performance measures in this case are calculated similarly as in Section 3.1, so we omit most of the calculations and present the final results.

The steady state distribution of the number of groups in the system is

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \pi_0,$$

$$\pi_0 = \left(\sum_{n=0}^{N} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n\right)^{-1} = \frac{N!}{e^{\frac{\lambda}{\mu}} \Gamma\left(N+1, \frac{\lambda}{\mu}\right)},$$

where $\Gamma(k, x) = \int_x^\infty t^{k-1} e^{-t} dt$ is the incomplete Gamma function. The probability of creating a new group in the system is

$$P_{new} = \sum_{n=0}^{N-1} \pi_n \frac{1}{n+1} = \pi_0 \sum_{n=0}^{N-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{(n+1)!}.$$

The LST and mean of the sojourn time in the system, both for a group leader and for an arbitrary customer, are

$$\widetilde{W}(s) = \frac{\sum_{n=0}^{N-1} \left(\frac{\lambda}{\mu+s}\right)^{n+1} \frac{1}{(n+1)!}}{\sum_{n=0}^{N-1} \left(\frac{\lambda}{\mu}\right)^{n+1} \frac{1}{(n+1)!}} = \frac{e^{\frac{\lambda}{\mu+s}} \Gamma(N+1, \frac{\lambda}{\mu+s}) - N!}{e^{\frac{\lambda}{\mu}} \Gamma(N+1, \frac{\lambda}{\mu}) - N!},$$

$$\mathbb{E}[W] = \frac{\lambda(1-\pi_N)}{\mu^2(1-\pi_0)},$$

$$\widetilde{W}^a(s) = \sum_{n=0}^{N-1} \pi_n \frac{1}{n+1} \sum_{k=1}^{n+1} \left(\frac{\mu}{\mu+s}\right)^k + \pi_N \frac{1}{N} \sum_{k=1}^{N} \left(\frac{\mu}{\mu+s}\right)^k,$$

$$\mathbb{E}[W^a] = \sum_{n=0}^{N-1} \pi_n \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{k}{\mu} + \pi_N \frac{1}{N} \sum_{k=1}^{N} \frac{k}{\mu} = \sum_{n=0}^{N-1} \pi_n \frac{n+2}{2\mu} + \pi_N \frac{N+1}{2\mu}$$

$$= \frac{1}{2\mu} \left(\mathbb{E}[X] - \pi_N + 2\right).$$

The mean number of customers present in the kth group, for $k = 1, 2, \ldots, N$, is given by

$$\mathbb{E}[L_k] = \sum_{j=0}^{N-k} \frac{q_{k+j}}{1-\alpha_{k+j}} \prod_{i=0}^{j-1} \frac{\beta_{k+1+i}}{1-\alpha_{k+i}},$$

where

$$\alpha_k = \sum_{j=k}^{N} \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!} \sigma_0,$$

$$\beta_k = \frac{\mu}{\lambda} \alpha_k,$$

$$q_k = \sigma_0 \sum_{j=k-1}^{N} \frac{1}{(j+1)!} \left(\frac{\lambda}{\mu}\right)^j,$$

$$\sigma_0 = \left[1 + \sum_{n=1}^{N} (\lambda + \mu) \frac{\lambda^{n-1}}{n! \mu^n} \right]^{-1}.$$

Finally, the mean busy period, starting with $1 \le n \le N$ groups, is

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_N] - \sum_{j=1}^{N-n-1} \frac{\lambda^j}{\mu^{j+1}} \sum_{i=1}^{N-n-j} \frac{(N-i+1-j)!}{(N-i+1)!} - \frac{N-n}{\mu}. \tag{3.10}$$

Setting $n = 1$ in (3.10) and using Eq. (2.18) which holds in this model too, we obtain an expression for $\mathbb{E}[\Theta_N]$, from which we finally get

$$\mathbb{E}[\Theta_n] = \frac{1-\pi_0}{\lambda \pi_0} + \sum_{j=1}^{N-2} \frac{\lambda^j}{\mu^{j+1}} \sum_{i=1}^{N-1-j} \frac{(N-i+1-j)!}{(N-i+1)!} - \sum_{j=1}^{N-n-1} \frac{\lambda^j}{\mu^{j+1}} \sum_{i=1}^{N-n-j} \frac{(N-i+1-j)!}{(N-i+1)!} + \frac{n-1}{\mu}.$$

4 Model 2: Geometric joining probability; finite N

The Geometric group-joining policy with infinite number of groups was analyzed in Perel and Yechiali (2014a). The finite case with at most N groups was only partially discussed there, and the following results were obtained. The steady-state probabilities of the number of groups in the system are

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n (1-p)^{\frac{n(n-1)}{2}} \pi_0, \quad 1 \le n \le N,$$

$$\pi_0 = \left(\sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n (1-p)^{\frac{n(n-1)}{2}} \right)^{-1}. \tag{4.1}$$

Let $D^{(k)}$ denote the total size of the group standing at the kth position ($1 \le k \le N$), *an instant after a service completion*. It was shown that, for $1 \le k \le N$,

$$\mathbb{E}\left[D^{(k)}\right] = \frac{\lambda}{\mu}(1-p)^{k-1} + \frac{\pi_k}{\sum_{j=k}^{N} \pi_j}. \tag{4.2}$$

In this section we extend the above results and derive: (i) the LST's and means of the sojourn times, both of a group leader and of an arbitrary customer; (ii) the mean groups' sizes right

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after a service completion or an arrival; and (iii) the LST and mean of the length of a busy period, starting with $n \geq 1$ groups.

4.1 Sojourn times

Let $G(z) = \sum_{n=0}^{N} \pi_n z^n$ be the Probability Generating Function (PGF) of X. Then, the probability of creating a new group in the system is given by

$$P_{new} = \sum_{n=0}^{N-1} \pi_n (1-p)^n = G(1-p) - \pi_N (1-p)^N. \tag{4.3}$$

Using relation (2.4) we get

$$G(1-p) = \frac{\mu}{\lambda}(1-\pi_0) + \pi_N (1-p)^N.$$

We then have,

$$\begin{aligned}\widetilde{W}(s) =& \frac{1}{P_{new}} \sum_{n=0}^{N-1} \pi_n (1-p)^n \left(\frac{\mu}{\mu+s}\right)^{n+1}\\ =& \frac{1}{P_{new}} \frac{\mu}{\mu+s} \left(G\left(\frac{(1-p)\mu}{\mu+s}\right) - \pi_N \frac{(1-p)\mu}{\mu+s} \right).\end{aligned}$$

Furthermore,

$$\mathbb{E}[W] = \frac{1}{P_{new}} \sum_{n=0}^{N-1} \pi_n (1-p)^n \left(\frac{n+1}{\mu}\right) = \frac{\mathbb{E}[X]}{\lambda P_{new}}. \tag{4.4}$$

To derive the mean and LST of W^a, we distinguishing between the events where a new arrival joins an existing group, and the event where he/she creates a new one. We write

$$\begin{aligned}\widetilde{W}^a(s) = \pi_0 \frac{\mu}{\mu+s} &+ \pi_1 \left(p\frac{\mu}{\mu+s} + (1-p)\left(\frac{\mu}{\mu+s}\right)^2 \right) + \ldots\\ &+ \pi_n \left(p\frac{\mu}{\mu+s} + (1-p)p\left(\frac{\mu}{\mu+s}\right)^2 + \ldots + (1-p)^{n-1} p \left(\frac{\mu}{\mu+s}\right)^n \right.\\ &\left. +(1-p)^n \left(\frac{\mu}{\mu+s}\right)^{n+1} \right) + \ldots\\ &+ \pi_N \left(p\frac{\mu}{\mu+s} + (1-p)p\left(\frac{\mu}{\mu+s}\right)^2 + \ldots + (1-p)^{N-2} p \left(\frac{\mu}{\mu+s}\right)^{N-1} \right.\\ &\left. + (1-p)^{N-1} \left(\frac{\mu}{\mu+s}\right)^N \right),\end{aligned}$$

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or, after some algebra,

$$\begin{aligned}
\widetilde{W}^a(s) &= \sum_{n=0}^{N-1} \pi_n (1-p)^n \left(\frac{\mu}{\mu+s}\right)^{n+1} + \sum_{n=0}^{N-1} \pi_n p \frac{\mu}{\mu+s} \sum_{k=0}^{n-1} \left(\frac{(1-p)\mu}{\mu+s}\right)^k \\
&\quad + \pi_N p \frac{\mu}{\mu+s} \sum_{k=0}^{N-2} \left(\frac{(1-p)\mu}{\mu+s}\right)^k + \pi_N (1-p)^{N-1} \left(\frac{\mu}{\mu+s}\right)^N \\
&= \sum_{n=0}^{N-1} \pi_n (1-p)^n \left(\frac{\mu}{\mu+s}\right)^{n+1} + \sum_{n=0}^{N-1} \pi_n \frac{\mu p}{\mu p+s} \left(1 - \left(\frac{(1-p)\mu}{\mu+s}\right)^n\right) \\
&\quad + \pi_N \frac{\mu p}{\mu p+s} \left(1 - \left(\frac{(1-p)\mu}{\mu+s}\right)^{N-1}\right) + \pi_N (1-p)^{N-1} \left(\frac{\mu}{\mu+s}\right)^N .
\end{aligned}$$

In the same manner, the mean waiting time of an arbitrary customer is calculated as

$$\begin{aligned}
\mathbb{E}[W^a] &= \sum_{n=0}^{N-1} \pi_n (1-p)^n \frac{n+1}{\mu} + \frac{1}{\mu p} \sum_{n=1}^{N-1} \pi_n \left(1 - (1-p)^n (1+np)\right) \\
&\quad + \pi_N \frac{1}{\mu p} \left(1 - (1-p)^{N-2}(1+(N-2)p)\right) + \pi_N (1-p)^{N-1} \frac{N}{\mu}.
\end{aligned}$$

4.2 Number of customers in the kth group

The one step transition probability matrix of the process $\{Y_m, m \geq 1\}$ defined in Sect. 2.3 is given by

$$Q = \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{\mu}{\lambda+\mu} & \frac{\lambda p}{\lambda+\mu} & \frac{\lambda(1-p)}{\lambda+\mu} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-(1-p)^2)}{\lambda+\mu} & \frac{\lambda(1-p)^2}{\lambda+\mu} & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda(1-(1-p)^{N-1})}{\lambda+\mu} & \frac{\lambda(1-p)^{N-1}}{\lambda+\mu} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}
\end{pmatrix}.$$

The calculation of the vector $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_N)$ leads to

$$\sigma_k = \sigma_0 (\lambda+\mu) \frac{\lambda^{k-1}}{\mu^k} (1-p)^{\frac{k(k-1)}{2}}, \tag{4.5}$$

where

$$\sigma_0 = \left(1 + (\lambda+\mu) \sum_{k=1}^{N} \frac{\lambda^{k-1}}{\mu^k} (1-p)^{\frac{k(k-1)}{2}}\right)^{-1}. \tag{4.6}$$

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The law of motion of the group sizes is

$$
(L_1, L_2, \ldots, L_N) \stackrel{d}{=} \begin{cases}
(1, 0, 0, \ldots, 0) & \text{w.p.} \quad \sigma_0 \\
(L_1 + 1, 0, \ldots, 0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_1 \\
(L_1, 1, 0, \ldots, 0) & \text{w.p.} \quad \frac{\lambda(1-p)}{\lambda+\mu}\sigma_1 \\
(0, 0, \ldots, 0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_1 \\
\vdots & \vdots \\
(L_1 + 1, L_2, \ldots, L_k, 0, \ldots, 0) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_k \\
\vdots & \vdots \\
(L_1, L_2, \ldots, L_k + 1, 0, \ldots, 0) & \text{w.p.} \quad \frac{\lambda(1-p)^{k-1} p}{\lambda+\mu}\sigma_k \\
(L_1, L_2, \ldots, L_k, 1, 0, \ldots, 0) & \text{w.p.} \quad \frac{\lambda(1-p)^k}{\lambda+\mu}\sigma_k \\
(L_2, L_3, \ldots, L_k, 0, \ldots, 0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_k \\
\vdots & \vdots \\
(L_1 + 1, L_2, \ldots, L_N) & \text{w.p.} \quad \frac{\lambda p}{\lambda+\mu}\sigma_N \\
\vdots & \vdots \\
(L_1, L_2, \ldots, L_N + 1) & \text{w.p.} \quad \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}\sigma_N \\
(L_2, L_3, \ldots, L_N, 0) & \text{w.p.} \quad \frac{\mu}{\lambda+\mu}\sigma_N
\end{cases} \tag{4.7}
$$

Using (4.7) we have,

$$
\mathbb{E}[L_1] = \sigma_0 + \frac{\lambda p}{\lambda + \mu} \sum_{j=1}^{N} \sigma_j + \mathbb{E}[L_1] \frac{\lambda}{\lambda + \mu} \sum_{j=1}^{N} \sigma_j + \mathbb{E}[L_2] \frac{\mu}{\lambda + \mu} \sum_{j=2}^{N} \sigma_j, \tag{4.8}
$$

and, for $k = 2, 3, \ldots, N-1$,

$$
\begin{aligned}
\mathbb{E}[L_k] = {} & \frac{\lambda(1-p)^{k-1}}{\lambda + \mu}\sigma_{k-1} + \frac{\lambda(1-p)^{k-1} p}{\lambda + \mu} \sum_{j=k}^{N} \sigma_j + \mathbb{E}[L_k] \frac{\lambda}{\lambda + \mu} \sum_{j=k}^{N} \sigma_j \\
& + \mathbb{E}\left[L_{k+1}\right] \frac{\mu}{\lambda + \mu} \sum_{j=k+1}^{N} \sigma_j.
\end{aligned} \tag{4.9}
$$

Finally, for $k = N$ we get

$$
\mathbb{E}[L_N] = \frac{\lambda(1-p)^{N-1}}{\lambda + \mu}(\sigma_{N-1} + \sigma_N) + \mathbb{E}[L_N] \frac{\lambda}{\lambda + \mu}\sigma_N, \tag{4.10}
$$

from which

$$
\mathbb{E}[L_N] = \frac{\lambda(1-p)^{N-1}(\sigma_{N-1} + \sigma_N)}{\lambda(1-\sigma_N) + \mu}. \tag{4.11}
$$

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Define

$$\alpha_k = \frac{\lambda}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j, \quad k = 1, 2, \ldots, N,$$

$$\beta_k = \frac{\mu}{\lambda+\mu}\sum_{j=k+1}^{N}\sigma_j, \;\; k = 1, 2, \ldots, N-1,$$

$$q_1 = \sigma_0 + \frac{\lambda p}{\lambda+\mu}\sum_{j=1}^{N}\sigma_j,$$

$$q_k = \frac{\lambda(1-p)^{k-1}}{\lambda+\mu}\sigma_{k-1} + \frac{\lambda(1-p)^{k-1}p}{\lambda+\mu}\sum_{j=k}^{N}\sigma_j, \;\; k = 2, 3, \ldots, N-1,$$

$$q_N = \frac{\lambda(1-p)^{N-1}}{\lambda+\mu}(\sigma_{N-1}+\sigma_N).$$

Then, after some algebra we obtain

$$\mathbb{E}[L_k] = \sum_{j=0}^{N-k}\frac{q_{k+j}}{1-\alpha_{k+j}}\prod_{i=0}^{j-1}\frac{\beta_{k+i}}{1-\alpha_{k+i}}, \quad k = 1, 2, \ldots, N, \tag{4.12}$$

where $\prod_{j=0}^{-1}(\cdot) \triangleq 1$, and $\sum_{j=0}^{-1}(\cdot) \triangleq 0$.

4.3 The busy period

As before, Θ_n ($n = 1, 2, \ldots, N$) denotes the time from a moment when there are n groups in the system until the first moment thereafter when no groups are present. The busy period is Θ_1. We now derive the LST of Θ_n, as well as a closed-form expression for $\mathbb{E}[\Theta_n]$.

4.3.1 The LST of Θ_n

Let $\widetilde{\Theta}_n(s)$ denote the LST of Θ_n. We now derive $\{\widetilde{\Theta}_n(s)\}_{n=1}^{N}$ by constructing and solving a set of N linear equations, as follows. First, we have that

$$\Theta_1 \stackrel{d}{=} Exp(\lambda(1-p)+\mu) + \begin{cases} 0 & w.p. \;\; \frac{\mu}{\lambda(1-p)+\mu} \\ \Theta_2 & w.p. \;\; \frac{\lambda(1-p)}{\lambda(1-p)+\mu} \end{cases},$$

which yields

$$\widetilde{\Theta}_1(s) = \frac{\mu}{\lambda(1-p)+\mu+s} + \frac{\lambda(1-p)}{\lambda(1-p)+\mu+s}\widetilde{\Theta}_2(s). \tag{4.13}$$

Second, for $n = 2, 3, \ldots, N-1$,

$$\Theta_n \stackrel{d}{=} Exp(\lambda(1-p)^n+\mu) + \begin{cases} \Theta_{n-1} & w.p. \;\; \frac{\mu}{\lambda(1-p)^n+\mu} \\ \Theta_{n+1} & w.p. \;\; \frac{\lambda(1-p)^n}{\lambda(1-p)^n+\mu} \end{cases}, \tag{4.14}$$

which leads to

$$\widetilde{\Theta}_n(s) = \frac{\mu}{\lambda(1-p)^n+\mu+s}\widetilde{\Theta}_{n-1}(s) + \frac{\lambda(1-p)^n}{\lambda(1-p)^n+\mu+s}\widetilde{\Theta}_{n+1}(s). \tag{4.15}$$

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Last, for $n = N$,

$$\Theta_N \overset{d}{=} Exp(\mu) + \Theta_{N-1}, \tag{4.16}$$

resulting in

$$\widetilde{\Theta}_N(s) = \frac{\mu}{\mu + s}\widetilde{\Theta}_{N-1}(s). \tag{4.17}$$

Equations (4.13)–(4.17) comprise a set of N linear equations which can be written in the following matrix form:

$$A(s) \cdot \vec{\Theta}(s) = \vec{b}, \tag{4.18}$$

where

$$A(s) = \begin{pmatrix} \lambda(1-p)+\mu+s & -\lambda(1-p) & 0 & \cdots\ \cdots & \cdots & 0 \\ -\mu & \lambda(1-p)^2+\mu+s & -\lambda(1-p)^2 & 0\ \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots\ \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots\ -\mu & \lambda(1-p)^{N-1}+\mu+s & -\lambda(1-p)^{N-1} \\ 0 & \cdots & \cdots & \cdots\ 0 & -\mu & \mu+s \end{pmatrix},$$

$\vec{\Theta}(s) = \left(\widetilde{\Theta}_1(s), \widetilde{\Theta}_2(s), \ldots, \widetilde{\Theta}_N(s)\right)^T$ is a column vector of the desired LST's, and $\vec{b} = (\mu, 0, 0, \ldots, 0)^T$. The solution for (4.18) is given by $\vec{\Theta}(s) = (A(s))^{-1}\vec{b}$, and since $\vec{b}$ is all zeros except from its first coordinate (which equals μ), we have that $\vec{\Theta}(s)$ equals the first column of $(A(s))^{-1}$ multiplied by μ. Note that $A(s)$ is a tridiagonal matrix. There is an increasing interest in tridiagonal matrices in many fields, where inversions of such matrices are required. Examples for recent works that present explicit formula for the elements of the inverse of a general tridiagonal matrix are Mallik (2001) and Kiliç (2008), and references there. Thus, once the inverse of $A(s)$ is calculated, the vector $\vec{\Theta}(s)$ is fully obtained, and the mean values of the busy periods, i.e. $\mathbb{E}[\Theta_n]$ for $n = 1, 2, \ldots, N$, can be derived using differentiation. However, a closed form expression for $\mathbb{E}[\Theta_n]$, convenient for numerical calculations, can be derived as shown in the next section.

4.3.2 Calculation of $\mathbb{E}[\Theta_n]$

From Eq. (4.16) we get

$$\mathbb{E}[\Theta_{N-1}] = \mathbb{E}[\Theta_N] - \frac{1}{\mu}. \tag{4.19}$$

Using Eq. (4.14) results in

$$\mathbb{E}[\Theta_n] = \frac{1}{\lambda(1-p)^n + \mu} + \frac{\lambda(1-p)^n}{\lambda(1-p)^n + \mu}\mathbb{E}[\Theta_{n+1}] + \frac{\mu}{\lambda(1-p)^n + \mu}\mathbb{E}[\Theta_{n-1}],$$

or equivalently,

$$(\lambda(1-p)^n + \mu)\mathbb{E}[\Theta_n] = 1 + \lambda(1-p)^n\mathbb{E}[\Theta_{n+1}] + \mu\mathbb{E}[\Theta_{n-1}]. \tag{4.20}$$

Substituting $n = N - 1$ in Eq. (4.20) leads to

$$\mathbb{E}[\Theta_{N-1}] = \frac{1}{\lambda(1-p)^{N-1} + \mu} + \frac{\lambda(1-p)^{N-1}}{\lambda(1-p)^{N-1} + \mu}\mathbb{E}[\Theta_N] + \frac{\mu}{\lambda(1-p)^{N-1} + \mu}\mathbb{E}[\Theta_{N-2}].$$

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Using the expression for $\mathbb{E}[\Theta_{N-1}]$ given in (4.19) and rearranging terms give

$$\mathbb{E}[\Theta_{N-2}] = \mathbb{E}[\Theta_N] - \frac{\lambda(1-p)^{N-1}}{\mu^2} - \frac{2}{\mu}. \tag{4.21}$$

Continuing further, substituting $n = N-2$ in Eq. (4.20) gives

$$\mathbb{E}[\Theta_{N-2}] = \frac{1}{\lambda(1-p)^{N-2}+\mu} + \frac{\lambda(1-p)^{N-2}}{\lambda(1-p)^{N-2}+\mu}\mathbb{E}[\Theta_{N-1}] + \frac{\mu}{\lambda(1-p)^{N-2}+\mu}\mathbb{E}[\Theta_{N-3}].$$

Using the expressions for $\mathbb{E}[\Theta_{N-2}]$ given in (4.21) and for $\mathbb{E}[\Theta_{N-1}]$ given in (4.19), and rearranging terms give

$$\mathbb{E}[\Theta_{N-3}] = \mathbb{E}[\Theta_N] - \frac{\lambda^2}{\mu^3}(1-p)^{N-1}(1-p)^{N-2} - \frac{\lambda}{\mu^2}\left((1-p)^{N-1} + (1-p)^{N-2}\right) - \frac{3}{\mu}. \tag{4.22}$$

Continuing, the structure of Eqs. (4.19) and (4.21)–(4.22) leads to the following general solution,

$$\mathbb{E}[\Theta_{N-j}] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{j-1}\frac{\lambda^i}{\mu^{i+1}}\sum_{k=1}^{j-i}(1-p)^{Ni-\frac{i(i+2k-1)}{2}} - \frac{j}{\mu}, \quad j = 0, 1, \ldots, N-1.$$

By setting $n = N - j$ and rewriting the power of the term $(1-p)$ we get

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{N-n-1}\frac{\lambda^i}{\mu^{i+1}}\sum_{k=1}^{N-n-i}(1-p)^{\frac{i(2N-2k-i+1)}{2}} - \frac{N-n}{\mu}, \quad n = 1, 2, \ldots, N, \tag{4.23}$$

where we define $\sum_{i=1}^{-1}(\cdot) = \sum_{i=1}^{0}(\cdot) = 0$.

Now, the second summation appearing in Eq. (4.23) is

$$\sum_{k=1}^{N-n-i}(1-p)^{\frac{i(2N-2k-i+1)}{2}} = (1-p)^{\frac{i(2N-i+1)}{2}}\sum_{k=1}^{N-n-i}(1-p)^{-ik}$$
$$= (1-p)^{\frac{i(2N-i+1)}{2}}\frac{(1-p)^{i(i-N+n)}-1}{1-(1-p)^i} = \frac{(1-p)^{\frac{i(2n+i+1)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}}{1-(1-p)^i},$$

so that Eq. (4.23) becomes

$$\mathbb{E}[\Theta_n] = \mathbb{E}[\Theta_N] - \sum_{i=1}^{N-n-1}\frac{\lambda^i\left((1-p)^{\frac{i(2n+i+1)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)} - \frac{N-n}{\mu}, \quad n = 1, 2, \ldots, N. \tag{4.24}$$

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Substituting $n = 1$ in Eq. (4.24), and using the expression for $\mathbb{E}[\Theta_1]$ given in equation (2.18), yield an expression for $\mathbb{E}[\Theta_N]$ in terms of π_0,

$$\mathbb{E}[\Theta_N] = \frac{1-\pi_0}{\lambda\pi_0} + \sum_{i=1}^{N-2} \frac{\lambda^i \left((1-p)^{\frac{i(i+3)}{2}} - (1-p)^{\frac{i(2N-i+1)}{2}}\right)}{\mu^{i+1}(1-(1-p)^i)} + \frac{N-1}{\mu}. \tag{4.25}$$

Thus, in view of (4.24), $\mathbb{E}[\Theta_n]$ is completely determined for all $1 \le n \le N$.

5 Numerical results and discussion

In this section we present numerical results summarized in tables, for models 1 and 2, as follows. Tables 1 and 2 deal with Model 1 with infinite number of groups, Tables 3, 4, 5, 6, 7 and 8 exhibit results related to Model 1 with various values of finite N, and Tables 9, 10, 11 and 12 relate to Model 2.

Table 1 presents values for $\mathbb{E}[L_k], k = 1, 2, \ldots, 10$ when $\mu = 1$, and $\lambda = 0.5, 1, 5, 10, 20$. As expected, as λ grows, the size of each group becomes larger. Also, as k increases, $\mathbb{E}[L_k]$ decreases, meaning that groups standing "far" from the server are smaller (on the average) than groups which are "close" to the server. Table 2 presents results for $\mathbb{E}[\Theta_n], k = 1, 2, \ldots, 10$ when $\mu = 1$, and $\lambda = 0.5, 1, 5, 10$.

Tables 3, 4, 5, 6, 7 and 8 show numerical results for Model 1 with finite number of groups, where N assumes values of 5, 10 and 20, and $\mu = 1$. Tables 3, 4 and 5 show that for small values of λ, $\mathbb{E}[L_k]$ are mostly the same, for any value of N. However, as λ increases, the difference between $\mathbb{E}[L_k]'s$ is more apparent. Also, in Tables 6, 7 and 8 it is seen that for small values of λ, $\mathbb{E}[\Theta_n]$ are very close, whereas for larger values of λ there is a significant difference between the values of $\mathbb{E}[\Theta_n]$.

The Geometric model is presented in Tables 9, 10, 11 and 12. In Table 9 ($N = 5$) we calculate the first moment of L_k and of $D^{(k)}$, $k = 1, 2, \ldots, 5$, as well as the first moment of Θ_n, $n = 1, 2, \ldots, 5$. Different values of λ and p are considered, while $\mu = 1$ in all calculations. The results show that $\mathbb{E}[L_1]$, the mean size of the group standing in the first position (the one being served) increases with p, since for larger values of p, a great number of customers concentrate in the first group. The size of the group in the second position behaves differently for various values of p. Specifically, when p increases from 0.01 to 0.2, $\mathbb{E}[L_2]$ slightly increases, while when p grows from 0.2 to 0.6, $\mathbb{E}[L_2]$ significantly decreases. This follows since $(1-p)p$, the probability of joining the second group, is increasing when $0 < p < 0.5$, and decreasing when $p > 0.5$. Furthermore, $\mathbb{E}[L_3]$, $\mathbb{E}[L_4]$ and $\mathbb{E}[L_5]$ decrease as p increases. We also observe that $\mathbb{E}\left[D^{(k)}\right]$ is larger than $\mathbb{E}[L_k]$. This follows since $\mathbb{E}\left[D^{(k)}\right]$ is calculated after a service completion, so $\mathbb{E}\left[D^{(k)}\right]$ contains all the customers that join this group during a single service period. In contrast, $\mathbb{E}[L_k]$ is calculated right after a Poissonian event, which may be either an arrival or a service completion. In addition, Table 9 shows that for all n, $\mathbb{E}[\Theta_n]$ drops considerably with the enlargement of p.

Table 10 presents results for $\mathbb{E}[L_k]$, $k = 1, 2, \ldots 10$, when $N = 10$. As expected, the values of $\mathbb{E}[L_k]$ decrease as the group's index k grows. However, for small p (e.g. $p = 0.01$), the mean size of the last group is slightly greater than the mean sizes of the groups in front of it, and the values of $\mathbb{E}[L_k]$ differ by small amounts. This follows since for small p, there are values of k such that $(1-p)^9 > (1-p)^k p$. That is, the probability of joining the last group is larger than the probability of joining groups $k+1, k+2, \ldots, N-1$.

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Table 1 Model 1 (unbounded queue)—numerical results for $\mathbb{E}[L_k]$, $k = 1, 2, \ldots, 10$ and $\mu = 1$

λ	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[L_3]$	$\mathbb{E}[L_4]$	$\mathbb{E}[L_5]$	$\mathbb{E}[L_6]$	$\mathbb{E}[L_7]$	$\mathbb{E}[L_8]$	$\mathbb{E}[L_9]$	$\mathbb{E}[L_{10}]$
0.5	0.5785	0.1066	0.0162	0.0019	0.0019	1.5×10^{-5}	1.1×10^{-6}	6.9×10^{-8}	3.8×10^{-9}	1.9×10^{-10}
1	0.6839	0.1962	0.0519	0.0118	0.0022	3.6×10^{-4}	5.1×10^{-5}	6.2×10^{-6}	6.8×10^{-7}	6.8×10^{-8}
5	2.2004	1.3198	0.7061	0.3621	0.1868	0.0972	0.0502	0.0252	0.0121	0.0056
10	4.9303	3.9343	2.9625	2.0686	1.3316	0.8042	0.4723	0.2783	0.1669	0.1018
20	11.2573	10.2573	9.2573	8.2574	7.2581	6.2610	5.2714	4.3019	3.3768	2.5325

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Table 2 Model 1 (unbounded queue)—numerical results for $\mathbb{E}[\Theta_n]$, $n = 1, 2, \ldots, 10$ and $\mu = 1$

λ	$\mathbb{E}[\Theta_1]$	$\mathbb{E}[\Theta_2]$	$\mathbb{E}[\Theta_3]$	$\mathbb{E}[\Theta_4]$	$\mathbb{E}[\Theta_5]$	$\mathbb{E}[\Theta_6]$	$\mathbb{E}[\Theta_7]$	$\mathbb{E}[\Theta_8]$	$\mathbb{E}[\Theta_9]$	$\mathbb{E}[\Theta_{10}]$
0.5	1.2974	2.4872	3.6258	4.7348	5.8245	6.9006	7.9667	9.0252	10.0776	11.125
1	1.7182	3.1548	4.4654	5.7033	6.8971	8.0604	9.2004	10.3238	11.4337	12.5328
5	29.4826	40.8757	47.1115	51.3002	54.4888	57.1152	59.3922	61.4353	63.3130	65.0682
10	2202.55	2642.86	2774.65	2826.97	2852.62	2867.42	2877.08	2884.01	2889.33	2893.67

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Table 3 Model 1 (finite queue)—numerical results for $\mathbb{E}[L_k]$ where $N = 5$ and $\mu = 1$

λ	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[L_3]$	$\mathbb{E}[L_4]$	$\mathbb{E}[L_5]$
0.5	0.5785	0.1066	0.0162	0.0019	1.9×10^{-4}
1	0.6842	0.1962	0.0518	0.0117	2.2×10^{-3}
5	2.2931	1.2489	0.5934	0.2674	0.1144
10	5.2912	3.4095	1.8030	0.7654	0.2806
15	8.5440	5.8344	3.3023	1.3851	0.4331

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Table 4 Model 1 (finite queue)—numerical results for $\mathbb{E}[L_k]$ where $N = 10$ and $\mu = 1$

λ	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[L_3]$	$\mathbb{E}[L_4]$	$\mathbb{E}[L_5]$	$\mathbb{E}[L_6]$	$\mathbb{E}[L_7]$	$\mathbb{E}[L_8]$	$\mathbb{E}[L_9]$	$\mathbb{E}[L_{10}]$
0.5	0.5785	0.1066	0.0162	0.0019	1.9×10^{-4}	1.6×10^{-5}	1.1×10^{-6}	6.9×10^{-8}	3.8×10^{-9}	1.9×10^{-10}
1	0.6839	0.1962	0.0519	0.0118	0.0022	3.6×10^{-4}	5.1×10^{-5}	6.3×10^{-6}	6.9×10^{-7}	6.8×10^{-8}
5	2.2084	1.3214	0.7049	0.3604	0.1853	0.0960	0.0491	0.0242	0.0112	0.0045
10	5.2627	4.0749	2.9292	1.9117	1.1297	0.6233	0.3353	0.1796	0.0945	0.0463
15	9.0399	7.4809	5.9280	4.4062	2.9879	1.8022	0.9656	0.4758	0.2236	0.0985

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Table 5 Model 1 (finite queue)—numerical results for $\mathbb{E}[L_k]$ where $N = 20$ and $\mu = 1$

λ	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[L_3]$	$\mathbb{E}[L_4]$	$\mathbb{E}[L_5]$	$\mathbb{E}[L_6]$	$\mathbb{E}[L_7]$	$\mathbb{E}[L_8]$	$\mathbb{E}[L_9]$	$\mathbb{E}[L_{10}]$
0.5	0.5785	0.1066	0.0162	0.0019	1.9×10^{-4}	1.6×10^{-5}	1.1×10^{-6}	6.9×10^{-8}	3.8×10^{-9}	1.9×10^{-10}
1	0.6839	0.1962	0.0519	0.0118	0.0022	3.6×10^{-4}	5.1×10^{-5}	6.3×10^{-6}	6.9×10^{-7}	6.8×10^{-8}
5	2.2004	1.3198	0.7061	0.3621	0.1867	0.0972	0.0502	0.0252	0.0121	0.0055
10	4.9331	3.9362	2.9636	2.0689	1.3315	0.8040	0.4720	0.2781	0.1667	0.1017
15	8.1494	7.1169	6.0851	5.0577	4.0466	3.0811	2.2107	1.4917	0.9571	0.5971

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Table 6 Model 1 (finite queue)—numerical results for $\mathbb{E}[\Theta_n]$ where $N = 5$ and $\mu = 1$

λ	$\mathbb{E}[\Theta_1]$	$\mathbb{E}[\Theta_2]$	$\mathbb{E}[\Theta_3]$	$\mathbb{E}[\Theta_4]$	$\mathbb{E}[\Theta_5]$
0.5	1.2974	2.4869	3.6245	4.7245	5.7245
1	1.7167	3.15	4.45	5.65	6.65
5	18.0833	24.9167	28.4167	30.4167	31.4167
10	147.7	177.0	185.5	188.5	189.5
15	608.5	689.5	705.5	709.5	710.5

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Table 7 Model 1 (finite queue)—numerical results for $\mathbb{E}[\Theta_n]$ where $N = 10$ and $\mu = 1$

λ	$\mathbb{E}[\Theta_1]$	$\mathbb{E}[\Theta_2]$	$\mathbb{E}[\Theta_3]$	$\mathbb{E}[\Theta_4]$	$\mathbb{E}[\Theta_5]$	$\mathbb{E}[\Theta_6]$	$\mathbb{E}[\Theta_7]$	$\mathbb{E}[\Theta_8]$	$\mathbb{E}[\Theta_9]$	$\mathbb{E}[\Theta_{10}]$
0.5	1.2974	2.4872	3.6258	4.7348	5.8245	6.9006	7.9668	9.0251	10.0751	11.0751
1	1.7182	3.1548	4.4654	5.7033	6.8971	8.0600	9.2003	10.3225	11.4225	12.4225
5	29.0761	40.3066	46.4448	50.5555	53.6661	56.1988	58.3446	60.1780	61.6780	62.6780
10	1284.13	1540.76	1617.44	1647.72	1662.36	1670.54	1675.57	1678.79	1680.79	1681.79
15	25,817.4	29,259.6	29,947.8	30,131.1	30,191.9	30,215.8	30,226.4	30,231.6	30,234.1	30,235.1

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Table 8 Model 1 (finite queue)—numerical results for $\mathbb{E}[\Theta_n]$ where $N = 20$ and $\mu = 1$

λ	$\mathbb{E}[\Theta_1]$	$\mathbb{E}[\Theta_2]$	$\mathbb{E}[\Theta_3]$	$\mathbb{E}[\Theta_4]$	$\mathbb{E}[\Theta_5]$	$\mathbb{E}[\Theta_6]$	$\mathbb{E}[\Theta_7]$	$\mathbb{E}[\Theta_8]$	$\mathbb{E}[\Theta_9]$	$\mathbb{E}[\Theta_{10}]$
0.5	1.2974	2.4872	3.6258	4.7348	5.8245	6.9006	7.9668	9.0251	10.0776	11.1250
1	1.7183	3.1549	4.4654	5.7033	6.8971	8.0600	9.2005	10.3238	11.4337	12.5328
5	29.4826	40.8757	47.1115	51.3002	54.4888	57.1152	59.3922	61.4353	63.3130	65.0682
10	2199.05	2638.66	2770.24	2822.47	2848.09	2862.86	2872.5	2879.41	2884.73	2889.05
15	199,852	226,499	231,828	233,249	233,722	233,911	233,999	234,045	234,072	234,090

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Table 9 Model 2—numerical results for $N = 5$, $\mu = 1$

Value of p	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[L_1]$	0.3224	0.5317	0.7901	1.6689	2.6655	3.1171	6.6855	9.5628	11.5512
$\mathbb{E}[L_2]$	0.2165	0.2505	0.1481	1.6221	1.6941	0.6519	6.5358	6.5652	2.8826
$\mathbb{E}[L_3]$	0.1583	0.1174	0.0182	1.5927	0.9955	0.1255	6.3916	4.1884	0.4367
$\mathbb{E}[L_4]$	0.1283	0.0505	0.0011	1.6486	0.5819	0.0213	6.3008	2.4301	0.0687
$\mathbb{E}[L_5]$	0.1826	0.0242	2.73×10^{-5}	2.2565	0.5505	0.0023	7.0287	1.7942	0.0129
$\mathbb{E}[D^{(1)}]$	1.2081	1.3729	1.6811	5.0014	5.0085	5.1931	15.0000	15.0000	15.0247
$\mathbb{E}[D^{(2)}]$	1.2501	1.2758	1.2543	4.9569	4.0343	2.4787	14.8503	12.0019	6.1518
$\mathbb{E}[D^{(3)}]$	1.3246	1.2208	1.0984	4.9348	3.3136	1.5348	14.7059	9.6179	2.8294
$\mathbb{E}[D^{(4)}]$	1.4803	1.2244	1.0390	5.0238	2.8881	1.2065	14.6194	7.8199	1.6825
$\mathbb{E}[D^{(5)}]$	1.9606	1.4096	1.0256	5.8030	3.0480	1.1280	15.4089	7.1440	1.3840
$\mathbb{E}[\Theta_1]$	4.8062	2.6815	1.4682	171.131	117.677	5.1775	49,196.0	6448.75	40.5324
$\mathbb{E}[\Theta_2]$	8.6508	4.7834	2.6387	856.996	146.846	7.2663	52,508.8	6986.07	47.1212
$\mathbb{E}[\Theta_3]$	11.5531	6.5051	3.7043	886.149	155.649	8.6273	52,734.1	7041.93	49.4498
$\mathbb{E}[\Theta_4]$	13.5137	7.9147	4.7299	891.952	158.697	9.7553	52,749.5	7049.08	50.8338
$\mathbb{E}[\Theta_5]$	14.5137	8.9147	5.7299	892.952	159.697	10.7553	52,750.5	7050.08	51.8338

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Table 10 Model 2—numerical results for $\mathbb{E}[L_k]$, where $N = 10$, $\mu = 1$

Value of p	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[L_1]$	0.2644	0.5308	0.7902	1.7420	3.2775	3.1175	6.8794	12.1098	11.5783
$\mathbb{E}[L_2]$	0.1908	0.2508	0.1481	1.6920	2.2837	0.6521	6.7294	9.1099	2.9039
$\mathbb{E}[L_3]$	0.1446	0.1179	0.0182	1.6425	1.5083	0.1255	6.5809	6.7099	0.4432
$\mathbb{E}[L_4]$	0.1135	0.0509	0.0010	1.5936	0.9311	0.0214	6.4339	4.7905	0.0699
$\mathbb{E}[L_5]$	0.0914	0.0191	2.71×10^{-5}	1.5423	0.5381	0.0022	6.2883	3.2585	0.0118
$\mathbb{E}[L_6]$	0.0748	0.0059	2.74×10^{-7}	1.4983	0.2992	1.07×10^{-4}	6.1443	2.0494	0.0014
$\mathbb{E}[L_7]$	0.0621	0.0015	1.12×10^{-9}	1.4555	0.1656	2.16×10^{-6}	6.0019	1.1388	7.84×10^{-5}
$\mathbb{E}[L_8]$	0.0519	3.14×10^{-4}	1.83×10^{-12}	1.4306	0.0916	1.76×10^{-8}	5.8649	0.5415	$1.88 \cdot 10^{-6}$
$\mathbb{E}[L_9]$	0.0448	5.21×10^{-5}	1.2×10^{-15}	1.4899	0.0493	5.76×10^{-11}	5.7839	0.2467	1.83×10^{-8}
$\mathbb{E}[L_{10}]$	0.0696	7.65×10^{-6}	3.14×10^{-19}	2.0807	0.0357	7.54×10^{-14}	6.5099	0.2244	7.20×10^{-11}

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Table 11 Model 2—numerical results for $\mathbb{E}[D^{(k)}]$, where $N = 10$, $\mu = 1$

Value of p	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[D^{(1)}]$	1.1168	1.3665	1.6811	5.0000	5.0014	5.1930	15.0000	15.0000	15.0242
$\mathbb{E}[D^{(2)}]$	1.1209	1.2629	1.2543	4.9500	4.0054	2.4784	14.8500	12.0000	6.1485
$\mathbb{E}[D^{(3)}]$	1.1277	1.1916	1.0984	4.9005	3.2175	1.5336	14.7015	9.6000	2.8186
$\mathbb{E}[D^{(4)}]$	1.1383	1.1418	1.0388	4.8516	2.6057	1.2013	14.5545	7.6803	1.6512
$\mathbb{E}[D^{(5)}]$	1.1546	1.1064	1.0154	4.8033	2.1461	1.0783	14.4089	6.1458	1.2436
$\mathbb{E}[D^{(6)}]$	1.1799	1.0809	1.0062	4.7566	1.8166	1.0309	14.2649	4.9242	1.0937
$\mathbb{E}[D^{(7)}]$	1.2209	1.0624	1.0025	4.7154	1.5949	1.0123	14.1226	3.9679	1.0372
$\mathbb{E}[D^{(8)}]$	1.2936	1.0498	1.0009	4.6978	1.4649	1.0049	13.9859	3.2622	1.0148
$\mathbb{E}[D^{(9)}]$	1.4453	1.0494	1.0004	4.7933	1.4373	1.0019	13.9092	2.8484	1.0059
$\mathbb{E}[D^{(10)}]$	1.9135	1.1342	1.0002	5.5676	1.6711	1.0013	14.7028	3.0133	1.0039

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Table 12 Model 2—numerical results for $\mathbb{E}[\Theta_n]$, where $N = 10$, $\mu = 1$

Value of p	$\lambda = 1$			$\lambda = 5$			$\lambda = 15$		
	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$	$p = 0.01$	$p = 0.2$	$p = 0.6$
$\mathbb{E}[\Theta_1]$	8.563	2.728	1.468	1,589,601.4	734.695	5.181	2.63808×10^{10}	2,974,866.9	41.399
$\mathbb{E}[\Theta_2]$	16.203	4.888	2.638	1,910,732.8	918.118	7.271	2.81573×10^{10}	3,222,772.4	48.132
$\mathbb{E}[\Theta_3]$	22.977	6.701	3.704	1,976,262.9	975.126	8.634	2.82781×10^{10}	3,248,595.8	50.521
$\mathbb{E}[\Theta_4]$	28.928	8.289	4.730	1,989,769.9	997.004	9.769	2.82864×10^{10}	3,251,958.1	51.968
$\mathbb{E}[\Theta_5]$	34.083	9.724	5.741	1,992,581.9	1007.198	10.822	2.8287016×10^{10}	3,252,505.2	53.131
$\mathbb{E}[\Theta_6]$	38.451	11.052	6.744	1,993,173.5	1012.809	11.842	2.828705×10^{10}	3,252,616.3	54.194
$\mathbb{E}[\Theta_7]$	42.029	12.302	7.746	1,993,298.5	1016.328	12.851	2.828706×10^{10}	3,252,644.3	55.219
$\mathbb{E}[\Theta_8]$	44.795	13.492	8.747	1,993,325.2	1018.729	13.854	2.828706×10^{10}	3,252,652.9	56.229
$\mathbb{E}[\Theta_9]$	46.708	14.626	9.747	1,993,330.7	1020.401	14.855	2.828706×10^{10}	3,252,655.9	57.233
$\mathbb{E}[\Theta_{10}]$	47.708	15.626	10.747	1,993,331.7	1021.401	15.855	2.828706×10^{10}	3,252,656.9	58.233

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In Table 11 the values for $\mathbb{E}\left[D^{(k)}\right]$ are presented, when $N = 10$. When $\frac{\lambda}{\mu}$ is large, π_0 is very small, and therefore, from Eq. (4.2), $\mathbb{E}\left[D^{(1)}\right]$ is very close to $\frac{\lambda}{\mu}$.

Table 12 exhibits the values of $\mathbb{E}[\Theta_n]$ when $N = 10$. For large values of λ and small p, $\mathbb{E}[\Theta_n]$ is extremely large. However, when increasing the value of p from 0.2 to 0.6, $\mathbb{E}[\Theta_n]$ drops drastically.

6 Appendix

6.1 Proof of Proposition 3.1

Proof We need to show that

$$\frac{1}{\mu} + \frac{\lambda}{2\mu^2} \leq \frac{\lambda}{\mu^2(1 - e^{-\frac{\lambda}{\mu}})}. \tag{6.1}$$

By setting $a = \frac{\lambda}{\mu}$, and some straightforward algebra, Eq. (6.1) is equivalent to

$$\frac{2-a}{2+a} \leq e^{-a}. \tag{6.2}$$

Equation (6.2) clearly holds for $a \geq 2$. We will prove that it also holds for $0 \leq a < 2$. Note that (6.2) can be written as

$$a + 2 + (a+2)e^a - 4e^a \geq 0.$$

Define $f(a) = a + 2 + (a+2)e^a - 4e^a$. We need to prove that $f(a) \geq 0$ for all $0 \leq a < 2$. Note that $f'(a) = 1 + e^a(a-1)$, and $f''(a) = ae^a \geq 0$. Therefore, $f'(a)$ is non-decreasing, and with $f'(0) = 0$ we have that $f'(a) \geq 0$ for $0 \leq a < 2$. This implies that $f(a)$ is also non-decreasing for $0 \leq a < 2$, and with $f(0) = 0$, the proof is completed. □

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