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# The Israeli queue with retrials

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**Abstract** The so-called "Israeli queue" (Boxma et al. in Stoch Model 24(4):604–625, 2008; Perel and Yechiali in Probab Eng Inf Sci, 2013; Perel and Yechiali in Stoch Model 29(3):353–379, 2013) is a multi-queue polling-type system with a single server. Service is given in batches, where the batch sizes are unlimited and the service time of a batch does not depend on its size. After completing service, the next queue to be visited by the server is the one with the most senior customer. In this paper, we study the Israeli queue with retrials, where the system is comprised of a "main" queue and an orbit queue. The main queue consists of at most $M$ groups, where a new arrival enters the main queue either by joining one of the existing groups, or by creating a new group. If an arrival cannot join one of the groups in the main queue, he goes to a retrial (orbit) queue. The orbit queue dispatches orbiting customers back to the main queue at a constant rate. We analyze the system via both probability generating functions and matrix geometric methods, and calculate analytically various performance measures and present numerical results.

**Keywords** The Israeli queue · Retrial queues · Polling systems · Unlimited-size batch service

**Mathematics Subject Classification** Primary 60K25 · Secondary 90B22

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## 1 Introduction

The “Israeli Queue” model was introduced in Boxma et al. [10, Sec. 5] when studying a multi-queue single-server polling system with unlimited-size batch service [9,10], where the next queue to be served is the one with the most senior customer (the customer who has been waiting for the longest time among all present customers). The term “Israeli Queue” originates from a real waiting line of individuals formed in order to buy tickets for a show. The associated queueing system is comprised of heads of groups, where each head can buy an unrestricted number of tickets for the group of individuals he (“he” stands for “she” as well) represents, while the purchasing time is assumed to be independent of the number of tickets bought. A new arrival either joins one of the existing groups if he knows the group’s head, or creates a new group, acting as its leader.

Unlimited batch service was studied by van der Wal and Yechiali [22] when analyzing a computer tape-reading problem in a system where large amounts of information are stored on tapes. It is assumed that the time to mount, read, and dismount the tape is independent of the amount of information read from the tape. The problem was formulated as a polling system, and the optimal visiting rules of the server were studied. Unlimited batch-service models were also considered in the literature as application to videotex, telex, and time division multiple access (TDMA) systems (e.g., [1,11] and [17]). Van Oyen and Teneketzis [21] formulated a central data base system and an automated guided vehicle (AGV) as a polling system with an infinite capacity batch service.

Recently, Perel and Yechiali [19] extended the Israeli queue model to the case where there is no bound on the number of different groups that can be present simultaneously in the system. They analyzed single-server models with finite and infinite number of groups, as well as models with multiple servers, and derived various performance measures. Perel and Yechiali [20] further studied a two-class single-server preemptive priority queueing model in which the high priority customers form a classical $M/M/1$ queue, while the low priority (class 2) customers form the unlimited-size batch service Israeli queue with a finite number of groups. They calculated various performance measures, such as the mean number of low priority groups in the system along with the mean size of a class 2 group; the covariance between the number of high priority customers and the number of low priority groups; sojourn times of a class 2 group leader, as well as of an arbitrary class 2 customer.

In this paper, we consider a single-server Israeli queue with at most $M$ groups (the main queue) and an infinite capacity orbit queue. The arrival process to the main queue is Poisson with rate $\lambda$, and the service time of a batch, independent of its size, is exponentially distributed with parameter $\mu$. Groups in the main queue are formed as follows: each group has a “leader” or a “head”—the first member of the group to arrive at the system. New arrivals see only the leader of each group. The probability for an arriving customer to know another group leader standing in line is $p$, with the same $p$ for all group leaders. We assume that an arriving customer can join the group in service while it is being served. Specifically, if there are $m$ groups in the system (including the one in service), $m = 1, 2, \ldots, M-1$, then the probability that a new arrival joins the $k-th$ group is $(1-p)^{k-1}p$, for $1 \leq k \leq m$. When $m = 0, 1, \ldots, M-1$, the probability of creating a new group (the ($m$+1)-st) is $(1-p)^m$. However, if $M$ groups are present in the main queue and an arriving customer does not know any of the group leaders, he joins a retrial orbit queue that dispatches individual orbiting customers back to the main queue at a constant Poisson rate $\gamma$ (whenever there is a positive number of orbiting customers). The order of dispatching customers from the orbit queue to the main queue follows the order of their arrival, namely a FCFS order, so that only the customer at the head of the orbit queue is allowed to try to access the main queue. If, upon a retrial, the main queue is full (i.e., consists of $M$ non-empty groups), the orbiting customer goes back to the orbit queue. If an orbiting customer finds the main queue with less than $M$ groups, he forms a new group, last in the line of groups, that is, a customer arriving from orbit does not look for a friend in the main queue, but rather creates a new group to which new arrivals can join. Inter-arrival times, retrial times, and service times are mutually independent. Recently, an $M/M/1$-type queueing model with customer interjections, where interjecting customers try to cut into the queue following a geometric distribution, has been studied by He and Chavoushi [15].

Retrial queues have been widely used to model a variety of problems in areas such as telephone switching systems, telecommunication networks, computer systems, and others. There exists an extensive literature on retrial queues (see, for example [2–8, 12–14, 23] and many references therein) in which various models are considered (single or multiple servers, constant or non-constant retrial rates, priority models, server breakdowns, or vacations) and a variety of mathematical techniques are utilized (e.g., regenerative approach, probability generating functions, matrix analytic methods, mean value analysis, and approximations) to analyze these models.

Several of the works mentioned above consider a main queue that can hold at most one customer, that is, there is no waiting room for arriving customers. In contrast, in our model, the main queue can hold up to $M \geq 1$ groups, where the size of each group is unrestricted. Avrachenkov and Yechiali [7] considered a queueing system with a finite buffer $M/M/1/K$ primary queue and an infinite buffer $M/M/1/\infty$ orbit queue, where customers arriving to a full buffer in the primary queue are blocked and go to an orbit queue. Explicit analytical results were derived for a buffer of size 1 and of size 2, and a necessary and sufficient stability condition was obtained. In particular, the stability condition obtained in the current paper, when setting $p = 0$, coincides with the one established in [7]. Falin [14] studied a retrial queue with batch arrivals in which batches of customers arrive at a single server. If the server is free at an arrival epoch, then one of the customers from the batch begins his service while the other members of the batch join an ordinary queue in front of the server and are served following some service discipline. If the server is busy at a batch arrival epoch, then all customers from the batch go to an orbit queue. Every such orbiting customer produces a process of repeated calls until he finds the server idle. In contrast to our model, the server in [14] does not serve the whole batch in one service period: the group is split into individuals, whether in front of the server or in orbit.

In most of the retrial models mentioned above, there is a single retrial queue. Avrachenkov et al. [6] recently studied a system with two input streams and two orbit queues and analyzed it via both matrix geometric methods and probability generating functions, while solving a Riemann–Hilbert boundary value problem.

The analysis of our "Israeli queue with retrials" model is presented as follows. In Sect. 2, we define the model, establish balance equations for the two-dimensional steady-state probabilities characterizing the system, and employ probability generating functions (PGFs) to analyze the model. This requires the calculation of certain boundary probabilities. Those probabilities are obtained by finding and characterizing the roots of a (M+1)-degree polynomial being the determinant of a certain matrix whose entries are functions of the system's parameters. In Sect. 3, we derive various performance measures, such as the mean number of groups in the main queue, the mean number of orbiting customers, the mean size of a group standing in the main queue, and the mean number of bypasses made by an arriving customer. In Sect. 4, we briefly discuss the multi-server version of the model. In Sect. 5, we use matrix geometric methods to further analyze the system, while in Sect. 6, we present numerical results. It is seen that the mean number of groups in the main queue decreases monotonically when $p$ increases, and the mean number of customers in the orbit queue descent rapidly when $p$ increases. If $p$ is too small, the orbit queue explodes (the analytical condition for stability is given in Eq. (2.11) and equivalently in Eq. (5.4)). Furthermore, the mean size of the served batch is monotonically increasing with $p$.

## 2 The model

### 2.1 Model description

We consider the model described in the Introduction, namely a single-server Israeli queue with at most $M$ groups (the main queue) and an infinite capacity $M/M/1$-type orbit queue. The outside arrival stream to the main queue is Poisson with rate $\lambda$, while a service time of a group, independent of its size, is exponentially distributed with parameter $\mu$. If the main queue is full, then a new arriving customer joins the orbit queue with probability $(1-p)^M$ and stands there last in line. The orbit queue dispatches individual orbiting customers back to the main queue at a constant Poisson rate $\gamma$ (whenever there is a positive number of orbiting customers), so that only the customer at the head of the orbit queue is allowed to try to access the main queue.
Let $L_1(t)$ be the total number of *groups* in the main queue at time $t$, and $L_2(t)$ the number of *customers* in the orbit queue at time $t$. Let $L_i = \lim_{t\to\infty} L_i(t)$, and $P_{mn} = \mathbb{P}(L_1 = m, L_2 = n)$, for $m = 0, 1, \ldots, M$ and $n \geq 0$. A transition-rate diagram of the two-dimensional continuous-time Markov process $(L_1, L_2)$ is depicted in Fig. 1.

### 2.2 Balance equations and generating functions

For $m = 0$, the following relations hold,

$$\lambda P_{00} = \mu P_{10}, \tag{2.1}$$

$$(\lambda + \gamma)\, P_{0n} = \mu P_{1n}, \quad n \geq 1. \tag{2.2}$$

For $1 \leq m \leq M-1$, we get

$$\big(\lambda(1-p)^m + \mu\big)\, P_{m0} = \lambda(1-p)^{m-1} P_{m-1,0} + \mu P_{m+1,0} + \gamma P_{m-1,1}, \tag{2.3}$$

$$\begin{aligned}\big(\lambda(1-p)^m + \mu + \gamma\big)\, P_{mn} = \lambda(1-p)^{m-1} P_{m-1,n} + \mu P_{m+1,n} \\ + \gamma P_{m-1,n+1}, \quad n \geq 1.\end{aligned} \tag{2.4}$$

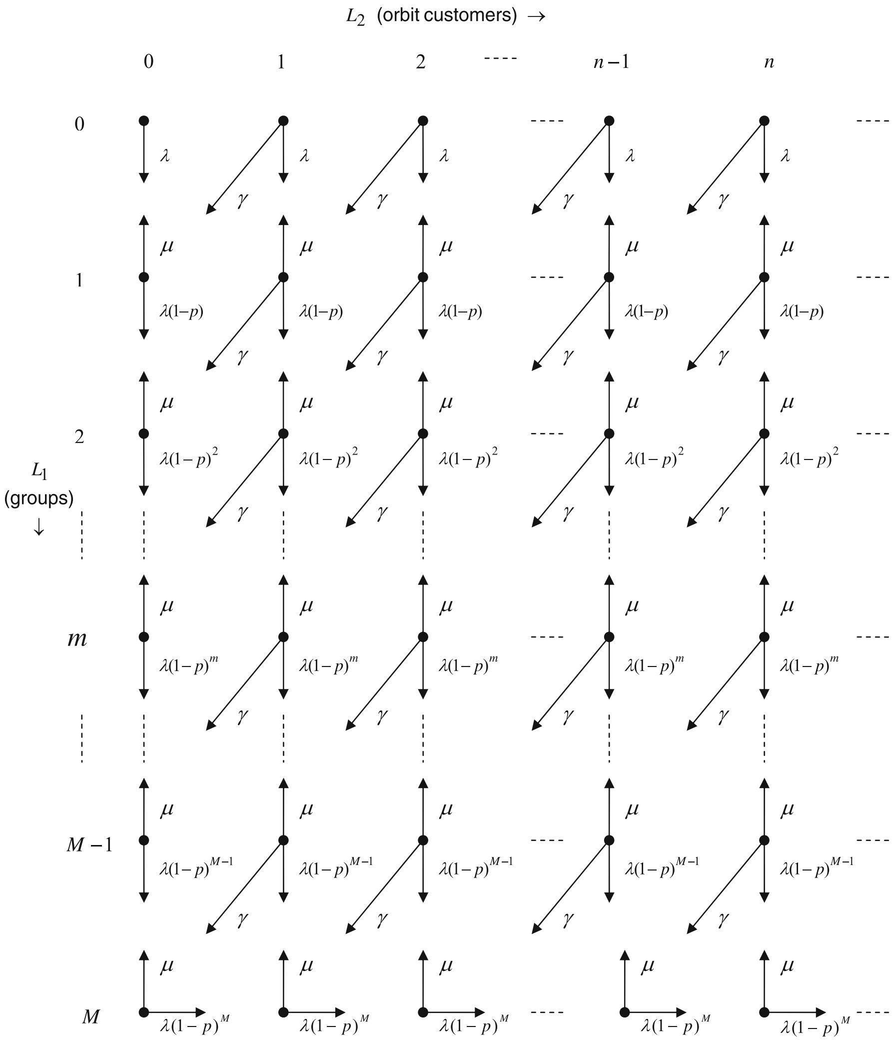


**Fig. 1** Transition-rate diagram of $(L_1, L_2)$

Lastly, for $m = M$, we have

$$\left(\lambda(1-p)^M + \mu\right) P_{M0} = \lambda(1-p)^{M-1} P_{M-1,0} + \gamma P_{M-1,1}, \tag{2.5}$$

$$\begin{aligned}\left(\lambda(1-p)^M + \mu\right) P_{Mn} &= \lambda(1-p)^{M-1} P_{M-1,n} + \lambda(1-p)^M P_{M,n-1} \\ &\quad + \gamma P_{M-1,n+1}, \quad n \geq 1.\end{aligned} \tag{2.6}$$

Now, for $0 \leq m \leq M$, define the $m$-th marginal PGF of the number of customers in orbit:

$$G_m(z) = \sum_{n=0}^{\infty} P_{mn} z^n.$$

Then, for $m = 0$, multiplying Eq. (2.2) by $z^n$ and summing over $n$, together with (2.1), gives

$$(\lambda + \gamma)\, G_0(z) - \mu G_1(z) = \gamma P_{00}. \tag{2.7}$$

For $1 \le m \le M-1$, multiplying Eq. (2.4) by $z^n$ and summing over $n$, together with (2.3), leads to

$$\begin{aligned} &\left(\lambda(1-p)^m + \mu + \gamma\right) zG_m(z) - \left(\lambda(1-p)^{m-1}z + \gamma\right) G_{m-1}(z) \\ &\quad - \mu z G_{m+1}(z) = \gamma\left(P_{m0}z - P_{m-1,0}\right). \end{aligned} \tag{2.8}$$

Finally, multiplying Eq. (2.6) by $z^n$ and summing over $n$, together with (2.5), results in

$$\left(\lambda(1-p)^M(1-z) + \mu\right) zG_M(z) - \left(\lambda(1-p)^{M-1}z + \gamma\right) G_{M-1}(z) = -\gamma P_{M-1,0}. \tag{2.9}$$

Define

$$\begin{aligned} \alpha_0(z) &= \lambda + \gamma, \\ \alpha_m(z) &= (\lambda(1-p)^m + \mu + \gamma)z, \quad 1 \le m \le M-1, \\ \alpha_M(z) &= (\lambda(1-p)^M(1-z) + \mu)z. \end{aligned}$$

The set of Eqs. (2.7), (2.8), and (2.9) can be written in a matrix form as

$$A(z) \cdot \vec{G}(z) = \vec{b}(z), \tag{2.10}$$

where

$$A(z) = \begin{pmatrix} \alpha_0(z) & -\mu & 0 & \cdots & \cdots & \cdots & 0 \\ -(\lambda z + \gamma) & \alpha_1(z) & -\mu z & 0 & \cdots & \cdots & \vdots \\ 0 & -(\lambda(1-p)z + \gamma) & \alpha_2(z) & -\mu z & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -(\lambda(1-p)^{M-2}z + \gamma) & \alpha_{M-1}(z) & -\mu z \\ 0 & \cdots & \cdots & 0 & 0 & -(\lambda(1-p)^{M-1}z + \gamma) & \alpha_M(z) \end{pmatrix},$$

$\vec{G}(z) = (G_0(z), G_1(z), \ldots, G_M(z))^T$ is a column vector (of size $M+1$) of the desired PGF's, and

$$\vec{b}(z) = \begin{pmatrix} b_0(z) \\ b_1(z) \\ \vdots \\ b_m(z) \\ \vdots \\ b_{M-1}(z) \\ b_M(z) \end{pmatrix} = \begin{pmatrix} \gamma P_{00} \\ \gamma\,(P_{10}z - P_{00}) \\ \vdots \\ \gamma\left(P_{m0}z - P_{m-1,0}\right) \\ \vdots \\ \gamma\left(P_{M-1,0}z - P_{M-2,0}\right) \\ -\gamma P_{M-1,0} \end{pmatrix}.$$

To obtain $G_m(z)$, we use Cramer's rule and write $G_m(z) = \frac{|A_m(z)|}{|A(z)|}$, $m = 0, 1, \ldots, M$, where $|A|$ is the determinant of a matrix $A$ and $A_m(z)$ is the matrix obtained from $A(z)$ by replacing its $m$-th column by $\vec{b}(z)$. The functions $G_m(z)$ are expressed in terms of $M$ unknown boundary probabilities, $P_{00}$, $P_{10}$,. . .,$P_{M-1,0}$, appearing in $\vec{b}(z)$. In order to derive these boundary probabilities, we utilize the roots of $|A(z)|$. We have the following:

**Theorem 2.1** *For any finite buffer size $M$, the polynomial $|A(z)|$ is of degree $M+1$. It possesses a root of multiplicity $\left\lfloor \frac{M}{2} \right\rfloor$ at $z = 0$ and a single root at $z = 1$. Furthermore, if the condition*

$$\lambda(1-p)^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) < \gamma\mu^M \sum_{m=0}^{M-1} \frac{1}{\mu^m} \prod_{k=0}^{m-1} (\lambda(1-p)^k + \gamma) \tag{2.11}$$

*holds, then $|A(z)|$ has exactly $M - \left\lfloor \frac{M}{2} \right\rfloor - 1$ roots in the open interval $(0, 1)$ and a single root in $(1, \infty)$. Else, $|A(z)|$ has exactly $M - \left\lfloor \frac{M}{2} \right\rfloor$ roots in $(0, 1)$.*

*Proof* Let $q_0(z) = 1$ and define the determinants of the minors of the diagonal of the matrix $A(z)$, starting from the upper left corner, as follows:

$$q_1(z) = \alpha_0(z) = \lambda + \gamma,\ q_2(z) = \begin{vmatrix} \alpha_0(z) & -\mu \\ -(\lambda z + \gamma) & \alpha_1(z) \end{vmatrix}, \ldots, q_{M+1}(z) = |A(z)|.$$

It can be verified that

$$\begin{aligned} &q_1(z) = \alpha_0(z)q_0(z), \\ &q_2(z) = \alpha_1(z)q_1(z) - \mu(\lambda z + \gamma)q_0(z), \\ &q_m(z) = \alpha_{m-1}(z)q_{m-1}(z) - \mu z(\lambda(1-p)^{m-2}z + \gamma)q_{m-2}(z), \quad m = 3, \ldots, M+1. \end{aligned} \tag{2.12}$$

We get the following properties:

1. $q_0(z)$ and $q_1(z)$ have no roots.
2. $q_m(z)$ is of degree $m - 1$, for $m = 1, 2, \ldots, M$.
3. $q_{M+1}(z) = |A(z)|$ is of degree $M + 1$.

4. $q_m(z)$ and $q_{m+1}(z)$ have no common roots in (0,1), because, if they do have such a common root, then it is also a root of $q_{m-1}(z)$, $q_{m-2}(z)$,…,$q_1(z)$, but $q_1(z)$ possesses no roots.
5. $q_1(0) > 0$, $q_2(0) < 0$.
6. $q_m(1) = \prod_{k=0}^{m-1}\left(\lambda(1-p)^k + \gamma\right) > 0$ for $m = 0, 1, \ldots, M$, where $\prod_{i=0}^{-1}(\cdot) \triangleq 1$.
7. $q_{M+1}(1) = 0$.
8. If $z^*$ is a root of $q_m(z)$, then $\mathrm{sign}(q_{m-1}(z^*) \cdot q_{m+1}(z^*)) = -1$.
9. For the polynomials $q_m(z)$, where $m = 1, 2, \ldots, M$, zero is a root of multiplicity $\left\lfloor \frac{m-1}{2} \right\rfloor$.
10. For $k \geq 1$, sign($q_{2k+1}(0^+)$)=$(-1)^k$ and sign($q_{2k+2}(0^+)$)=$(-1)^{k+1}$ (proven by induction).

We now consider the roots of the polynomials $q_m(z)$, for $m = 1, 2, \ldots, M+1$. $q_1(z) = \lambda + \gamma$ and therefore has no roots. $q_2(z)$ is of degree 1 and has a single root in (0,1). We shall denote it by $z_{2,1}$. Next, $q_3(0) = 0$, $q_3(1) > 0$. Since $q_1(z_{2,1}) > 0$, $q_3(z_{2,1}) < 0$ (by property 8). Therefore, since $q_3(z)$ is of degree 2, it has a single root in ($z_{2,1}$,1). We shall denote this root by $z_{3,1}$. Further, $q_4(z)$ is of degree 3, $q_4(0) = 0$, so two roots are left to be determined. We have that $q_4(0^+) > 0$, and since $q_2(z_{3,1}) > 0$, it follows from property 8 that $q_4(z_{3,1}) < 0$. With the fact that $q_4(1) > 0$, we conclude that $q_4(z)$ has the roots $z_{4,1} \in (0, z_{3,1})$ and $z_{4,2} \in (z_{3,1}, 1)$. Continuing with $q_5(z)$, its degree is 4 and it has a root at $z = 0$ with multiplicity 2. Now, $q_5(0^+) > 0$, $q_5(z_{4,1}) > 0$, $q_5(z_{4,2}) < 0$, and $q_5(1) > 0$. This implies that $q_5(z)$ has the roots $z_{5,1} \in (z_{4,1}, z_{4,2})$ and $z_{5,2} \in (z_{4,2}, 1)$. Proceeding further, we conclude that $q_M(z)$ has a root at $z = 0$ with multiplicity $\lfloor \frac{M-1}{2} \rfloor$ and exactly $M - \lfloor \frac{M-1}{2} \rfloor - 1$ roots in the open interval (0, 1). Our interest is in the roots of $q_{M+1}(z) = |A(z)|$. From Eq. (2.12), the degree of $q_{M+1}(z)$ is $M+1$. It has a root of multiplicity $\left\lfloor \frac{M}{2} \right\rfloor$ at $z = 0$, a single root at $z = 1$, and $M - \left\lfloor \frac{M}{2} \right\rfloor - 1$ roots in the open interval (0, 1). Note that since $z_{M,M-1-\lfloor \frac{M-1}{2} \rfloor} \in (z_{M-1,M-2-\lfloor \frac{M-2}{2} \rfloor}, 1)$ (meaning that the largest root of $q_M(z)$ is between the largest root of $q_{M-1}(z)$ and 1), and since $q_{M-1}(1) > 0$, then $q_{M+1}(z_{M,M-1-\lfloor \frac{M-1}{2} \rfloor}) < 0$. So, the last root of $q_{M+1}(z)$, denoted by $z^*_{M+1}$, might be either in $(z_{M,M-1-\lfloor \frac{M-1}{2} \rfloor}, 1)$ or in $(1, \infty)$. We shall now prove that if condition (2.11) holds, then the last root is in $(1, \infty)$, else it is in $(z_{M,M-1-\lfloor \frac{M-1}{2} \rfloor}, 1)$. First, note that since $q_M(1) > 0$ and $q_M(z)$ has no roots in $[1, \infty)$, then $q_M(\infty) > 0$. It follows from (2.12) that $q_{M+1}(\infty) < 0$. Next, in the Appendix, we show that the polynomials $q_m(z)$ are of the following form:

$$q_m(z) = z^{m-1} \prod_{k=0}^{m-1} (\lambda(1-p)^k + \gamma) + (1-z)h_m^{(M)}(z), \quad m = 1, 2, \ldots, M,$$
$$q_{M+1}(z) = (1-z)h_{M+1}^{(M)}(z),$$

where $h_m^{(M)}(z)$ are functions discussed and explored in the Appendix. Now,

$$q'_{M+1}(1) = -h_{M+1}^{(M)}(1).$$

From all of the above, we conclude that if $h^{(M)}_{M+1}(1) < 0$, then $q'_{M+1}(1) > 0$ and therefore, since $q_{M+1}(\infty) < 0$, $z^*_{M+1} \in (1, \infty)$. Otherwise, if $h^{(M)}_{M+1}(1) > 0$, then $q'_{M+1}(1) < 0$, which yields $z^*_{M+1} \in (z_{M,M-\lfloor \frac{M-1}{2} \rfloor -1}, 1)$.

As proven in the Appendix,

$$h^{(M)}_{M+1}(1) = \lambda(1-p)^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) - \gamma\mu^M \sum_{m=0}^{M-1} \frac{1}{\mu^m} \prod_{k=0}^{m-1} (\lambda(1-p)^k + \gamma).$$

If $h^{(M)}_{M+1}(1) < 0$, then the condition in (2.11) holds, so that $z^*_{M+1} \in (1, \infty)$, else $z^*_{M+1} \in (z_{M,M-\lfloor \frac{M-1}{2} \rfloor -1}, 1)$. This completes the proof.

*Remark* In Sect. 5, we provide an analysis via matrix geometric method. The stability condition derived using this method, given in the sequel by Eq. (5.4), is exactly the condition given by (2.11), that is, we may say that $z^*_{M+1}$ is in $(1, \infty)$ iff the system is stable.

We now assume that the system is stable and explain how to derive the boundary probabilities $P_{00}$, $P_{10}$,…,$P_{M-1,0}$. First, since for all $m = 0, 1, \ldots, M$, $G_m(z) = \frac{|A_m(z)|}{|A(z)|}$ is a PGF, every root of $|A(z)|$ is a root of $|A_m(z)|$. From Theorem 2.1, we conclude that

$$|A_m(z_{M+1,k})| = 0, \quad z_{M+1,k} \in (0, 1),\ k = 1, 2, \ldots, M - \left\lfloor \frac{M}{2} \right\rfloor - 1, \tag{2.13}$$

$$\frac{d^k}{dz^k} |A_m(z)| \bigg|_{z=0} = 0, \quad k = 1, 2, \ldots, \left\lfloor \frac{M}{2} \right\rfloor - 1, \tag{2.14}$$

where Eq. (2.14) follows from the fact that $q_{M+1}(z)$ has a root of multiplicity $\left\lfloor \frac{M}{2} \right\rfloor$ at $z = 0$. Second, we utilize the roots $z_{M+1,k} \in (0, 1)$, for $k = 1, 2, \ldots, M - \left\lfloor \frac{M}{2} \right\rfloor - 1$, as follows: we substitute $z_{M+1,1}$ in $|A_0(z)|$, $z_{M+1,2}$ in $|A_1(z)|$, and so on, so that the last root, $z_{M+1,M-\lfloor \frac{M}{2} \rfloor -1}$, is substituted in $|A_{M-\lfloor \frac{M}{2} \rfloor -2}(z)|$. This provides us with $M - \left\lfloor \frac{M}{2} \right\rfloor - 1$ equations relating between the boundary probabilities. Another set of equations is derived from the derivatives of $|A_m(z)|$ as follows. We substitute $z = 0$ in the $\left(\left\lfloor \frac{M}{2} \right\rfloor - 1\right)$-th derivative of $|A_{M-k}(z)|$, for $k = 0, 1, \ldots, \lfloor \frac{M}{2} \rfloor - 1$. This gives us $\left\lfloor \frac{M}{2} \right\rfloor$ more equations, bringing the total to $M - \left\lfloor \frac{M}{2} \right\rfloor - 1 + \left\lfloor \frac{M}{2} \right\rfloor = M - 1$. The last relation used is the normalization equation, that is,

$$\sum_{m=0}^{M} P_{m\bullet} = \sum_{m=0}^{M} G_m(1) = \left( \sum_{m=0}^{M} \frac{|A_m(z)|}{|A(z)|} \right) \bigg|_{z=1} = 1.$$

For example, let $M = 8$. We need to determine 8 boundary probabilities, $P_{00}$, $P_{10}, \ldots, P_{70}$. $|A(z)| = q_9(z)$ is of degree 9. Under stability conditions, it possesses a root at $z = 0$ with multiplicity 4, three roots in $(0, 1)$, a single root at $z = 1$, and a single root in $(1, \infty)$ (which is not utilized for the calculation of the boundary probabilities). We need to solve the following system:

$$\begin{aligned}
&|A_0(z_{9,1})| = 0, \quad |A_1(z_{9,2})| = 0, \quad |A_2(z_{9,3})| = 0,\\
&\frac{d^3}{dz^3}|A_m(z)|\bigg|_{z=0} = 0, \;\; m = 5, 6, 7, 8,\\
&\left(\sum_{m=0}^{8} \frac{|A_m(z)|}{|A(z)|}\right)\bigg|_{z=1} = 1.
\end{aligned} \tag{2.15}$$

The set (2.15) yields 8 linear equations in the 8 sought for boundary probabilities.

In general, when all the boundary probabilities are known, the set of PFGs $\{G_m(z)\}_{m=0}^{M}$ is completely determined. In addition, the marginal distribution of the size of the main queue is given by

$$P_{m\bullet} = \sum_{n=0}^{\infty} P_{mn} = P(L_1 = m) = G_m(1), \;\; m = 0, 1, \ldots, M.$$

Alternatively, the marginal probabilities $\{P_{m\bullet}\}_{m=0}^{M}$ can be calculated by applying horizontal "cuts" on Fig. 1, as follows:

$$(\lambda(1-p)^m + \gamma)P_{m\bullet} - \gamma P_{m0} = \mu P_{m+1,\bullet}\,, \quad m = 0, 1, \ldots, M-1, \tag{2.16}$$

so that for all $1 \leq m \leq M$, $P_{m\bullet}$ can be expressed in terms of $P_{0\bullet}$, $P_{00}$, $P_{10}$,. . .,$P_{m-1,0}$.

Solving iteratively, Eq. (2.16) gives

$$\begin{aligned}
P_{m\bullet} = &\frac{1}{\mu^m}\prod_{k=0}^{m-1}(\lambda(1-p)^k + \gamma)P_{0\bullet}\\
&-\gamma\sum_{j=0}^{m-1}\frac{1}{\mu^{m-j}}\prod_{k=j+1}^{m-1}(\lambda(1-p)^k + \gamma)P_{j0}, \quad m=1, 2, \ldots, M,
\end{aligned} \tag{2.17}$$

where $P_{0\bullet}$ is derived by using $\sum_{m=0}^{M} P_{m\bullet} = 1$. We thus have

$$P_{0\bullet} = \frac{1 + \sum_{m=0}^{M}\gamma\sum_{j=0}^{m-1}\frac{1}{\mu^{m-j}}\prod_{k=j+1}^{m-1}(\lambda(1-p)^k + \gamma)P_{j0}}{\sum_{m=0}^{M}\frac{1}{\mu^m}\prod_{k=0}^{m-1}(\lambda(1-p)^k + \gamma)}, \tag{2.18}$$

with the notations $\sum_{k=0}^{-1}(\cdot) \triangleq 0$ and $\prod_{k=0}^{-1}(\cdot) \triangleq 1$. Furthermore, summation of Eq. (2.16) over $m$ leads to

$$\sum_{m=0}^{M-1} \lambda(1-p)^m P_{m\bullet} + \gamma(1 - P_{M\bullet} - P_{\bullet 0} + P_{M0}) = \mu(1 - P_{0\bullet}), \tag{2.19}$$

meaning that the creation rate of new groups in the main queue (by external arrivals or by orbit customers) is equal to the emptying rate of groups from the main queue. In a similar manner, vertical "cuts" on Fig. 1 yield

$$\lambda(1-p)^M P_{M\bullet} = \gamma(1 - P_{M\bullet} - P_{\bullet 0} + P_{M0}), \tag{2.20}$$

implying that the entrance and departure rate to and from the orbit queue are equal. Combining Eqs. (2.19) and (2.20) yields

$$\sum_{m=0}^{M} \lambda(1-p)^m P_{m\bullet} = \mu(1 - P_{0\bullet}). \tag{2.21}$$

## 3 Performance measures

### 3.1 Mean queue lengths

Let $\mathbb{E}[L_1]$ and $\mathbb{E}[L_2]$ denote the mean total number of groups in the main queue, and the mean total number of customers in the orbit queue, respectively. Then,

$$\mathbb{E}[L_1] = \sum_{m=0}^{M} m P_{m\bullet} = \sum_{m=0}^{M} m G_m(1), \tag{3.1}$$

$$\mathbb{E}[L_2] = \sum_{n=0}^{\infty} n P_{\bullet n} = \sum_{m=0}^{M} G'_m(1). \tag{3.2}$$

The expression given for $\mathbb{E}[L_2]$ can be rewritten in a simpler form, easier for performing computations. Specifically, we multiply Eq. (2.7) by $z$ and sum it together with Eqs. (2.8) (for $m = 1, 2, \ldots, M-1$) and (2.9). This gives

$$\lambda(1-p)^M z G_M(z) - \gamma \sum_{m=0}^{M-1} G_m(z) = -\gamma \sum_{m=0}^{M-1} P_{m0}, \tag{3.3}$$

or equivalently

$$\sum_{m=0}^{M} G_m(z) = \left( \frac{\lambda(1-p)^M z}{\gamma} + 1 \right) G_M(z) + \sum_{m=0}^{M-1} P_{m0}. \tag{3.4}$$

Taking the derivatives on both sides of (3.4) and letting $z = 1$ results in

$$\mathbb{E}[L_2] = \sum_{m=0}^{M} G'_m(1) = \frac{\lambda(1-p)^M}{\gamma} P_{M\bullet} + G'_M(1) \left( \frac{\lambda(1-p)^M}{\gamma} + 1 \right). \tag{3.5}$$

### 3.2 Mean size of a batch

We now calculate the mean size of the group standing at the $m$-th position ($1 \leq m \leq N$), right after the moment of service completion, and before the group moves forward to the $(m-1)$-st position (or leaves the system, when $m = 1$). We define the following variables, for $1 \leq m \leq M$:

- $\xi^{(k)}$ = number of customers who have joined the group in the $k$-th position during a single service period, assuming that the $k$-th group exists.
- $D_i^{(m)}$ = size of the batch standing in the $m$-th position at the moment of service completion, given that it was formed in the $i$-th position, $m \leq i \leq M$.
- $D^{(m)}$ = size of the batch standing in the $m$-th position at the moment of service completion.

Since the mean number of arrivals during a service period is $\frac{\lambda}{\mu}$, and the probability that an arrival joins the $k$-th group (when it exists) is $(1-p)^{k-1}p$, we get

$$\mathbb{E}\left[\xi^{(k)}\right] = \frac{\lambda}{\mu}(1-p)^{k-1}p, \quad 1 \leq k \leq M. \tag{3.6}$$

Let $\nu_{i,m}$ denote the event that the group standing in the $m$-th position was created in the $i$-th position (so that $i \geq m$), whether by an external arrival (that does not know any of the present group leaders), or by a customer arriving from the orbit queue. By the definition of $D_i^{(m)}$, we have that $D_i^{(m)} = 1 + \sum_{k=1}^{i-m+1} \xi^{(i+1-k)} = 1 + \sum_{k=m}^{i} \xi^{(k)}$ with probability $\mathbb{P}(\nu_{i,m})$. In order to calculate $\mathbb{P}(\nu_{i,m})$, we condition on whether the orbit queue is empty or not, and if not, we further condition on the type of customer that forms the new group (either a new customer or one coming from the orbit queue). We have

$$\mathbb{P}(\nu_{i,m}) = \frac{P_{i-1,0}(1-p)^{i-1} + \left(P_{i-1,\bullet} - P_{i-1,0}\right)\left(\frac{\lambda}{\lambda+\gamma}(1-p)^{i-1} + \frac{\gamma}{\lambda+\gamma}\right)}{\sum_{j=m}^{M}\left[P_{j-1,0}(1-p)^{j-1} + \left(P_{j-1,\bullet} - P_{j-1,0}\right)\left(\frac{\lambda}{\lambda+\gamma}(1-p)^{j-1} + \frac{\gamma}{\lambda+\gamma}\right)\right]}. \tag{3.7}$$

Rewriting Eq. (3.7) gives

$$\begin{aligned}\mathbb{P}(\nu_{i,m}) &= \frac{P_{i-1,\bullet}(\lambda(1-p)^{i-1} + \gamma) - \gamma P_{i-1,0}(1-(1-p)^{i-1})}{\sum_{j=m}^{M}\left[P_{j-1,\bullet}(\lambda(1-p)^{j-1} + \gamma) - \gamma P_{j-1,0}(1-(1-p)^{j-1})\right]} \\ &= \frac{\gamma P_{i-1,0}(1-p)^{i-1} + \mu P_{i\bullet}}{\sum_{j=m}^{M}\left[\gamma P_{j-1,0}(1-p)^{j-1} + \mu P_{j\bullet}\right]},\end{aligned} \tag{3.8}$$

where in the last equality in (3.8) we used Eq. (2.16).

From (3.8) and (3.6), we get

$$\begin{aligned}\mathbb{E}\left[D^{(m)}\right] &= \sum_{i=m}^{M} \mathbb{E}\left[D_i^{(m)}\right] \mathbb{P}(\nu_{i,m}) \\ &= \sum_{i=m}^{M} \left(1 + \sum_{k=m}^{i} \frac{\lambda}{\mu}(1-p)^{k-1} p\right) \frac{\gamma P_{i-1,0}(1-p)^{i-1} + \mu P_{i\bullet}}{\sum_{j=m}^{M} \left[\gamma P_{j-1,0}(1-p)^{j-1} + \mu P_{j\bullet}\right]} \\ &= 1 + \frac{\lambda}{\mu}(1-p)^{m-1} - \frac{\lambda}{\mu} \frac{\sum_{i=m}^{M} \left[\gamma P_{i-1,0}(1-p)^{i-1} + \mu P_{i\bullet}\right](1-p)^{i}}{\sum_{j=m}^{M} \left[\gamma P_{j-1,0}(1-p)^{j-1} + \mu P_{j\bullet}\right]},\end{aligned} \tag{3.9}$$

where $P_{i\bullet}$ are given in (2.17) and (2.18).

In particular, the mean size of the served batch at a moment of service completion is given by

$$\begin{aligned}\mathbb{E}\left[D^{(1)}\right] &= \sum_{i=1}^{M} \mathbb{E}\left[D_i^{(1)}\right] \mathbb{P}(\nu_{i,1}) \\ &= 1 + \frac{\lambda}{\mu} - \frac{\lambda}{\mu} \cdot \frac{\sum_{i=1}^{M} \left[\gamma P_{i-1,0}(1-p)^{i-1} + \mu P_{i\bullet}\right](1-p)^{i}}{\sum_{j=1}^{M} \gamma P_{j-1,0}(1-p)^{j-1} + \mu(1 - P_{0\bullet})}.\end{aligned} \tag{3.10}$$

For the case where $M = 1$, we get from Eq. (3.10): $\mathbb{E}\left[D^{(1)}\right] = 1 + \frac{\lambda}{\mu} p$. Indeed, $D^{(1)}$ is composed of the leader of the group plus all arrivals that joined him during his service duration.

On the other hand, when $p = 1$, there is only one group and $\mathbb{E}\left[D^{(1)}\right] = 1 + \frac{\lambda}{\mu}$, that is, the mean size of a served group is composed of the group's leader and the mean number of arrivals during his service period.

### 3.3 Number of bypasses and the position of a new arriving customer

In our model, it can happen that a newly arriving customer will be served before customers that have arrived before him. For example, suppose the system is in state $(m, n)$ ($m$ groups in the main queue and $n$ customers in the orbit queue), and suppose that a new arrival joins the main queue in position $i \le m$. We then say that the number of bypasses made by this new arrival is $m - i + n$, that is, he passes $m - i$ group leaders and $n$ orbit customers. Let $Y$ denote the number of bypasses. Then,

$$\mathbb{P}(Y = 0) = P_{00} + \sum_{m=1}^{M-1} P_{m0}(1-p)^{m-1} + P_{M0}(1-p)^{M-1} p + P_{M\bullet}(1-p)^{M}, \tag{3.11}$$

where the first 3 expressions in (3.11) describe the case that a new arrival either opens a new group in the main queue or joins the last group there, and the last expression means that he joins the last position in the orbit queue (and therefore no bypasses are made).

For $Y = k \geq 1$, we distinguish between two cases: $k < M$ and $k \geq M$. For each case, we consider two possibilities: $(i)$ the newly arriving customer forms a new group, or $(ii)$ the newly arriving customer joins an existing group. More specifically, the first possibility is that the system is in state $(m, k)$ for $m = 0, 1 \ldots, M-1$ and $k \geq 1$, and a new arrival does not know any of the first $m$ group leaders. Consequently, he forms a new group and therefore bypasses only the $k$ customers present in the orbit. The second possibility is that the system is in state $(m, k-j)$, $1 \leq m \leq M$, $j \leq Min\{m-1, k\}$, and a new arrival knows the group leader in the $(m-j)$-th position, so that $j$ groups in the main queue are being bypassed. Therefore, the numbers of bypasses is $j + (k-j) = k$. We now calculate $\mathbb{P}(Y = k)$ for each of the cases $k < M$ and $k \geq M$, as follows:

For $k < M$,

$$\mathbb{P}(Y = k) = \sum_{m=0}^{M-1} P_{mk}(1-p)^m + p \sum_{m=1}^{k} \sum_{j=0}^{m-1} P_{m,k-j}(1-p)^{m-j-1} + p \sum_{m=k+1}^{M} \sum_{j=0}^{k} P_{m,k-j}(1-p)^{m-j-1}. \tag{3.12}$$

The first term in the right-hand side of Eq. (3.12) stands for a new arrival that forms a new group and hence bypasses only the customers in the orbit queue. The second and third terms represent the case where both the groups in the main queue and the customers in the orbit queue are being bypassed, distinguished by either $m \leq k$ or $m > k$.

Similarly, for $k \geq M$,

$$\mathbb{P}(Y = k) = \sum_{m=0}^{M-1} P_{mk}(1-p)^m + p \sum_{m=1}^{M} \sum_{j=0}^{m-1} P_{m,k-j}(1-p)^{m-j-1}. \tag{3.13}$$

It follows that the mean number of bypasses, $\mathbb{E}[Y] = \sum_{k=0}^{\infty} k\mathbb{P}(Y = k)$, can only be numerically calculated up to a certain accuracy, using truncation. However, an explicit expression for $\mathbb{E}[Y]$ can be obtained as follows. Let $X$ denote the position (group's number) that a new arrival enters to in the main queue, given that he is not blocked. Since the probability of joining the orbit queue is $P_{M\bullet}(1-p)^M$, the distribution function of $X$ is given by

$$\mathbb{P}(X = i) = \frac{P_{i-1,\bullet}(1-p)^{i-1} + \sum_{m=i}^{M} P_{m\bullet}(1-p)^{i-1} p}{1 - P_{M\bullet}(1-p)^M}, \quad i = 1, \ldots, M, \tag{3.14}$$

and

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{1-P_{M\bullet}(1-p)^M}\left[\sum_{i=1}^{M} i P_{i-1,\bullet}(1-p)^{i-1} + \sum_{m=1}^{M} P_{m\bullet}\sum_{i=1}^{m} i(1-p)^{i-1}p\right]\\ &= \frac{1}{1-P_{M\bullet}(1-p)^M}\left[\sum_{i=1}^{M} i P_{i-1,\bullet}(1-p)^{i-1} + \sum_{m=1}^{M} P_{m\bullet}\frac{1-(1-p)^m(1+mp)}{p}\right]\\ &= \frac{1}{1-P_{M\bullet}(1-p)^M}\left[\left(1-\sum_{m=0}^{M} P_{m\bullet}(1-p)^m\right)\frac{1}{p}\right.\\ &\qquad \left. + \sum_{i=1}^{M} i(1-p)^{i-1}\left[P_{i-1,\bullet}-P_{i\bullet}(1-p)\right]\right]. \end{aligned} \tag{3.15}$$

Define $Z = Max\{0, L_1 - X\}$ to be the number of groups (in the main queue only) being bypassed by a new arriving customer that joins the main queue. The distribution function of the random variable $Z$ is given by

$$\mathbb{P}(Z=0) = \frac{1}{1-P_{M\bullet}(1-p)^M}\left(P_{0\bullet}+\sum_{m=1}^{M-1} P_{m\bullet}(1-p)^{m-1}+P_{M\bullet}(1-p)^{M-1}p\right),$$

$$\mathbb{P}(Z=j) = \frac{1}{1-P_{M\bullet}(1-p)^M}\sum_{m=j+1}^{M} P_{m\bullet}(1-p)^{m-j-1}p, \quad j=1,2,\ldots,M-1.$$

This implies

$$\begin{aligned}\mathbb{E}[Z] &= \frac{1}{1-P_{M\bullet}(1-p)^M}\sum_{j=1}^{M-1} j \sum_{m=j+1}^{M} P_{m\bullet}(1-p)^{m-j-1}p\\ &= \frac{1}{1-P_{M\bullet}(1-p)^M}\sum_{m=2}^{M} P_{m\bullet}\sum_{j=1}^{m-1} j(1-p)^{m-j-1}p\\ &= \frac{1}{1-P_{M\bullet}(1-p)^M}\frac{1}{p}\sum_{m=2}^{M} P_{m\bullet}(mp-1+(1-p)^m). \end{aligned} \tag{3.16}$$

Straightforward algebra on Eq. (3.16) leads to

$$\mathbb{E}[Z] = \frac{1}{1-P_{M\bullet}(1-p)^M}\left(\mathbb{E}[L_1]-\frac{1}{p}\left(1-\sum_{m=0}^{M} P_{m\bullet}(1-p)^m\right)\right). \tag{3.17}$$

To calculate $\mathbb{E}[Y]$, we condition on whether the new arrival joins the main queue or the orbit queue. We get

$$
\begin{aligned}
\mathbb{E}[Y] &= \mathbb{P}(\text{A new arrival enters the orbit queue}) \cdot 0 \\
&\quad + \mathbb{P}(\text{A new arrival enters the main queue}) \cdot (\mathbb{E}\,[Z] + \mathbb{E}[L_2]) \\
&= (1 - P_{M\bullet}(1-p)^M)\,(\mathbb{E}\,[Z] + \mathbb{E}[L_2]) \\
&= \mathbb{E}[L_1] - \frac{1}{p}\left(1 - \sum_{m=0}^{M} P_{m\bullet}(1-p)^m\right) + (1 - P_{M\bullet}(1-p)^M)\mathbb{E}[L_2].
\end{aligned}
\tag{3.18}
$$

## 4 $c \geq 1$ servers

The above single-server model can be readily extended to a multi-server system with $1 \leq c \leq M$ servers. The corresponding transition-rate diagram will look similar to Fig. 1 with the modification that in state $(L_1, L_2)$, the service rate in the main queue is $L_1\mu$ for $L_1 \leq c$, and $c\mu$ for $c < L_1 \leq M$. The resulting balance equations will have the same structure and will lead to a set of equations regarding the PGFs similar to Eq. (2.10) with the appropriate matrix $A(z)$ and the vector $\vec{b}(z)$. We will not elaborate further on this extension.

## 5 Matrix geometric approach

Following Neuts [18], we construct a quasi-birth-and-death (QBD) process, with $M+1$ phases and an infinite number of levels. State $(n, m)$ indicates that there are $m$ different groups in the main queue and $n$ customers in the orbit queue, $m = 0, 1, \ldots, M, n \geq 0$. We arrange these states in a lexicographic order. Then, the infinitesimal generator of the QBD, denoted by $Q$, is given by

$$
Q = \begin{pmatrix}
B & A_0 & \mathbf{0} & \mathbf{0} & \cdots \\
A_2 & A_1 & A_0 & \mathbf{0} & \cdots \\
\mathbf{0} & A_2 & A_1 & A_0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
$$

where $B$, $A_0$, $A_1$, and $A_2$ are all square matrices of order $M+1$, as follows:

$$
B = \begin{pmatrix}
-\lambda & \lambda & 0 & \cdots & \cdots & 0 \\
\mu & -(\lambda(1-p)+\mu) & \lambda(1-p) & 0 & \cdots & \vdots \\
0 & \mu & -(\lambda(1-p)^2+\mu) & \lambda(1-p)^2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \mu & -(\lambda(1-p)^{M-1}+\mu) & \lambda(1-p)^{M-1} \\
0 & 0 & 0 & 0 & \mu & -(\lambda(1-p)^M+\mu)
\end{pmatrix},
$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & \lambda(1-p)^M \end{pmatrix}.$$

$$A_1 = \begin{pmatrix} -(\lambda+\gamma) & \lambda & 0 & \cdots & \cdots & 0 \\ \mu & -(\lambda(1-p)+\mu+\gamma) & \lambda(1-p) & 0 & \cdots & \vdots \\ 0 & \mu & -(\lambda(1-p)^2+\mu+\gamma) & \lambda(1-p)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu & -(\lambda(1-p)^{M-1}+\mu+\gamma) & \lambda(1-p)^{M-1} \\ 0 & 0 & 0 & 0 & \mu & -(\lambda(1-p)^M+\mu) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 0 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & \gamma & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \gamma \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Define $A = A_0 + A_1 + A_2$. Then,

$$A = \begin{pmatrix} -(\lambda+\gamma) & \lambda+\gamma & 0 & \cdots & \cdots & 0 \\ \mu & -(\lambda(1-p)+\mu+\gamma) & \lambda(1-p)+\gamma & 0 & \cdots & \vdots \\ 0 & \mu & -(\lambda(1-p)^2+\mu+\gamma) & \lambda(1-p)^2+\gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu & -(\lambda(1-p)^{M-1}+\mu+\gamma) & \lambda(1-p)^{M-1}+\gamma \\ 0 & 0 & 0 & 0 & \mu & -\mu \end{pmatrix}.$$

The matrix $A$ can be looked upon as the infinitesimal generator of a finite buffer $M/M/1$-type queue with service rate $\mu$, and with state-dependent arrival rate $\lambda(1-p)^m + \gamma$ for state $m$, $m = 0, 1, \ldots, M-1$.

Let $\vec{x} = (x_0, x_1, \ldots, x_M)$ be the stationary vector of the irreducible matrix $A$, i.e.,

$$\begin{cases} \vec{x}A = 0, \\ \vec{x} \cdot \vec{e} = 1 \end{cases} \tag{5.1}$$

A product-form solution to (5.1) is given by

$$x_m = \frac{1}{\mu^m} \prod_{j=0}^{m-1} \left(\lambda(1-p)^j + \gamma\right) x_0, \quad m = 1, 2, \ldots, M, \tag{5.2}$$

$$x_0 = \left( \sum_{m=0}^{M} \frac{1}{\mu^m} \prod_{j=0}^{m-1} \left( \lambda(1-p)^j + \gamma \right) \right)^{-1}. \tag{5.3}$$

The stability condition [18], $\vec{x} A_0 \vec{e} < \vec{x} A_2 \vec{e}$, translates into

$$\lambda(1-p)^M x_M < \gamma(1 - x_M). \tag{5.4}$$

By substituting in Eq. (5.4) the expression for $x_M$ given in (5.2), we obtain again the stability condition (2.11), that is, the system is stable iff the determinant of the matrix $A(z)$ presented in Sect. 2 has a root in $(1, \infty)$.

For the sake of consistency in defining steady-state probabilities in models analyzed via matrix geometric methods, we define for all $n \geq 0$ the steady-state probability vector $\vec{\pi}_n = (\pi_{n0}, \pi_{n1}, \ldots, \pi_{nM})$. Note that $\pi_{nm} = P_{mn}$ and $\vec{\pi}_n \cdot \vec{e} = P_{\bullet n} = P(L_2 = n)$. Then (see [18]),

$$\vec{\pi}_n = \vec{\pi}_0 R^n, \quad n \geq 0,$$

where $R$ is the minimal nonnegative solution of the matrix quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0. \tag{5.5}$$

The vector $\vec{\pi}_0$ is derived by solving the following linear system,

$$\begin{aligned} &\vec{\pi}_0 (B + RA_2) = \vec{0}, \\ &\vec{\pi}_0 \, [I - R]^{-1} \cdot \vec{e} = 1. \end{aligned} \tag{5.6}$$

Algorithms for the computation of the matrix $R$ are suggested in various works; see, for exmaple, [16,18] and [5]. In our case, we shall use Theorem 8.5.2 in [16], which considers a special form of the matrix $A_0$, as follows: if the QBD is positive recurrent and $A_0 = c \cdot r$, where $c$ is a column vector and $r$ is a row vector normalized by $r\mathbf{1} = 1$, then the matrix $R$ is given by $R = c \cdot \xi$, where $\xi = r\,(-A_1 - \eta A_2)^{-1}$. $\eta$ is the spectral radius of $R$, i.e., it is the eigenvalue of the matrix $R$ that has the largest absolute value. Indeed, in our model, the matrix $A_0$ may be represented as

$$A_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda(1-p)^M \end{pmatrix} \cdot (0, \ldots, 0, 1) = c \cdot r.$$

Following [16], we have that $\eta$ is the unique solution of the scalar equation $z = r\,(-A_1 - zA_2)^{-1}\, c$ in $(0, 1)$. Now, since $R = c \cdot \xi$ and all elements of $c$ are zeros except for the last one, then all first $M$ rows of $R$ are zeros except for the last row. For example, for $M = 5$, $\lambda = 2$, $\mu = 2$, $\gamma = 1$, and $p = 0.02$, we get that $\eta = 0.97597$ and

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.14102 & 0.21153 & 0.31476 & 0.46392 & 0.67661 & 0.97597 \end{pmatrix}.$$

## 6 Numerical results

In Tables 1, 2, and 3 below, we present numerical results for each one of the cases $M = 3$, $M = 5$, and $M = 8$, for different values of $\lambda$, $\mu$, $\gamma$, and $p$. We calculate the measures whose formulas were obtained in the previous sections. Note that the smallest values of $p$ in these tables are slightly higher than the $p$ value causing the system to be unstable.

The tables exhibit that, as expected, $\mathbb{E}[L_1]$ decreases monotonically when $p$ increases. In addition, there is a drastic decrease in $\mathbb{E}[L_2]$ as $p$ increases. This is well demonstrated in Fig. 2, in which $\mathbb{E}[L_2]$ is plotted as a function of $p$. The system's parameters considered there are $M = 3$, $\lambda = \mu = 2$, and $\gamma = 1$. The value of $p$ in which $\mathbb{E}[L_2]$ tends to infinity is about $p = 0.0727895$. Below, this value of $p$ the system is unstable. In Table 4, we provide exact values of $\mathbb{E}[L_2]$ for various values of $p$. The steep decrease in $\mathbb{E}[L_2]$ is apparent with a minor increase in $p$ above its stability value. In addition, the mean size of the served batch, $\mathbb{E}[D^{(1)}]$, is monotonically increasing when $p$ increases, but the mean number of bypasses, $\mathbb{E}[Y]$, decreases when $p$ increases. This follows since, when $p$ increases, the number of orbiting jobs drops.

**Table 1** Numerical results for $M = 3$

| $\mu = 2, \gamma = 1$ | | $\mathbb{E}[L_1]$ | $\mathbb{E}[L_2]$ | $\mathbb{E}[D^{(1)}]$ | $\mathbb{E}[Y]$ |
|---|---|---|---|---|---|
| $\lambda = 2$ | $p = 0.075$ | 1.9101 | 139.0710 | 1.1564 | 97.2087 |
| | $p = 0.1$ | 1.7866 | 9.4201 | 1.1972 | 7.2252 |
| | $p = 0.2$ | 1.4337 | 0.9843 | 1.3328 | 1.0477 |
| | $p = 0.3$ | 1.2065 | 0.2618 | 1.4396 | 0.4324 |
| | $p = 0.5$ | 0.9153 | 0.0232 | 1.6070 | 0.1823 |
| $\lambda = 4$ | $p = 0.325$ | 2.1327 | 109.7110 | 2.1483 | 95.2510 |
| | $p = 0.35$ | 2.0299 | 5.8917 | 2.1813 | 5.7313 |
| | $p = 0.4$ | 1.8504 | 1.3843 | 2.2460 | 1.7506 |
| | $p = 0.5$ | 1.5636 | 0.2485 | 2.3654 | 0.6504 |
| | $p = 0.7$ | 1.1470 | 0.0091 | 2.5829 | 0.2805 |
| $\lambda = 6$ | $p = 0.435$ | 2.2419 | 232.4990 | 3.1346 | 213.1300 |
| | $p = 0.45$ | 2.1828 | 8.4841 | 3.1524 | 8.5183 |
| | $p = 0.5$ | 2.0022 | 1.3188 | 3.2139 | 1.8840 |
| | $p = 0.7$ | 1.4356 | 0.0300 | 3.4607 | 0.4498 |
| | $p = 0.8$ | 1.2059 | 0.0027 | 3.5973 | 0.3062 |

**Table 2** Numerical results for $M = 5$

| $\mu = 2, \gamma = 1$ | | $\mathbb{E}[L_1]$ | $\mathbb{E}[L_2]$ | $\mathbb{E}[D^{(1)}]$ | $\mathbb{E}[Y]$ |
|---|---|---|---|---|---|
| $\lambda = 2$ | $p = 0.015$ | 3.5181 | 958.507 | 1.0543 | 648.489 |
| | $p = 0.02$ | 3.3867 | 37.6015 | 1.0696 | 26.6751 |
| | $p = 0.1$ | 2.1970 | 0.6046 | 1.2366 | 0.8115 |
| | $p = 0.2$ | 1.5919 | 0.0548 | 1.3625 | 0.3111 |
| | $p = 0.5$ | 0.9246 | $3.6 \times 10^{-5}$ | 1.6092 | 0.1675 |
| $\lambda = 4$ | $p = 0.176$ | 3.8373 | 570.3940 | 2.0353 | 485.6580 |
| | $p = 0.18$ | 3.7748 | 32.0374 | 2.0391 | 28.4827 |
| | $p = 0.2$ | 3.4920 | 4.1915 | 2.0611 | 4.6828 |
| | $p = 0.3$ | 2.5421 | 0.1724 | 2.1785 | 0.9057 |
| | $p = 0.6$ | 1.3540 | $4.1 \times 10^{-5}$ | 2.4784 | 0.3599 |
| $\lambda = 6$ | $p = 0.255$ | 3.9741 | 789.6740 | 3.0248 | 714.9410 |
| | $p = 0.26$ | 3.9115 | 21.8246 | 3.0270 | 21.2260 |
| | $p = 0.3$ | 3.4686 | 1.4058 | 3.0563 | 2.5665 |
| | $p = 0.4$ | 2.6661 | 0.0839 | 3.1473 | 1.0433 |
| | $p = 0.6$ | 1.7388 | $2.5 \times 10^{-4}$ | 3.3454 | 0.5706 |

**Table 3** Numerical results for $M = 8$

| $\mu = 2, \gamma = 1$ | | $\mathbb{E}[L_1]$ | $\mathbb{E}[L_2]$ | $\mathbb{E}[D^{(1)}]$ | $\mathbb{E}[Y]$ |
|---|---|---|---|---|---|
| $\lambda = 2$ | $p = 0.0023$ | 6.2042 | 1,828.0822 | 1.0144 | 1, 224.7346 |
| | $p = 0.003$ | 6.1119 | 178.0613 | 1.0131 | 121.3286 |
| | $p = 0.01$ | 5.3390 | 12.9351 | 1.0528 | 10.1176 |
| | $p = 0.1$ | 2.3508 | 0.0332 | 1.2537 | 0.3528 |
| | $p = 0.3$ | 1.2753 | $2.7 \times 10^{-6}$ | 1.4558 | 0.2317 |
| $\lambda = 4$ | $p = 0.102$ | 6.6039 | 1,373.1455 | 2.0047 | 1, 159.5788 |
| | $p = 0.105$ | 6.4496 | 30.1096 | 2.0042 | 27.5141 |
| | $p = 0.11$ | 6.2082 | 9.8736 | 2.0053 | 10.3268 |
| | $p = 0.2$ | 3.7204 | 0.0447 | 2.0929 | 1.1522 |
| | $p = 0.5$ | 1.6183 | $3.6 \times 10^{-9}$ | 2.3786 | 0.4592 |
| $\lambda = 6$ | $p = 0.154$ | 6.7515 | 1,019.2930 | 3.0021 | 918.0628 |
| | $p = 0.155$ | 6.7142 | 78.7758 | 3.0014 | 73.4208 |
| | $p = 0.2$ | 5.3532 | 0.5313 | 3.0133 | 2.5281 |
| | $p = 0.3$ | 3.6098 | 0.0052 | 3.0748 | 1.3657 |
| | $p = 0.5$ | 2.1278 | $1.2 \times 10^{-7}$ | 3.2419 | 0.7447 |

## 7 Conclusions

In this paper, we studied the Israeli queue with at most $M$ groups in the main queue, where blocked customers form a separate retrial (orbit) queue with constant retrial rate. The two-dimensional continuous-time Markov process describing the system is analyzed via both probability generating functions and matrix geometric methods.

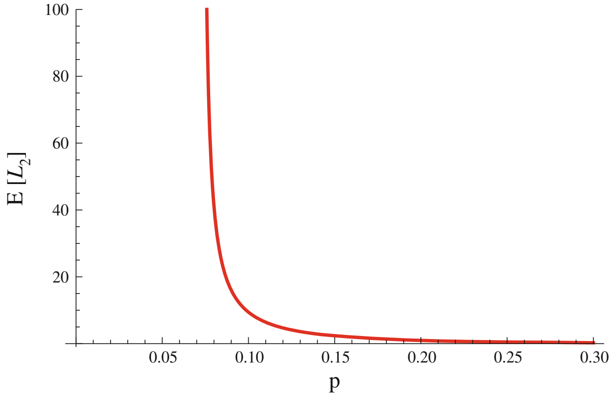


**Fig. 2** $\mathbb{E}[L_2]$ as a function of $p$, with $M = 3$, $\lambda = \mu = 2$ and $\gamma = 1$

**Table 4** Explicit values of $\mathbb{E}[L_2]$ as a function of $p$, with $M = 3$, $\lambda = \mu = 2$ and $\gamma = 1$

| $p$ | 0.0727895 | 0.07279 | 0.0728 | 0.073 | 0.075 | 0.08 | 0.1 | 0.3 | 0.5 | 0.7 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{E}[L_2]$ | 899,741.62 | 368,805.54 | 28,806.47 | 1,479.80 | 139.071 | 41.10 | 9.42 | 0.2618 | 0.0232 | 0.0009 |

Various performance measures are analytically calculated and numerical results are presented. The stability condition, depending on the value $p$ (being the probability that a new arrival knows a group leader) and on the system's other parameters, is established. Numerical examples exhibit how the mean number of orbiting customers, the mean number of groups, and the mean number of bypasses drop with increasing values of $p$, while the mean size of a served group increases with $p$.

## 8 Appendix

**Proposition 8.1** *For a given $1 \leq M < \infty$, the polynomials $q_m(z)$ are of the following form:*

$$q_m(z) = z^{m-1} \prod_{k=0}^{m-1} (\lambda(1-p)^k + \gamma) + (1-z)h_m^{(M)}(z), \quad m = 0, 1, \ldots, M,$$

$$q_{M+1}(z) = (1-z)h_{M+1}^{(M)}(z),$$

*where*

$$h_0^{(M)}(z) = h_1^{(M)}(z) = 0,$$

$$h_m^{(M)}(z) = h_{m-1}^{(M)}(z)(\lambda(1-p)^{m-1} + \mu + \gamma)z - h_{m-1}^{(M)}(z)\mu z(\lambda(1-p)^{m-2}z + \gamma) - \mu\gamma z^{m-2} \prod_{k=0}^{m-3} (\lambda(1-p)^k + \gamma), \quad m = 2, 3, \ldots, M,$$

*and*

$$h_{M+1}^{(M)}(z) = h_M^{(M)}(z)(\lambda(1-p)^M(1-z)+\mu)z - h_{M-1}^{(M)}(z)\mu z(\lambda(1-p)^{M-1}z+\gamma)$$
$$+ z^M\lambda(1-p)^M \prod_{k=0}^{M-1}(\lambda(1-p)^k+\gamma) - \mu\gamma z^{M-1}\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma).$$

*Proof* The proof is conducted by induction over $m$. First, $q_0(z) = 1$ and $q_1(z) = \lambda+\gamma$. Therefore, $h_0^{(M)}(z) = h_1^{(M)}(z) = 0$. Next,

$$\begin{aligned}
q_2(z) &= \alpha_1(z)q_1(z) - \mu z(\lambda z+\gamma)q_0(z)\\
&= (\lambda(1-p)+\gamma)z(\lambda+\gamma) - \mu(\lambda z+\gamma)\\
&= z\prod_{k=0}^{1}(\lambda(1-p)^k+\gamma) - \mu\gamma(1-z),
\end{aligned}$$

so that $h_2^{(M)}(z) = -\mu\gamma$. Assume now that the proposition holds for any $m = 2, 3, \ldots, M-1$. For $m+1$, we have

$$\begin{aligned}
q_{m+1}(z) &= \alpha_m(z)q_m(z) - \mu z(\lambda(1-p)^{m-1}z+\gamma)q_{m-1}(z)\\
&= (\lambda(1-p)^m+\mu+\gamma)z\left(z^{m-1}\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma) + (1-z)h_m^{(M)}(z)\right)\\
&\quad - \mu z(\lambda(1-p)^{m-1}z+\gamma)\left(z^{m-2}\prod_{k=0}^{m-2}(\lambda(1-p)^k+\gamma)+(1-z)h_{m-1}^{(M)}(z)\right)\\
&= z^m\prod_{k=0}^{m}(\lambda(1-p)^k+\gamma) + \mu z^m\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma)\\
&\quad + h_m^{(M)}(z)(\lambda(1-p)^m+\mu+\gamma)z(1-z)\\
&\quad - \mu z^m\lambda(1-p)^{m-1}\prod_{k=0}^{m-2}(\lambda(1-p)^k+\gamma) - \mu z^{m-1}\gamma\prod_{k=0}^{m-2}(\lambda(1-p)^k+\gamma)\\
&\quad - h_{m-1}^{(M)}(z)\mu z(\lambda(1-p)^{m-1}z+\gamma)(1-z).
\end{aligned} \tag{8.1}$$

Note that

$$\mu z^m\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma) - \mu z^m\lambda(1-p)^{m-1}\prod_{k=0}^{m-2}(\lambda(1-p)^k+\gamma)$$
$$-\mu z^{m-1}\gamma\prod_{k=0}^{m-2}(\lambda(1-p)^k+\gamma)$$

$$= \mu z^m \lambda(1-p)^{m-1} \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma) + \mu z^m \gamma \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma)$$
$$- \mu z^m \lambda(1-p)^{m-1} \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma) - \mu z^{m-1} \gamma \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma)$$
$$= -\mu\gamma z^{m-1} \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma)(1-z). \tag{8.2}$$

Substituting (8.2) in (8.1) results in

$$q_{m+1}(z) = z^m \prod_{k=0}^{m} (\lambda(1-p)^k + \gamma) + (1-z)\Big(h_m^{(M)}(z)(\lambda(1-p)^m + \mu + \gamma)z$$
$$- h_{m-1}^{(M)}(z)\mu z(\lambda(1-p)^{m-1}z + \gamma) - \mu\gamma z^{m-1} \prod_{k=0}^{m-2} (\lambda(1-p)^k + \gamma)\Big). \tag{8.3}$$

Now, for $m = M+1$, we get

$$\begin{aligned} q_{M+1}(z) &= \alpha_M(z) q_M(z) - \mu z(\lambda(1-p)^{M-1}z + \gamma) q_{M-1}(z) \\ &= (\lambda(1-p)^M(1-z) + \mu)z\left(z^{M-1} \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) + (1-z)h_M^{(M)}(z)\right) \\ &\quad - \mu z(\lambda(1-p)^{M-1}z + \gamma)\left(z^{M-2} \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma) + (1-z)h_{M-1}^{(M)}(z)\right) \\ &= z^M \lambda(1-p)^M(1-z) \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) + \mu z^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) \\ &\quad + h_M^{(M)}(z)(\lambda(1-p)^M(1-z) + \mu)z(1-z) \\ &\quad - z^M \lambda(1-p)^{M-1} \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma) \\ &\quad - \mu z^{M-1}\gamma \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma) - h_{M-1}^{(M)}(z)\mu z(\lambda(1-p)^{M-1}z + \gamma)(1-z), \end{aligned} \tag{8.4}$$

which after some algebra leads to

$$q_{M+1}(z) = (1-z)\Big(h_M^{(M)}(z)(\lambda(1-p)^M(1-z) + \mu)z$$
$$- h_{M-1}^{(M)}(z)\mu z(\lambda(1-p)^{M-1}z + \gamma)$$

$$+ z^M \lambda(1-p)^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma)$$
$$\left. - \mu\gamma z^{M-1} \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma)\right). \quad (8.5)$$

This completes the proof.

**Proposition 8.2** *For all* $1 \le M < \infty$,

$$h_{M+1}^{(M)}(1) = \lambda(1-p)^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) - \gamma\mu^M \sum_{m=0}^{M-1} \frac{1}{\mu^m} \prod_{k=0}^{m-1} (\lambda(1-p)^k + \gamma). \quad (8.6)$$

*Proof* By induction over $M$. First, assume $M = 1$. Recall from Proposition 8.1 that for all $1 \le M < \infty$, $h_0^{(M)}(z) = h_1^{(M)}(z) = 0$. Then, from Proposition 8.1, we get

$$h_2^{(1)}(1) = \lambda(1-p)(\lambda + \gamma) - \mu\gamma,$$

which coincides with Eq. (8.6) when $M = 1$.

Assume now that the proposition holds for $M - 1$. Using Proposition 8.1 for $M$, we get

$$h_{M+1}^{(M)}(1) = h_M^{(M)}(1)\mu - h_{M-1}^{(M)}(1)\mu(\lambda(1-p)^{M-1} + \gamma)$$
$$+ \lambda(1-p)^M \prod_{k=0}^{M-1} (\lambda(1-p)^k + \gamma) - \mu\gamma \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma). \quad (8.7)$$

In addition, we use the following two relations

(*a*) $h_{M-1}^{(M)}(z) = h_{M-1}^{(M-1)}(z)$,

(*b*) $h_M^{(M-1)}(1) = h_M^{(M)}(1) - h_{M-1}^{(M-1)}(1)(\lambda(1-p)^{M-1} + \gamma) + \lambda(1-p)^{M-1} \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma)$,

□

so that Eq. (8.7) translates to

$$h_{M+1}^{(M)}(1) = \mu\Bigg(h_M^{(M-1)}(1) + h_{M-1}^{(M)}(1)(\lambda(1-p)^{M-1} + \gamma)$$
$$- \lambda(1-p)^{M-1} \prod_{k=0}^{M-2} (\lambda(1-p)^k + \gamma)\Bigg)$$

$$-h_{M-1}^{(M)}(1)\mu(\lambda(1-p)^{M-1}+\gamma)+\lambda(1-p)^M\prod_{k=0}^{M-1}(\lambda(1-p)^k+\gamma)$$
$$-\mu\gamma\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma). \tag{8.8}$$

Using the validity for $M-1$ yields

$$\begin{aligned}
h_{M+1}^{(M)}(1) &= \mu\Bigg(\lambda(1-p)^{M-1}\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma)-\gamma\mu^{M-1}\sum_{m=0}^{M-2}\frac{1}{\mu^m}\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma)\\
&\quad-\lambda(1-p)^{M-1}\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma)\Bigg)+\lambda(1-p)^M\prod_{k=0}^{M-1}(\lambda(1-p)^k+\gamma)\\
&\quad-\mu\gamma\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma)\\
&=\lambda(1-p)^M\prod_{k=0}^{M-1}(\lambda(1-p)^k+\gamma)-\gamma\mu^M\sum_{m=0}^{M-2}\frac{1}{\mu^m}\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma)\\
&\quad-\mu\gamma\prod_{k=0}^{M-2}(\lambda(1-p)^k+\gamma)\\
&=\lambda(1-p)^M\prod_{k=0}^{M-1}(\lambda(1-p)^k+\gamma)-\gamma\mu^M\sum_{m=0}^{M-1}\frac{1}{\mu^m}\prod_{k=0}^{m-1}(\lambda(1-p)^k+\gamma). \qquad (8.9)
\end{aligned}$$

This completes the proof.

## References

1. Ammar, M.H., Wong, J.W.: On the optimality o cyclic transmission in teletext systems. IEEE Trans. Commun. COM **35**(1), 68–73 (1987)
2. Artalejo, J.R.: Accessible bibliography on retrial queues. Math. Comput. Model. **30**, 223–233 (1999)
3. Artalejo, J.R., Gómez-Corral, A.: On a single server queue with negative arrivals and request repeated. J. Appl. Probab. **36**, 907–918 (1999)
4. Artalejo, J.R., Gómez-Corral, A., Neuts, M.F.: Analysis of multiserver queues with constant retrial rate. Eur. J. Oper. Res. **135**(3), 569–581 (2001)
5. Artalejo, J.R., Gómez-Corral, A.: Retrial Queueing Systems: A Computational Approach. Springer, Berlin (2008)
6. Avrachenkov, K., Nain, P., Yechiali, U.: A retrial system with two input streams and two orbit queues. Queueing Syst. (to appear) (2013)
7. Avrachenkov, K., Yechiali, U.: Retrial networks and their application to the internet data traffic. Probab. Eng. Inf. Sci. **22**(4), 519–536 (2008)
8. Avrachenkov, K., Yechiali, U.: On tandem blocking queues with a common retrial queue. Comput. Oper. Res. **37**(7), 1174–1180 (2009)
9. Boxma, O.J., van der Wal, Y., Yechiali, U.: Polling with gated batch service. In: Proceedings of the Sixth International Conference on "Analysis of Manufacturing Systems", Lunteren, The Netherlands, pp. 155–159 (2007)
10. Boxma, O.J., van der Wal, Y., Yechiali, U.: Polling with batch service. Stoch. Model. **24**(4), 604–625 (2008)

11. Dykeman, H.D., Ammar, M.H., Wong, J.W.: Scheduling algorithms for videotex systems under broadcast delivery. In: Proceedings of the International Conference on Communications (ICC'86), pp. 1847–1851 (1986)
12. Falin, G.I.: A survey of retrial queues. Queueing Syst. **7**, 127–167 (1990)
13. Falin, G.I., Templeton, J.G.C.: Retrial Queues. Chapman and Hall, London (1997)
14. Falin, G.I.: A single-server batch arrival queue with returning customers. Eur. J. Oper. Res. **201**, 786–790 (2010)
15. He, Q.-M., Chavoushi, A.A.: Analysis of queueing systems with customer interjections. Queueing Syst. **73**, 79–104 (2013)
16. Latouche, G., Ramaswami, V.: Introduction to matrix analytic methods in stochastic modeling. In: ASA-SIAM Series on Statistics and Applied Probability, SIAM, Philadelphia (1999)
17. Liu, Z., Nain, P.: Optimal scheduling in some multiqueue single-server systems. IEEE Trans. Autom. Control **37**(2), 247–252 (1992)
18. Neuts, M.F.: Matrix-Geometric Solutions in Stochastic Models. An Algorithmic Approach. The Johns Hopkins University Press, Baltimore (1981)
19. Perel, N., Yechiali, U.: The Israeli queue with infinite number of groups. Probab. Eng. Inf. Sci. (to appear) (2013)
20. Perel, N., Yechiali, U.: The Israeli queue with priorities. Stoch. Model. **29**(3), 353–379 (2013)
21. van Oyen, M.P., Teneketzis, D.: Optimal batch service of a polling system under partial informtion. Methods Model. OR **44**(3), 401–419 (1996)
22. van der Wal, J., Yechiali, U.: Dynamic visit-order rules for batch-service polling. Probab. Eng. Inf. Sci. **17**(3), 351–367 (2003)
23. Yang, T., Templeton, J.G.C.: A survey on retrial queues. Queueing Syst. **2**, 201–233 (1987)