Finite Two-Queue Systems where Customers of Each Queue Are the Severs of the Other Queue

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Abstract

We consider systems comprised of two interlacing finite queues where customers of each queue are the servers of the other queue. Examples of such systems can be found in file sharing programs, SETI@Home project, etc (see e.g. [1]). Denoting by L_i the number of customers in queue i, $i = 1, 2$, we study three models, distinguished by the way in which service is rendered to each queue. In Model 1, queue 1 (Q_1) operates as a finite-buffer **multi-server** Markovian queue with Poisson arrival rate λ_1 and exponential service time, for each costumer, with mean $1/\mu_1$, such that, at any moment, the L_2 customers currently present at queue 2 are the potential servers of the costumers of Q_1. The overall capacity of Q_1 is N customers, dictating that the actual number of active servers at Q_1 is $Min(N, L_2)$. We denote such a queue as $M(\lambda_1)/M(\mu_1)/Min(N,L_2)/N$. Queue 2 ($Q_2$) operates as a finite-buffer **single-server** Markovian queue with Poisson arrival rate λ_2 and exponential service time, as follows: The L_1 customers in Q_1 join hands together to form a single server with combined rate $\mu_2 L_1$. The overall capacity of Q_2 is K. We denote such a queue as $M(\lambda_2)/M(\mu_2 L_1)/1/K$. In Model 2, Q_1 operates as in Model 1, but Q_2 operates as a finite-buffer **multi-server** $M(\lambda_2)/M(\mu_2)/Min(K,L_1)/K$ system. In Model 3, Q_1 is a finite-buffer **single-server** $M(\lambda_1)/M(\mu_1 L_2)/1/N$ queue, while Q_2 is also a finite-buffer **single-server** $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue. The resulting systems are formulated as finite non-homogeneous quasi birth-and-death (QBD) processes. For each model we derive its steady state probability distribution function, and its related first

moment. We achieve this by making use of the special three diagonal structure of the QBD generator matrix.

We further show that in Model 1, the carried load by Q_1 is larger than its counterpart in Q_2, independent of the queues capacities, while in Model 2 the carried loads are equal. In Model 3 we show that, for both queues, the effective arrival rate is smaller than the realized service rate. Numerical examples are presented.

1 Introduction

Consider a system comprised of two finite connected and dependent queues, where customers of each queue render service to the customers of the other queue. We study 3 models as follows:

In Model 1 (Section 2) we assume that one queue, Q_1, operates as a multi-server, finite-buffer, $M(\lambda_1)/M(\mu_1)/Min(N,L_2)/N$ system with Poisson arrival rate λ_1, and exponential service time with mean $1/\mu_1$ for each individual customer. The potential servers at Q_1 are the L_2 customers present in Q_2. That is, each customer present in Q_2 *individually* acts as a server for the customers in Q_1, such that, at any given moment, the actual number of active servers is $Min(N,L_2)$. Q_1 has a limited overall capacity of size N. The other queue, Q_2, operates as a single-server finite-buffer $M(\lambda_2)/M(\mu_2 L_1)/1/K$ system with Poisson arrival rate λ_2, but with dynamically changing service rate, $\mu_2 L_1$. That is, the L_1 customers present in Q_1 join hands together and form a *single* server, having a combined service rate of $\mu_2 L_1$ for the customers in Q_2. In other words, the service rate at Q_2 changes according to the queue-size fluctuations of Q_1.

In Model 2 (Section 3) we assume that Q_1 operates as in Model 1, namely as an $M(\lambda_1)/M(\mu_1)/Min(N,L_2)/N$ system, but Q_2 operates as a finite-buffer, multi-server (rather than a single-server) $M(\lambda_2)/M(\mu_2)/Min(K,L_1)/K$ system. That is, each customer present in Q_1 *individually* acts as a potential server for the customers in Q_2.

In Model 3 (Section 4) we assume that Q_1 operates as a finite-buffer, single-server (rather than a multi-server) $M(\lambda_1)/M(\mu_1 L_2)/1/N$ system, served by the customers of the other queue, while Q_2 operates as in Model 1, namely as an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue, drawing its servers from its opposite queue.

We formulate each of the 3 models described above as a finite non-homogeneous quasi birth-and-death (QBD) process and study its steady-state behavior. Using De Nitto Personè & Grassi [2] method and utilizing the special structure of the generator matrix of the QBD processes we calculate each system's steady-state probabilities. We further calculate numerically the mean total number of customers in each queue, $E[L_1]$ and $E[L_2]$, as well as the probability of blocking at Q_i, $i=1, 2$. We show that in Model 1, the carried load of Q_1, namely $\left(1-P(Q_1 \text{ is blocked})\right)\lambda_1/\mu_1$, is always larger than the carried load of Q_2, being $\left(1-P(Q_2 \text{ is blocked})\right)\lambda_2/\mu_2$, while in Model 2, the carried loads of the queues are equal. In Model 3, for both queues the effective arrival rate, $\lambda_i\left(1-P(Q_i \text{ is blocked})\right)$, is smaller than the realized service rate, $\mu_i E[L_j]$, i = 1, 2, $j \neq i$ (as is the case for finite-buffer single-server Markovian queue). Finally, numerical examples are presented and the models are compared.

A first step in the analysis of queues where customers act as servers has been recently presented in Perel & Yechiali [7], where one queue, Q_1, operates as an $M(\lambda_1)/M(\mu_1)/1/N$ system and only the customers of queue 1 act as servers for the unbounded capacity Q_2. This scenario has been extended in Perel & Yechiali [8] to the case where costumers of both queues act as servers, each group serving the other queue. The present work further extends the scope of the analysis to the case where capacities of both queues are finite.

QBD processes have been used extensively to model a variety of systems, mostly representing occasions of unbounded populations where the service and inter-arrival times are given by phase-type distributions (see e.g. Neuts [6], and Latouche & Ramaswami [5]). Hajek [4] studies finite homogeneous QBD processes, while in [2] finite non homogeneous QBD processes are considered. The QBD processes representing our models are different from those treated in [2]. Nevertheless, we use some of the ideas presented in [2] to study our 'customers as servers' models.

2 Model 1

As described in the Introduction, Q_1 is a multi-server finite-buffer $M(\lambda_1)/M(\mu_1)/Min(N,L_2)/N$ system, while Q_2 is a single-server finite-buffer $M(\lambda_2)/M(\mu_2 L_1)/1/K$ queue. The arrival processes, as well as the service processes in both queues, are mutually independent. For this model we distinguish between two cases: (i) $N<K$ and (ii) $N \geq K$.

2.1 Balance Equations

Let L_i denote the total number of customers in Q_i, $i=1,\ 2$. Then, the pair (L_1, L_2) defines a non reducible continuous-time finite Markov process. For case (i), where $N<K$, the transition-rate diagram is depicted in Figure 2.1. Let $P_{nm} = P\left(L_1 = n, L_2 = m\right)$, $0 \leq n \leq N$ and $0 \leq m \leq K$ denote the system's stationary probabilities. Then, the set of balance equations is given as follows:

$\underline{n=0}$:

$$\begin{cases} m=1: \quad (\lambda_1+\lambda_2)P_{01} = \mu_1 P_{11} \\ 2 \leq m \leq K-1: \quad (\lambda_1+\lambda_2)P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m} \\ m=K: \quad \lambda_1 P_{0K} = \lambda_2 P_{0,K-1} + \mu_1 P_{1K} \end{cases} \tag{2.1}$$

$\underline{n=1}$:

$$\begin{cases} m=0: \quad (\lambda_1+\lambda_2)P_{10} = \mu_2 P_{11} \\ m=1: \quad (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{11} = \lambda_1 P_{0,1} + \lambda_2 P_{10} + \mu_1 P_{21} + \mu_2 P_{12} \\ 2 \leq m \leq K-1: \quad (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + 2\mu_1 P_{2m} + \mu_2 P_{1,m+1} \\ m=K: \quad (\lambda_1+\mu_1+\mu_2)P_{1K} = \lambda_1 P_{0K} + \lambda_2 P_{1,K-1} + 2\mu_1 P_{2K} \end{cases} \tag{2.2}$$

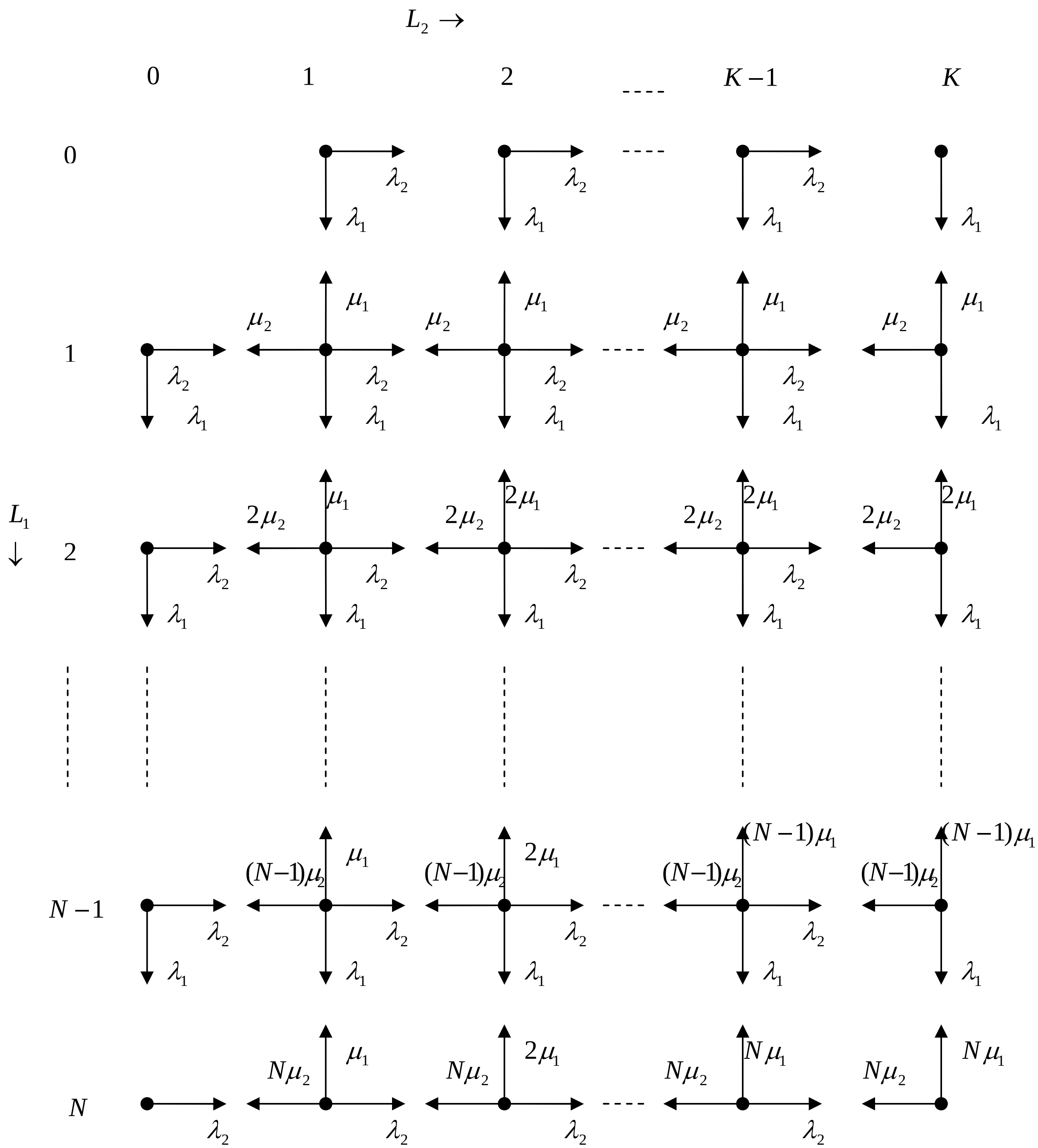

Figure 2.1: Transition-rate diagram of (L_1, L_2) for Model 1, where $N < K$.

$\underline{2 \le n \le N-1}$:

$$\begin{cases} m=0: \ (\lambda_1+\lambda_2)P_{n0} = \lambda_1 P_{n-1,0} + n\mu_2 P_{n1} \\ 1 \le m \le n: \ (\lambda_1+\lambda_2+m\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \\ n+1 \le m \le K-1: (\lambda_1+\lambda_2+n\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \\ m=K: (\lambda_1+n\mu_1+n\mu_2)P_{nK} = \lambda_1 P_{n-1,K} + \lambda_2 P_{n,K-1} + (n+1)\mu_1 P_{n+1,K} \end{cases} \tag{2.3}$$

$\underline{n=N}$:

$$\begin{cases} m=0: \ \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{N1} \\ 1 \le m \le N: \ (\lambda_2+m\mu_1+N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \\ N+1 \le m \le K-1: (\lambda_2+N\mu_1+N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \\ m=K: (N\mu_1+N\mu_2)P_{NK} = \lambda_1 P_{N-1,K} + \lambda_2 P_{N,K-1} \end{cases} \tag{2.4}$$

Define (where $P_{00}=0$) the marginal probabilities:

$$P(L_1=n) \equiv P_{n\bullet} = \sum_{m=0}^{K} P_{nm} \quad \text{for} \ \ 0 \le n \le N$$

$$P(L_2=m) \equiv P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \quad \text{for} \ \ 0 \le m \le K$$

Then, for every $0 \le m \le K-1$, summing equations (2.1) - (2.4) over n yields

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} E[L_1 \mid L_2 = m+1] \tag{2.5}$$

By summing (2.5) over m we get

$$\lambda_2 \sum_{m=0}^{K-1} P_{\bullet m} = \mu_2 \sum_{m=0}^{K-1} P_{\bullet m+1} E[L_1 \mid L_2 = m+1] \tag{2.6}$$

Therefore,

$$\lambda_2\left(1-P_{\bullet K}\right) = \mu_2\left(E[L_1] - P_{\bullet 0} E[L_1 \mid L_2 = 0]\right) = \mu_2\left(E[L_1] - \sum_{n=1}^{N} nP_{n0}\right).$$

That is,

$$E[L_1] = \left(1-P_{\bullet K}\right)\lambda_2/\mu_2 + \sum_{n=1}^{N} nP_{n0} \tag{2.7}$$

That is, the mean queue size at Q_1 is the sum of the carried load at Q_2 plus the expected number of customers in Q_1 when Q_2 is empty.

Furthermore, by summing equations (2.1) - (2.4) over m we get, for every $0 \le n \le N-1$,

$$\lambda_1 P_{n\bullet} = (n+1)\mu_1 P_{n+1,\bullet} - \mu_1 \sum_{m=0}^{n} (n+1-m) P_{n+1,m} \tag{2.8}$$

Summing equation (2.8) over *n* yields

$$\sum_{n=0}^{N-1} \lambda_1 P_{n\bullet} = \mu_1 \sum_{n=0}^{N-1} (n+1) P_{n+1,\bullet} - \mu_1 \sum_{n=0}^{N-1} \sum_{m=0}^{n} (n+1-m) P_{n+1,m}$$

Hence,

$$\lambda_1 (1 - P_{N\bullet}) = \mu_1 E[L_1] - \mu_1 \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \, ,$$

or,

$$E[L_1] = \left(1 - P_{N\bullet}\right) \lambda_1 / \mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} \tag{2.9}$$

Equating equation (2.7) and (2.9) results in

$$E[L_1] = \left(1 - P_{N\bullet}\right) \lambda_1 / \mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m} = \left(1 - P_{\bullet K}\right) \lambda_2 / \mu_2 + \sum_{n=1}^{N} n P_{n0}$$

or,

$$\left(1 - P_{N\bullet}\right) \lambda_1 / \mu_1 + \sum_{n=2}^{N} \sum_{m=1}^{n-1} (n-m) P_{n,m} = \left(1 - P_{\bullet K}\right) \lambda_2 / \mu_2 \, . \tag{2.10}$$

Equation (2.10) implies that the carrying load of Q_1, $\left(1 - P_{N\bullet}\right) \lambda_1 / \mu_1$, is smaller than the carrying load of Q_2, being $\left(1 - P_{\bullet K}\right) \lambda_2 / \mu_2$.

This follows since Q_2 is more effective, as all its potential servers are always occupied (as a combined single server), while in Q_1 not all its potential servers can always be utilized since Q_1 is operated as a multi-server queue.

2.2 Deriving $\left(P_{nm}\right)_{0 \le n \le N \, , \; 0 \le m \le K}$

Our model can be described as a queueing system with $N+1$ 'phases', where phase n indicates that the service rate in Q_2 is $n\mu_2$. State (m,n) denotes that there are m jobs in Q_2, $0 \le m \le K$, and the system is in phase n, $0 \le n \le N$. We construct a finite non-homogeneous QBD process with generator Q, given by

$$Q=\begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \dots & \dots & \dots & \dots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \dots & \dots & \dots & \vdots \\ \mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \dots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & A_2 & A_1^N & A_0 & \mathbf{0} & \vdots \\ \vdots & \vdots & \vdots & \ddots & A_2 & A_1^N & A_0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & A_0 \\ \mathbf{0} & \dots & \dots & \dots & \dots & \mathbf{0} & A_2 & A_1 \end{pmatrix},$$

where $\mathbf{0}$ is a matrix of zeros, and starting from the upper diagonal, A_0^0, A_0; $A_1^0, A_1^1, \dots, A_1^N, A_1$; A_2^1, A_2 are the following matrices: A_0^0 is of size $N\times(N+1)$; A_0 is of size $(N+1)\times(N+1)$; A_1^0 is of size $N\times N$; $A_1^1,\dots,A_1^N$ and A_1 are each of size $(N+1)\times(N+1)$; A_2^1 is of size $(N+1)\times N$; and A_2 is of size $(N+1)\times(N+1)$. They are given by

$$A_0^0=\begin{pmatrix} 0 & \lambda_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \lambda_2 \end{pmatrix}$$

$$A_0 = diag\left(\lambda_2\right)$$

$$A_1^0=\begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

For all $1\le m\le N$,

$$\left(A_1^m\right)_{ij}=\begin{cases}-(\lambda_1+\lambda_2) & j=i=0\\ -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) & j=i,\ i=1,\dots,m\\ -(\lambda_1+\lambda_2+m\mu_1+i\mu_2) & j=i,\ i=m+1,\dots,N-1\\ -(\lambda_2+m\mu_1+N\mu_2) & j=i=N\\ \lambda_1 & j=i+1,\ i=0,1,\dots,N-1\\ i\mu_1 & j=i-1,\ i=1,\dots,m\\ m\mu_1 & j=i-1,\ i=m+1,\dots,N\\ 0 & otherwise\end{cases}$$

$$A_1=\begin{pmatrix}-\lambda_1 & \lambda_1 & 0 & \dots & \dots & 0\\ \mu_1 & -(\lambda_1+\mu_1+\mu_2) & \lambda_1 & 0 & \dots & \vdots\\ 0 & 2\mu_1 & -(\lambda_1+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots\\ \vdots & 0 & 3\mu_1 & -(\lambda_1+3\mu_1+3\mu_2) & \lambda_1 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1\\ 0 & \cdots & 0 & 0 & N\mu_1 & -(N\mu_1+N\mu_2)\end{pmatrix}$$

$$A_2^1=\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ \mu_2 & 0 & \dots & \dots & \vdots\\ 0 & 2\mu_2 & 0 & \dots & \vdots\\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & \dots & \dots & 0 & N\mu_2\end{pmatrix}\qquad A_2=\begin{pmatrix}0 & \dots & \dots & \dots & \dots & 0\\ \vdots & \mu_2 & 0 & \dots & \dots & 0\\ \vdots & 0 & 2\mu_2 & \ddots & \dots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & \dots & \dots & \dots & N\mu_2\end{pmatrix}$$

Define the steady-state probability vectors $\vec{P}_0=\left(P_{10},\dots,P_{N0}\right)$ and $\vec{P}_m=\left(P_{0m},P_{1m},\dots,P_{Nm}\right)$ for all $1\le m\le K$. Then, the steady state probability vectors satisfy:

$$\begin{aligned}&\vec{P}_0A_1^0+\vec{P}_1A_2^1=\vec{0}\\ &\vec{P}_0A_0^0+\vec{P}_1A_1^1+\vec{P}_2A_2=\vec{0}\\ &\vec{P}_1A_0+\vec{P}_2A_1^2+\vec{P}_3A_2=\vec{0}\\ &\vdots\\ &\vec{P}_{N-1}A_0+\vec{P}_NA_1^N+\vec{P}_{N+1}A_2=\vec{0}\\ &\vdots\\ &\vec{P}_{K-2}A_0+\vec{P}_{K-1}A_1^N+\vec{P}_KA_2=\vec{0}\\ &\vec{P}_{K-1}A_0+\vec{P}_KA_1=\vec{0}\end{aligned}\tag{2.11}$$

Clearly, one can solve directly (numerically) the set (2.11) (including the normalization equation, $\sum_{m=0}^{K}\vec{P}_{\bullet m}\cdot\vec{e}=1$). This requires computational effort. We wish to present an alternative algorithmic-type method to ease the required computational effort (for computational issues see e.g., [2], Elhafsi & Molle [3], and [5]). For this aim we borrow from the ideas presented in [2] and modify them to our purposes. We claim:

Theorem 2.1. If A_0 is a non-singular matrix, then the following equations hold:

$$\vec{P}_{N-i}=-\vec{P}_N A_2 A_0^{-1} C_{11}(i-2)+\vec{P}_{N-1}\left(C_{21}(i-2)-A_1^{N-1}A_0^{-1}C_{11}(i-2)\right)\quad,\quad 2\le i\le N-1 \tag{2.12}$$

$$\vec{P}_{K-j}=-\vec{P}_K A_2 A_0^{-1} D_{11}{}^{(j-2)}+\vec{P}_{K-1}\left(D_{21}{}^{(j-2)}-A_1^{N}A_0^{-1}D_{11}{}^{(j-2)}\right)\quad,\quad 2\le j\le K-N+1 \tag{2.13}$$

where $C_{11}(i-2)$ and $C_{21}(i-2)$ are the $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ product matrix $C(i-2)$ defined as

$$C(i)=\begin{cases} I_{2(N+1)} & ,\ i=0 \\ C(i-1)\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2A_0^{-1} & \mathbf{0}\end{pmatrix} & ,\ i>0\end{cases}. \tag{2.14}$$

$D_{11}{}^{(j-2)}$ and $D_{21}{}^{(j-2)}$ are $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ power matrix $D^{(j-2)}$ defined as

$$D=\begin{pmatrix} -A_1^{N}A_0^{-1} & I_{N+1} \\ -A_2A_0^{-1} & \mathbf{0}\end{pmatrix}\ ,\quad D^{(j)}=\begin{cases} I_{2(N+1)} & ,\ j=0 \\ D^{(j-1)}D & ,\ j>0\end{cases} \tag{2.15}$$

where, I_n is the n-dimensional identity matrix.

Proof. We will proceed by induction. First, we show that (2.13) holds for every $2\le j\le K-N+1$ and then that (2.12) holds for every $2\le i\le N-1$.

By the definition of the matrix $D^{(j)}$, we have:

$$D^{(j)}=D^{(j-1)}D=\begin{pmatrix} D_{11}^{(j-1)} & D_{12}^{(j-1)} \\ D_{21}^{(j-1)} & D_{22}^{(j-1)}\end{pmatrix}\begin{pmatrix} -A_1^{N}A_0^{-1} & I_{N+1} \\ -A_2A_0^{-1} & \mathbf{0}\end{pmatrix}=\begin{pmatrix} -D_{11}^{(j-1)}A_1^{N}A_0^{-1}-D_{12}^{(j-1)}A_2A_0^{-1} & D_{11}^{(j-1)} \\ -D_{21}^{(j-1)}A_1^{N}A_0^{-1}-D_{22}^{(j-1)}A_2A_0^{-1} & D_{21}^{(j-1)}\end{pmatrix}$$

Therefore we get:

$$D_{11}^{(j)}=-D_{11}^{(j-1)}A_1^{N}A_0^{-1}-D_{12}^{(j-1)}A_2A_0^{-1}\quad,\quad D_{12}^{(j)}=D_{11}^{(j-1)} \tag{2.16}$$

$$D_{21}^{(j)}=-D_{21}^{(j-1)}A_1^{N}A_0^{-1}-D_{22}^{(j-1)}A_2A_0^{-1}\quad,\quad D_{22}^{(j)}=D_{21}^{(j-1)}.$$

For $j=2$, provided that A_0 is non-singular, we get from equation (2.11):

$$\vec{P}_{K-2} = -\vec{P}_K A_2 A_0^{-1} - \vec{P}_{K-1} A_1^N A_0^{-1}.$$

From (2.15), $D_{11}^{(0)} = I_{N+1}$ and $D_{21}^{(0)} = \mathbf{0}$. Therefore,

$$\vec{P}_{K-2} = -\vec{P}_K A_2 A_0^{-1} D_{11}^{(0)} + \vec{P}_{K-1}\left(D_{21}^{(0)} - A_1^N A_0^{-1} D_{11}^{(0)}\right).$$

Assume the proposition holds for $j-1$, we now show that it holds for $j \le K-N+1$. From (2.11) we have:

$$\vec{P}_{K-j} = -\vec{P}_{K-j+2} A_2 A_0^{-1} - \vec{P}_{K-j+1} A_1^N A_0^{-1}.$$

Substituting the values of $\vec{P}_{K-j+2}$ and $\vec{P}_{K-j+1}$, we get:

$$\begin{aligned}
\vec{P}_{K-j} &= -\left(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-4)} + \vec{P}_{K-1}\left(D_{21}^{(j-4)} - A_1^N A_0^{-1} D_{11}^{(j-4)}\right)\right) A_2 A_0^{-1} \\
&\quad -\left(-\vec{P}_K A_2 A_0^{-1} D_{11}^{(j-3)} + \vec{P}_{K-1}\left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)}\right)\right) A_1^N A_0^{-1} \\
&= \vec{P}_K A_2 A_0^{-1}\left(D_{11}^{(j-4)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1}\right) \\
&\quad -\vec{P}_{K-1}\left(\left(D_{21}^{(j-4)} - A_1^N A_0^{-1} D_{11}^{(j-4)}\right) A_2 A_0^{-1} + \left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)}\right) A_1^N A_0^{-1}\right)
\end{aligned} \tag{2.17}$$

Substituting (2.16) in (2.17) we have:

$$\begin{aligned}
\vec{P}_{K-j} &= \vec{P}_K A_2 A_0^{-1}\left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1}\right) \\
&\quad -\vec{P}_{K-1}\left(\left(D_{22}^{(j-3)} - A_1^N A_0^{-1} D_{12}^{(j-3)}\right) A_2 A_0^{-1} + \left(D_{21}^{(j-3)} - A_1^N A_0^{-1} D_{11}^{(j-3)}\right) A_1^N A_0^{-1}\right) \\
&= \vec{P}_K A_2 A_0^{-1}\left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1}\right) \\
&\quad -\vec{P}_{K-1}\left(D_{22}^{(j-3)} A_2 A_0^{-1} + D_{21}^{(j-3)} A_1^N A_0^{-1} - A_1^N A_0^{-1}\left(D_{12}^{(j-3)} A_2 A_0^{-1} + D_{11}^{(j-3)} A_1^N A_0^{-1}\right)\right) \\
&= -\vec{P}_K A_2 A_0^{-1} D_{11}{}^{(j-2)} + \vec{P}_{K-1}\left(D_{21}{}^{(j-2)} - A_1^N A_0^{-1} D_{11}{}^{(j-2)}\right)
\end{aligned}$$

By the definition of the matrix $C(i)$, we have:

$$\begin{aligned}
C(i) = C(i-1)\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2 A_0^{-1} & 0 \end{pmatrix} &= \begin{pmatrix} C_{11}(i-1) & C_{12}(i-1) \\ C_{21}(i-1) & C_{22}(i-1) \end{pmatrix}\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2 A_0^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} -C_{11}(i-1)A_1^{N-i-1}A_0^{-1} - C_{12}(i-1)A_2A_0^{-1} & C_{11}(i-1) \\ -C_{21}(i-1)A_1^{N-i-1}A_0^{-1} - C_{22}(i-1)A_2A_0^{-1} & C_{21}(i-1) \end{pmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
C_{11}(i) &= -C_{11}(i-1)A_1^{N-i-1}A_0^{-1} - C_{12}(i-1)A_2A_0^{-1}, \quad C_{12}(i) = C_{11}(i-1) \\
C_{21}(i) &= -C_{21}(i-1)A_1^{N-i-1}A_0^{-1} - C_{22}(i-1)A_2A_0^{-1}, \quad C_{22}(i) = C_{21}(i-1)
\end{aligned} \tag{2.18}$$

For $i=2$, provided that A_0 is non-singular, we get from equation (2.11):

$$\vec{P}_{N-2} = -\vec{P}_N A_2 A_0^{-1} - \vec{P}_{N-1} A_1^{N-1} A_0^{-1}.$$

From (2.14), $C_{11}(0) = I_{N+1}$ and $C_{21}(0) = \mathbf{0}$. Therefore,

$$\vec{P}_{N-2} = -\vec{P}_N A_2 A_0^{-1} C_{11}(0) + \vec{P}_{N-1}\left(C_{21}(0) - A_1^{N-1} A_0^{-1} C_{11}(0)\right).$$

Assume the proposition holds for $i-1$, we will show that it holds for $i \le N-1$. From (2.11) we have:

$$\vec{P}_{N-i} = -\vec{P}_{N-i+2} A_2 A_0^{-1} - \vec{P}_{N-i+1} A_1^{N-i+1} A_0^{-1}.$$

Substituting the values of $\vec{P}_{N-i+2}$ and $\vec{P}_{N-i+1}$, we get:

$$\begin{aligned}
\vec{P}_{N-i} &= -\left(-\vec{P}_N A_2 A_0^{-1} C_{11}(i-4) + \vec{P}_{N-1}\left(C_{21}(i-4) - A_1^{N-1} A_0^{-1} C_{11}(i-4)\right)\right) A_2 A_0^{-1} \\
&\quad -\left(-\vec{P}_N A_2 A_0^{-1} C_{11}(i-3) + \vec{P}_{N-1}\left(C_{21}(i-3) - A_1^{N-1} A_0^{-1} C_{11}(i-3)\right)\right) A_1^{N-i+1} A_0^{-1} \\
&= \vec{P}_N A_2 A_0^{-1}\left(C_{11}(i-4) A_2 A_0^{-1} + C_{11}(i-3) A_1^{N-i+1} A_0^{-1}\right) \\
&\quad -\vec{P}_{N-1}\left(\left(C_{21}(i-4) - A_1^{N-1} A_0^{-1} C_{11}(i-4)\right) A_2 A_0^{-1} + \left(C_{21}(i-3) - A_1^{N-1} A_0^{-1} C_{11}(i-3)\right) A_1^{N-i+1} A_0^{-1}\right)
\end{aligned} \tag{2.19}$$

Substituting (2.18) in (2.19) we have:

$$\begin{aligned}
\vec{P}_{N-i} &= \vec{P}_N A_2 A_0^{-1}\left(C_{12}(i-3) A_2 A_0^{-1} + C_{11}(i-3) A_1^{N-i+1} A_0^{-1}\right) \\
&\quad -\vec{P}_{N-1}\left(\left(C_{22}(i-3) - A_1^{N-1} A_0^{-1} C_{12}(i-3)\right) A_2 A_0^{-1} + \left(C_{21}(i-3) - A_1^{N-1} A_0^{-1} C_{11}(i-3)\right) A_1^{N-i+1} A_0^{-1}\right) \\
&= \vec{P}_N A_2 A_0^{-1}\left(C_{12}(i-3) A_2 A_0^{-1} + C_{11}(i-3) A_1^{N-i+1} A_0^{-1}\right) \\
&\quad -\vec{P}_{N-1}\left(C_{22}(i-3) A_2 A_0^{-1} + C_{21}(i-3) A_1^{N-i+1} A_0^{-1} - A_1^{N-1} A_0^{-1}\left(C_{12}(i-3) A_2 A_0^{-1} + C_{11}(i-3) A_1^{N-i+1} A_0^{-1}\right)\right) \\
&= -\vec{P}_N A_2 A_0^{-1} C_1(i-2) + \vec{P}_{N-1}\left(C_{21}(i-2) - A_1^{N-1} A_0^{-1} C_1(i-2)\right)
\end{aligned}$$

This completes the proof.

From Theorem 2.1 we have that $\vec{P}_m$, $1 \le m \le K-2$, are expressed in terms of the boundary probability vectors $\vec{P}_{N-1}$, $\vec{P}_N$, $\vec{P}_{K-1}$ and $\vec{P}_K$. Therefore, the solution of (2.11) can be calculated by solving only the following reduced linear system:

$$\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0} \\
&\vec{P}_{N-1} + \vec{P}_K A_2 A_0^{-1} D_{11}^{(K-N-1)} - \vec{P}_{K-1}\left(D_{21}^{(K-N-1)} - A_1^N A_0^{-1} D_{11}^{(K-N-1)}\right) = \vec{0} \\
&\vec{P}_N + \vec{P}_K A_2 A_0^{-1} D_{11}^{(K-N-2)} - \vec{P}_{K-1}\left(D_{21}^{(K-N-2)} - A_1^N A_0^{-1} D_{11}^{(K-N-2)}\right) = \vec{0} \\
&\vec{P}_{K-1} A_0 + \vec{P}_K A_2 = \vec{0}
\end{aligned} \tag{2.20}$$

From (2.12) we can express $\vec{P}_1$ and $\vec{P}_2$ in terms of $\vec{P}_{N-1}$ and $\vec{P}_N$. Hence, the system (2.20) becomes

$$\vec{P}_0 A_1^0 - \vec{P}_N A_2 A_0^{-1} C_{11}(N-3) A_2^1 + \vec{P}_{N-1}\left(C_{21}(N-3) - A_1^{N-1} A_0^{-1} C_{11}(N-3)\right) A_2^1 = \vec{0} \tag{2.21}$$

$$\vec{P}_0 A_0^0 - \vec{P}_N A_2 A_0^{-1}\left(C_{11}(N-3) A_1^1 + C_{11}(N-4) A_2\right)$$

$$+\vec{P}_{N-1}\left(\left(C_{21}(N-3) - A_1^{N-1} A_0^{-1} C_{11}(N-3)\right) A_1^1 + \left(C_{21}(N-3) - A_1^{N-1} A_0^{-1} C_{11}(N-3)\right) A_2\right) = \vec{0}$$

$$\vec{P}_{N-1} + \vec{P}_K A_2 A_0^{-1} D_{11}{}^{(K-N-1)} - \vec{P}_{K-1}\left(D_{21}{}^{(K-N-1)} - A_1^N A_0^{-1} D_{11}{}^{(K-N-1)}\right) = \vec{0}$$

$$\vec{P}_N + \vec{P}_K A_2 A_0^{-1} D_{11}{}^{(K-N-2)} - \vec{P}_{K-1}\left(D_{21}{}^{(K-N-2)} - A_1^N A_0^{-1} D_{11}{}^{(K-N-2)}\right) = \vec{0}$$

$$\vec{P}_{K-1} A_0 + \vec{P}_K A_2 = \vec{0}.$$

The normalization equation now becomes

$$\vec{P}_0\vec{e} - \vec{P}_N A_2 A_0^{-1} \sum_{i=1}^{N-2} C_{11}(N-i-2)\vec{e} + \vec{P}_{N-1}\left(\sum_{i=1}^{N-2} C_{21}(N-i-2) - A_1^{N-1} A_0^{-1} \sum_{i=1}^{N-2} C_{11}(N-i-2)\right)\vec{e} \tag{2.22}$$

$$-\vec{P}_K\left(A_2 A_0^{-1} \sum_{j=N-1}^{K-2} D_{11}{}^{(K-j-2)} + I_{N+1}\right)\vec{e} + \vec{P}_{K-1}\left(\sum_{j=N-1}^{K-2} D_{21}{}^{(K-j-2)} - A_1^N A_0^{-1} \sum_{j=N-1}^{K-2} D_{11}{}^{(K-j-2)} + I_{N+1}\right)\vec{e} = 1$$

Therefore, instead of solving tediously the set of linear equations (2.11), it is enough to calculate the matrices $C(i)$, $0 \le i \le N-3$ and $D^{(j)}$, $0 \le j \le K-N-1$, and solve the set of linear equations (2.21) and (2.22).

We thus obtain the set of sought for probability vectors $\vec{P}_0$, $\vec{P}_1$, …, $\vec{P}_K$.

Moreover, in our case, due to the structure of A_0 and A_2, the computational effort can be further reduced as follows: $A_0^{-1} = diag\left(1/\lambda_2\right)$ and $A_2 = \mu_1 \vec{Z} I_{(N+1)}$, where $\vec{Z} = \left(0,1,2,...,N\right)$. Thus, $A_2 A_0^{-1} = \frac{\mu_1}{\lambda_2}\vec{Z} I_{(N+1)}$. It follows that

$$C(i) = \begin{cases} I_{2(N+1)} & , \quad i = 0 \\ C(i-1)\begin{pmatrix} -\frac{1}{\lambda_2} A_1^{N-i-1} & I_{N+1} \\ -\frac{\mu_1}{\lambda_2}\vec{Z} I_{(N+1)} & \mathbf{0} \end{pmatrix} & , \quad i > 0 \end{cases}$$

$$D = \begin{pmatrix} -\frac{1}{\lambda_2} A_1^N & I_{N+1} \\ -\frac{\mu_1}{\lambda_2}\vec{Z} I_{(N+1)} & \mathbf{0} \end{pmatrix}.$$

The mean total number of customers in Q_2, $E[L_2]$, is given by:

$$E[L_2]=\sum_{m=1}^{K} m\vec{P}_m\vec{e} = -\vec{P}_N A_2 A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\vec{e}$$
$$+\vec{P}_{N-1}\left(\sum_{i=1}^{N-2} iC_{21}(N-i-2) - A_1^{N-1}A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\right)\vec{e}$$
$$-\vec{P}_K\left(A_2A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}{}^{(K-j-2)} + KI_{N+1}\right)\vec{e}$$
$$+\vec{P}_{K-1}\left(\sum_{j=N-1}^{K-2} jD_{21}{}^{(K-j-2)} - A_1^N A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}{}^{(K-j-2)} + (K-1)I_{N+1}\right)\vec{e}.$$

Equation (2.7) can now be expressed as $E[L_1]=\left(1-\vec{P}_K\vec{e}\right)\lambda_2/\mu_2+\vec{P}_0\vec{Z}_0$, where $\vec{Z}_0=\left(1,2,...,N\right)$.

Note: For Model 1, the analysis of case (*ii*) where $N \geq K$ is very much similar to case (*i*) and we exclude it from the presentation. We note, however, that in this case the QBD process generator matrix Q is given by

$$Q=\begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \dots & \dots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \dots & \vdots \\ \mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & A_2 & A_1^{K-1} & A_0 \\ \mathbf{0} & \dots & \dots & \mathbf{0} & A_2 & A_1^K \end{pmatrix}$$

where the matrices $A_0^0, A_0;\ A_1^0, A_1^1, \dots, A_1^{K-1};\ A_2^1, A_2$ are the same as in case (*i*), but the matrix A_1^K is given by

$$\left(A_1^K\right)_{ij}=\begin{cases} -\lambda_1 \quad j=i=0 \\ -(\lambda_1+i\mu_1+i\mu_2) \quad j=i,\ i=1,...,K \\ -(\lambda_1+K\mu_1+i\mu_2) \quad j=i,\ i=K+1,...,N-1 \\ -(K\mu_1+N\mu_2) \quad j=i=N \\ \lambda_1 \quad j=i+1,\ i=0,1,...,N-1 \\ i\mu_1 \quad j=i-1,\ i=1,...,K \\ K\mu_1 \quad j=i-1,\ i=K+1,...,N \\ 0 \quad otherwise \end{cases}$$

Hence the set (2.11) becomes

$$
\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0} \\
&\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2 = \vec{0} \\
&\vdots \\
&\vec{P}_{K-2} A_0 + \vec{P}_{K-1} A_1^{K-1} + \vec{P}_K A_2 = \vec{0} \\
&\vec{P}_{K-1} A_0 + \vec{P}_K A_1^K = \vec{0}
\end{aligned}
$$

In such a case equations (2.12) and (2.13) are merged, and Theorem 2.1 states: If A_0 is a non-singular matrix then, for all $2 \le i \le K-1$, the following equation holds:

$$\vec{P}_{K-i} = -\vec{P}_K A_2 A_0^{-1} C_{11}(i-2) + \vec{P}_{K-1}\left(C_{21}(i-2) - A_1^{K-1} A_0^{-1} C_{11}(i-2)\right).$$

3 Model 2

In this model we modify the service scheme for Q_2: while Q_1 remains an $M(\lambda_1)/M(\mu_1)/Min(N,\ L_2)/N$ system, Q_2 operates now as a multi-server $M(\lambda_2)/M(\mu_2)/Min(K,L_1)/K$ queue in which <u>each</u> customer present in Q_1 <u>individually</u> acts as a server for the customers in Q_2.

As a result of symmetry, it is enough to analyze the case where $N \le K$.

3.1 Balance Equations

Let L_i denote the total number of customers in Q_i, $i = 1,\ 2$. Then, the pair (L_1, L_2) defines a non reducible continuous-time Markov process. The transition-rate diagram for the case $N \le K$ is shown in Figure 3.1. As before, let $P_{nm} = P\left(L_1 = n,\ L_2 = m\right)$, $0 \le n \le N$ and $0 \le m \le K$, denote the system's stationary probabilities. Then, the set of balance equations is:

$\underline{n=0}$:

$$
\begin{cases}
m = 1: \quad (\lambda_1 + \lambda_2) P_{01} = \mu_1 P_{11} \\
2 \le m \le K-1: \quad (\lambda_1 + \lambda_2) P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m} \\
m = K: \quad \lambda_1 P_{0K} = \lambda_2 P_{0,K-1} + \mu_1 P_{1K}
\end{cases}
\tag{3.1}
$$

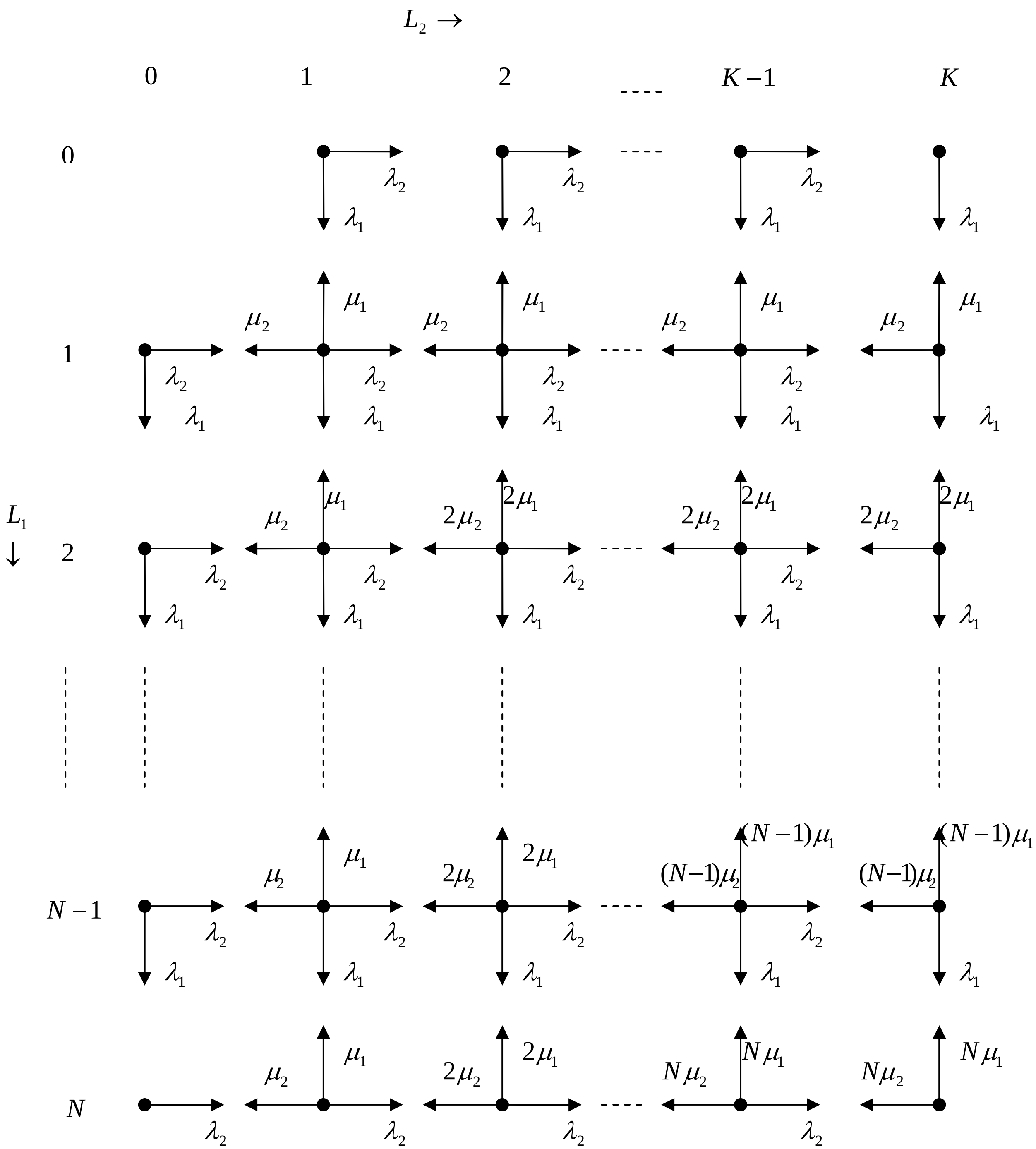

Figure 3.1: Transition-rate diagram of (L_1, L_2) for Model 2 where $N \le K$.

$\underline{n=1}$:

$$\begin{cases} m=0: \ (\lambda_1+\lambda_2)P_{10}=\mu_2 P_{11} \\ m=1: \ (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{11}=\lambda_1 P_{01}+\lambda_2 P_{10}+\mu_1 P_{21}+\mu_2 P_{12} \\ 2\le m\le K-1: \ (\lambda_1+\lambda_2+\mu_1+\mu_2)P_{1m}=\lambda_1 P_{0m}+\lambda_2 P_{1,m-1}+2\mu_1 P_{2m}+\mu_2 P_{1,m+1} \\ m=K: \ (\lambda_1+\mu_1+\mu_2)P_{1K}=\lambda_1 P_{0K}+\lambda_2 P_{1,K-1}+2\mu_1 P_{2K} \end{cases} \tag{3.2}$$

$\underline{2\le n\le N-1}$:

$$\begin{cases} m=0: \ (\lambda_1+\lambda_2)P_{n0}=\lambda_1 P_{n-1,0}+\mu_2 P_{n1} \\ 1\le m<n: \ (\lambda_1+\lambda_2+m\mu_1+m\mu_2)P_{nm}=\lambda_1 P_{n-1,m}+\lambda_2 P_{n,m-1}+m\mu_1 P_{n+1,m}+(m+1)\mu_2 P_{n,m+1} \\ n\le m\le K-1: (\lambda_1+\lambda_2+n\mu_1+n\mu_2)P_{nm}=\lambda_1 P_{n-1,m}+\lambda_2 P_{n,m-1}+(n+1)\mu_1 P_{n+1,m}+n\mu_2 P_{n,m+1} \\ m=K: (\lambda_1+n\mu_1+n\mu_2)P_{nK}=\lambda_1 P_{n-1,K}+\lambda_2 P_{n,K-1}+(n+1)\mu_1 P_{n+1,K} \end{cases} \tag{3.3}$$

$\underline{n=N}$:

$$\begin{cases} m=0: \ \lambda_2 P_{N0}=\lambda_1 P_{N-1,0}+\mu_2 P_{N1} \\ 1\le m<N: \ (\lambda_2+m\mu_1+m\mu_2)P_{Nm}=\lambda_1 P_{N-1,m}+\lambda_2 P_{N,m-1}+(m+1)\mu_2 P_{N,m+1} \\ N\le m\le K-1: (\lambda_2+N\mu_1+N\mu_2)P_{Nm}=\lambda_1 P_{N-1,m}+\lambda_2 P_{N,m-1}+N\mu_2 P_{N,m+1} \\ m=K: (N\mu_1+N\mu_2)P_{NK}=\lambda_1 P_{N-1,K}+\lambda_2 P_{N,K-1} \end{cases} \tag{3.4}$$

The marginal probabilities are defined as

$$P\left(L_1=n\right)\equiv P_{n\bullet}=\sum_{m=0}^{K}P_{nm} \quad \text{for} \ \ 0\le n\le N$$

$$P\left(L_2=m\right)\equiv P_{\bullet m}=\sum_{n=0}^{N}P_{nm} \quad \text{for} \ \ 0\le m\le K$$

Similar to section 2.1, by algebraic manipulations we arrive at

$$\lambda_2\sum_{m=0}^{K-1}P_{\bullet m}=\mu_2\sum_{m=0}^{K-1}P_{\bullet m+1}E[L_1\,|\,L_2=m+1]-\mu_2\sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m}\ ,$$

or,

$$E[L_1]=\left(1-P_{\bullet K}\right)\lambda_2/\mu_2+\sum_{n=1}^{N}\sum_{m=0}^{n}(n-m)P_{nm}\ . \tag{3.5}$$

In addition,

$$\lambda_1(1-P_{N\bullet})=\mu_1 E[L_1]-\mu_1\sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m}\ ,$$

or,

$$E[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m} \tag{3.6}$$

Equating equations (3.5) and (3.6) results in

$$E[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m} = (1 - P_{\bullet K})\lambda_2/\mu_2 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{n,m}\ ,$$

implying that

$$(1 - P_{N\bullet})\lambda_1/\mu_1 = (1 - P_{\bullet K})\lambda_2/\mu_2\ . \tag{3.7}$$

Equation (3.7) reveals an interesting result: the carried load of Q_1, namely $(1 - P_{N\bullet})\lambda_1/\mu_1$, is equal to the carried load of Q_2, $(1 - P_{\bullet K})\lambda_2/\mu_2$, independent of the capacities *N* and *K*.

3.2 Deriving $(P_{nm})_{0 \le n \le N\ ,\ 0 \le m \le K}$

The above set of balance equations can be solved numerically by standard methods. However, we again wish to utilize the method presented in the previous section to reduce calculation effort.

The system of balance equations (3.1) - (3.4) can be described as a queueing system with $N+1$ phases, where phase n indicates that there are n servers available. We construct a finite QBD process with generator Q, given by

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \vdots \\ \mathbf{0} & A_2^2 & A_1^2 & A_0 & \mathbf{0} & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & A_2^N & A_1^N & A_0 & \mathbf{0} & \vdots \\ \vdots & \vdots & \vdots & \ddots & A_2^N & A_1^N & A_0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & A_0 \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} & A_2^N & A_1 \end{pmatrix},$$

where, A_0^0, A_0; $A_1^0, A_1^1, \ldots, A_1^N$ and A_1; $A_2^1, A_2^2, \ldots, A_2^N$ are the following matrices: A_0^0 is of size $N \times (N+1)$; A_0 is of size $(N+1)\times(N+1)$; A_1^0 is of size $N \times N$; $A_1^1, \ldots, A_1^N$ and A_1

are each of size $(N+1)\times(N+1)$; A_2^1 is of size $(N+1)\times N$, and $A_2^2,\ldots, A_2^N$ are each of size $(N+1)\times(N+1)$. They are given by

$$A_0^0=\begin{pmatrix} 0 & \lambda_2 & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_2 \end{pmatrix}$$

$$A_0 = diag\left(\lambda_2\right)$$

$$A_1^0=\begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \ldots & \ldots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \ldots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

For all $1\le m\le N$,

$$\left(A_1^m\right)_{ij}=\begin{cases} -(\lambda_1+\lambda_2) \quad j=i=0 \\ -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) \quad j=i,\ i=1,\ldots,m \\ -(\lambda_1+\lambda_2+m\mu_1+m\mu_2) \quad j=i,\ i=m+1,\ldots,N-1 \\ -(\lambda_2+m\mu_1+m\mu_2) \quad j=i=N \\ \lambda_1 \quad j=i+1,\ i=0,1,\ldots,N-1 \\ i\mu_1 \quad j=i-1,\ i=1,\ldots,m \\ m\mu_1 \quad j=i-1,\ i=m+1,\ldots,N \\ 0 \quad otherwise \end{cases}$$

$$A_1=\begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \ldots & \ldots & 0 \\ \mu_1 & -(\lambda_1+\mu_1+\mu_2) & \lambda_1 & 0 & \ldots & \vdots \\ 0 & 2\mu_1 & -(\lambda_1+2\mu_1+2\mu_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & 3\mu_1 & -(\lambda_1+3\mu_1+3\mu_2) & \lambda_1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & N\mu_1 & -(N\mu_1+N\mu_2) \end{pmatrix}$$

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \dots & \dots & \vdots \\ 0 & \mu_2 & 0 & \dots & \vdots \\ \vdots & \ddots & \mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mu_2 \end{pmatrix}$$

For all $2 \le m \le N$,

$$\left(A_2^m\right)_{ij} = \begin{cases} i\mu_2 & j = i = 0,1,...,m-1 \\ m\mu_2 & j = i = m,...,N \\ 0 & otherwise \end{cases}$$

Define the steady-state probability vectors $\vec{P}_0 = \left(P_{10}, \dots, P_{N0}\right)$ and $\vec{P}_m = \left(P_{0m}, P_{1m}, \dots, P_{Nm}\right)$ for all $1 \le m \le K$. Then, the steady state probability vectors satisfy:

$$\begin{aligned} &\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\ &\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2^2 = \vec{0} \\ &\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2^3 = \vec{0} \\ &\vdots \\ &\vec{P}_{N-2} A_0 + \vec{P}_{N-1} A_1^{N-1} + \vec{P}_N A_2^N = \vec{0} \\ &\vec{P}_{N-1} A_0 + \vec{P}_N A_1^N + \vec{P}_{N+1} A_2^N = \vec{0} \\ &\vdots \\ &\vec{P}_{K-2} A_0 + \vec{P}_{K-1} A_1^N + \vec{P}_K A_2^N = \vec{0} \\ &\vec{P}_{K-1} A_0 + \vec{P}_K A_1 = \vec{0} \end{aligned}$$

We claim the following:

Theorem 3.1. If A_0 is a non-singular matrix, then the following equations hold:

$$\begin{aligned} &\vec{P}_{N-i} = -\vec{P}_N A_2^N A_0^{-1} C_{11}(i-2) + \vec{P}_{N-1}\left(C_{21}(i-2) - A_1^{N-1} A_0^{-1} C_{11}(i-2)\right) \quad , \quad 2 \le i \le N-1 \\ &\vec{P}_{K-j} = -\vec{P}_K A_2^N A_0^{-1} D_{11}{}^{(j-2)} + \vec{P}_{K-1}\left(D_{21}{}^{(j-2)} - A_1^N A_0^{-1} D_{11}{}^{(j-2)}\right) \quad , \quad 2 \le j \le K-N+1 \end{aligned}$$

where $C_{11}(i-2)$ and $C_{21}(i-2)$ are $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ product matrix $C(i-2)$ defined as

$$C(i) = \begin{cases} I_{2(N+1)} & , \quad i = 0 \\ C(i-1)\begin{pmatrix} -A_1^{N-i-1} A_0^{-1} & I_{N+1} \\ -A_2^{N-i} A_0^{-1} & \mathbf{0} \end{pmatrix} & , \quad i > 0 \end{cases}.$$

$D_{11}{}^{(j-2)}$ and $D_{21}{}^{(j-2)}$ are $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ power matrix $D^{(j-2)}$ defined as

$$D=\begin{pmatrix}-A_1^N A_0^{-1} & I_{N+1}\\ -A_2^N A_0^{-1} & \mathbf{0}\end{pmatrix},\quad D^{(j)}=\begin{cases}I_{2(N+1)} & ,\ j=0\\ D^{(j-1)}D & ,\ i>0\end{cases}.$$

Proof. The proof is similar to the proof of Theorem 2.1.

Again, one calculates the matrices C and D and then obtains sequentially the probability vectors $\vec{P}_m$, $0\le m\le K$ as functions of $\vec{P}_{K-1}$ and $\vec{P}_K$. Together with the normalization equation $\sum_{m=0}^{K}\vec{P}_{\bullet m}\cdot\vec{e}=1$, all required probability vectors are obtained.

The mean total number of customers in Q_2, $E[L_2]$, is given by:

$$\begin{aligned}E[L_2]=&-\vec{P}_N A_2^N A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\vec{e}\\ &+\vec{P}_{N-1}\left(\sum_{i=1}^{N-2} iC_{11}(N-i-2)\vec{e}-A_1^{N-1}A_0^{-1}\sum_{i=1}^{N-2} iC_{11}(N-i-2)\vec{e}\right)\\ &-\vec{P}_K\left(A_2^N A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}{}^{(K-j-2)}\vec{e}+KI_{N+1}\right)\\ &+\vec{P}_{K-1}\left(\sum_{j=N-1}^{K-2} jD_{11}{}^{(K-j-2)}\vec{e}-A_1^N A_0^{-1}\sum_{j=N-1}^{K-2} jD_{11}{}^{(K-j-2)}\vec{e}+(K-1)I_{N+1}\right)\end{aligned}$$

Equation (3.5) can now be written as $E[L_1]=\left(1-\vec{P}_K\vec{e}\right)\lambda_2/\mu_2+\vec{P}_0\vec{Z}_0+\sum_{m=1}^{N-1}\vec{P}_m\vec{Z}_m$, where $\vec{Z}_0=\left(1,2,...,N\right)$, and for all $1\le m\le N-1$, $\left(\vec{Z}_m\right)_i=\begin{cases}0, & 0\le i\le m\\ i-m, & m+1\le i\le N\end{cases}$.

4 Model 3

In this model Q_1 is an $M(\lambda_1)/M(\mu_1 L_2)/1/N$ system, and Q_2 is similarly an $M(\lambda_2)/M(\mu_2 L_1)/1/K$ system. That is, all customers present in Q_i, $i=1,\ 2$, join hands together and form a single server that serves the customers of Q_j, $j=2,\ 1$, where the combined service rate depends on the queue length L_i and equals $\mu_j L_i$.

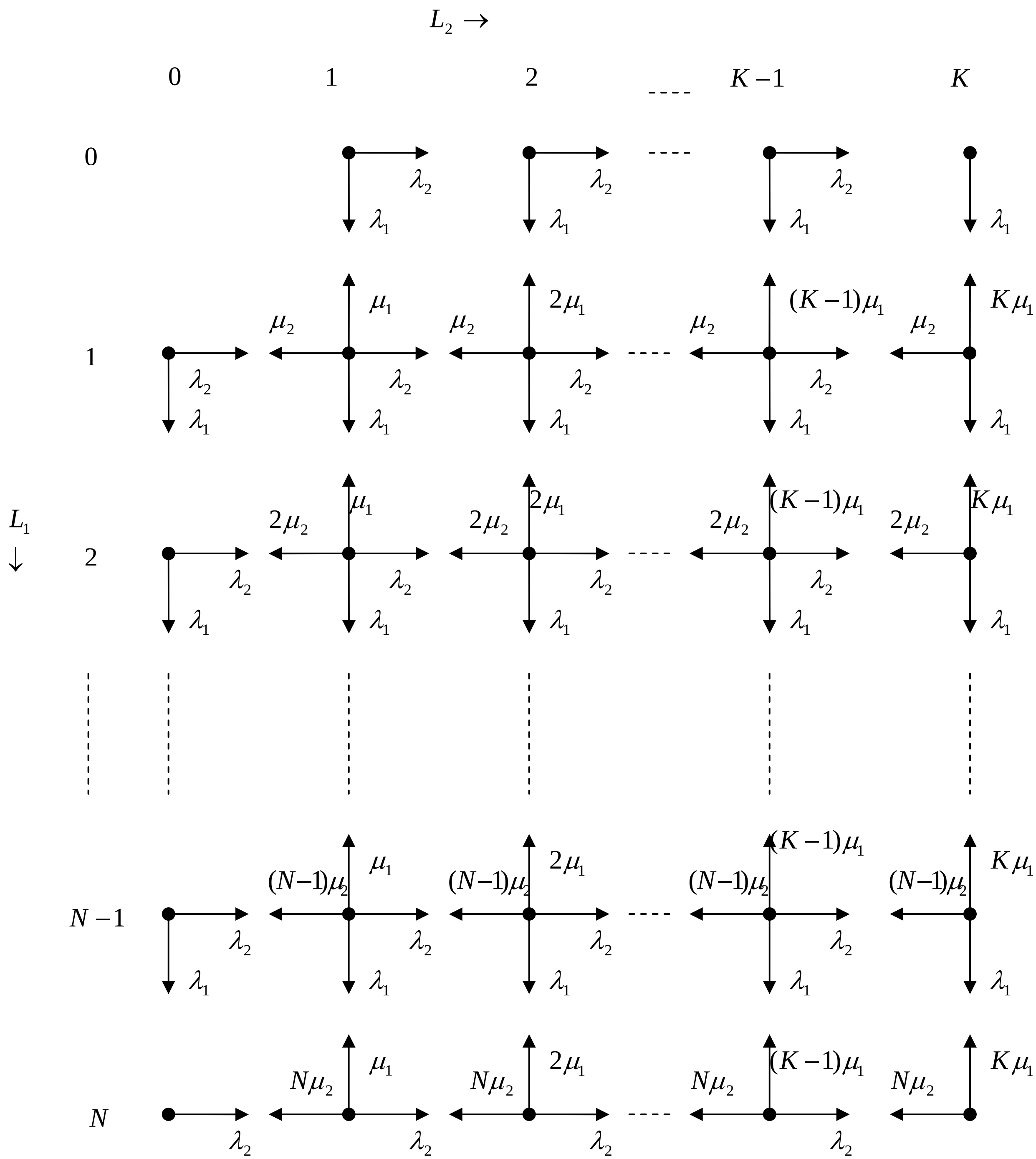

Figure 4.1: Transition-rate diagram of (L_1, L_2) for Model 3.

4.1 Balance Equations

With L_1, L_2 and P_{nm} as before ($0 \le n \le N$, $0 \le m \le K$), Figure 4.1 is the corresponding transition-rate diagram. The set of balance equations is given as follows:

$\underline{n = 0}$:

$$\begin{cases} m = 1: \ \ (\lambda_1 + \lambda_2) P_{01} = \mu_1 P_{11} \\ 2 \le m \le K - 1: \ \ (\lambda_1 + \lambda_2) P_{0m} = \lambda_2 P_{0,m-1} + m\mu_1 P_{1m} \\ m = K: \ \ \lambda_1 P_{0K} = \lambda_2 P_{0,K-1} + K\mu_1 P_{1K} \end{cases} \tag{4.1}$$

$\underline{n = 1}$:

$$\begin{cases} m = 0: \ \ (\lambda_1 + \lambda_2) P_{10} = \mu_2 P_{11} \\ 1 \le m \le K - 1: \ \ (\lambda_1 + \lambda_2 + m\mu_1 + \mu_2) P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + m\mu_1 P_{2m} + \mu_2 P_{1,m+1} \\ m = K: \ \ (\lambda_1 + K\mu_1 + \mu_2) P_{1K} = \lambda_1 P_{0K} + \lambda_2 P_{1,K-1} + K\mu_1 P_{2K} \end{cases} \tag{4.2}$$

$\underline{2 \le n \le N-1}$:

$$\begin{cases} m = 0: \ \ (\lambda_1 + \lambda_2) P_{n0} = \lambda_1 P_{n-1,0} + n\mu_2 P_{n1} \\ 1 \le m \le K - 1: \ \ (\lambda_1 + \lambda_2 + m\mu_1 + n\mu_2) P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \\ m = K: \ \ (\lambda_1 + K\mu_1 + n\mu_2) P_{nK} = \lambda_1 P_{n-1,K} + \lambda_2 P_{n,K-1} + K\mu_1 P_{n+1,K} \end{cases} \tag{4.3}$$

$\underline{n = N}$:

$$\begin{cases} m = 0: \ \ \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{N1} \\ 1 \le m \le K - 1: \ \ (\lambda_2 + m\mu_1 + N\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \\ m = K: \ \ (K\mu_1 + N\mu_2) P_{NK} = \lambda_1 P_{N-1,K} + \lambda_2 P_{N,K-1} \end{cases} \tag{4.4}$$

Again, by algebraic manipulations we arrive at

$$\lambda_2 \sum_{m=0}^{K-1} P_{\bullet m} = \mu_2 \sum_{m=0}^{K-1} P_{\bullet m+1} E[L_1 \mid L_2 = m+1] \tag{4.5}$$

Therefore, $\lambda_2 \left(1 - P_{\bullet K}\right) = \mu_2 \left(E[L_1] - P_{\bullet 0} E[L_1 \mid L_2 = 0]\right) = \mu_2 \left(E[L_1] - \sum_{n=1}^{N} nP_{n0} \right)$,

meaning that

$$E[L_1] = \left(1 - P_{\bullet K}\right) \lambda_2 / \mu_2 + \sum_{n=1}^{N} nP_{n0} \tag{4.6}$$

Furthermore,

$$\lambda_1(1-P_{N\bullet}) = \mu_1 E[L_2] - \mu_1 \sum_{m=1}^{K} mP_{0,m}$$

That is,

$$E[L_2] = (1-P_{N\bullet})\,\lambda_1 / \mu_1 + \sum_{m=1}^{K} mP_{0,m} \tag{4.7}$$

Equation (4.6) (and similarly equation (4.7)) shows that the effective arrival rate in Q_2, $\lambda_2\left(1-P_{\bullet K}\right)$, is smaller than the mean service rate there, being $\mu_2 E[L_1]$. This coincides with the finite-buffer single-server $M(\lambda)/M(\mu)/1/K$ queue, where always $\lambda_{eff} \equiv \lambda\left(1-P_K\right) < \mu$.

4.2 Deriving $(P_{nm})_{0\le n\le N,\ 0\le m\le K}$

The corresponding generator for the finite non homogeneous QBD process is

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \dots & \dots & \mathbf{0} \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & A_1^{K-1} & A_0 \\ \mathbf{0} & \dots & \dots & \dots & A_2 & A_1^K \end{pmatrix}$$

where, $A_0^0, A_0;\ A_1^0, A_1^1, \dots, A_1^K;\ A_2^1, A_2$ are the following matrices: A_0^0 is of size $N\times(N+1)$; A_0 is of size $(N+1)\times(N+1)$; A_1^0 is of size $N\times N$; $A_1^1,\dots, A_1^K$ are each of size $(N+1)\times(N+1)$; A_2^1 is of size $(N+1)\times N$, and A_2 is of size $(N+1)\times(N+1)$. They are given by

$$A_0^0 = \begin{pmatrix} 0 & \lambda_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \lambda_2 \end{pmatrix}$$

$$A_0 = diag\left(\lambda_2\right)$$

$$A_1^0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \dots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda_1 \\ 0 & \cdots & 0 & 0 & 0 & -\lambda_2 \end{pmatrix}$$

For all $1 \le m \le K-1$

$$\left(A_1^m\right)_{ij} = \begin{cases} -(\lambda_1+\lambda_2) & j=i=0 \\ -(\lambda_1+\lambda_2+m\mu_1+i\mu_2) & j=i,\ i=1,\dots,N-1 \\ -(\lambda_2+m\mu_1+N\mu_2) & j=i=N \\ \lambda_1 & j=i+1,\ i=0,1,\dots,N-1 \\ m\mu_1 & j=i-1,\ i=1,\dots,N \\ 0 & otherwise \end{cases}$$

$$\left(A_1^K\right)_{ij} = \begin{cases} -\lambda_1 & j=i=0 \\ -(\lambda_1+m\mu_1+i\mu_2) & j=i,\ i=1,\dots,N-1 \\ -(m\mu_1+N\mu_2) & j=i=N \\ \lambda_1 & j=i+1,\ i=0,1,\dots,N-1 \\ m\mu_1 & j=i-1,\ i=1,\dots,N \\ 0 & otherwise \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & \dots & \dots & \vdots \\ 0 & 2\mu_2 & 0 & \dots & \vdots \\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & N\mu_2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \mu_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & 2\mu_2 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & N\mu_2 \end{pmatrix}$$

The steady state probability vectors satisfy:

$$\begin{aligned} &\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\ &\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0} \\ &\vec{P}_1 A_0 + \vec{P}_2 A_1^2 + \vec{P}_3 A_2 = \vec{0} \\ &\vdots \\ &\vec{P}_{K-2} A_0 + \vec{P}_{K-1} A_1^{K-1} + \vec{P}_K A_2 = \vec{0} \\ &\vec{P}_{K-1} A_0 + \vec{P}_K A_1^K = \vec{0} \end{aligned}$$

Theorem 4.1. If A_0 is a non-singular matrix, then the following equation holds:

$$\vec{P}_{K-i} = -\vec{P}_K A_2 A_0^{-1} C_{11}(i-2) + \vec{P}_{K-1}\left(C_{21}(i-2) - A_1^{K-1} A_0^{-1} C_{11}(i-2)\right) \quad , \quad 2 \le i \le K-1,$$

where $C_{11}(i-2)$ and $C_{21}(i-2)$ are the $(N+1)\times(N+1)$ sub-matrices of the $2(N+1)\times 2(N+1)$ product matrix $C(i-2)$ defined as

$$C(i) = \begin{cases} I_{2(N+1)} & , \quad i = 0 \\ C(i-1)\begin{pmatrix} -A_1^{N-i-1}A_0^{-1} & I_{N+1} \\ -A_2 A_0^{-1} & \mathbf{0} \end{pmatrix} & , \quad i > 0 \end{cases}.$$

Proof. The proof is similar to the proof of the second part of Theorem 2.1.

The mean total number of customers in Q_2, $E[L_2]$, is given by:

$$E[L_2] = -\vec{P}_K\left(A_2 A_0^{-1} \sum_{i=1}^{K-2} i C_{11}(K-i-2)\vec{e} + K I_{N+1} \right)$$
$$+\vec{P}_{K-1}\left(\sum_{i=1}^{K-2} i C_{11}(K-i-2)\vec{e} - A_1^{K-1} A_0^{-1} \sum_{i=1}^{K-2} i C_{11}(K-i-2)\vec{e} + (K-1) I_{N+1} \right)$$

Equation (4.6) can now be expressed as $E[L_1] = \left(1 - \vec{P}_K \vec{e}\right)\lambda_2/\mu_2 + \vec{P}_0 \vec{Z}_0$.

5 Numerical Examples

Table 5.1 exhibits some numerical results for Models 1, 2, and 3, using the same set of parameters for all models: $\lambda_1 = 1$, $\mu_1 = 1$, $\lambda_2 = 1$ and $\mu_2 = 1$ for $N = 2$ and $K = 4$.

	Model 1	Model 2	Model 3
Prob. vectors	$\vec{P}_0 = (0.0243, 0.1864)$ $\vec{P}_1 = (0.0243, 0.0486, 0.0811)$ $\vec{P}_2 = (0.0445, 0.0648, 0.0446)$ $\vec{P}_3 = (0.0606, 0.0768, 0.0386)$ $\vec{P}_4 = (0.1651, 0.1045, 0.0357)$	$\vec{P}_0 = (0.0264, 0.1147)$ $\vec{P}_1 = (0.0264, 0.0529, 0.0882)$ $\vec{P}_2 = (0.0485, 0.0705, 0.0485)$ $\vec{P}_3 = (0.066, 0.0835, 0.042)$ $\vec{P}_4 = (0.1797, 0.1137, 0.0389)$	$\vec{P}_0 = (0.0103, 0.08)$ $\vec{P}_1 = (0.0103, 0.0207, 0.0348)$ $\vec{P}_2 = (0.0323, 0.0272, 0.0193)$ $\vec{P}_3 = (0.0825, 0.0442, 0.0173)$ $\vec{P}_4 = (0.4972, 0.1037, 0.0201)$
$E[L_1]$	1.0918	1.0117	0.5493
$E[L_2]$	2.2112	2.4062	3.1394

Table 5.1 numerical results

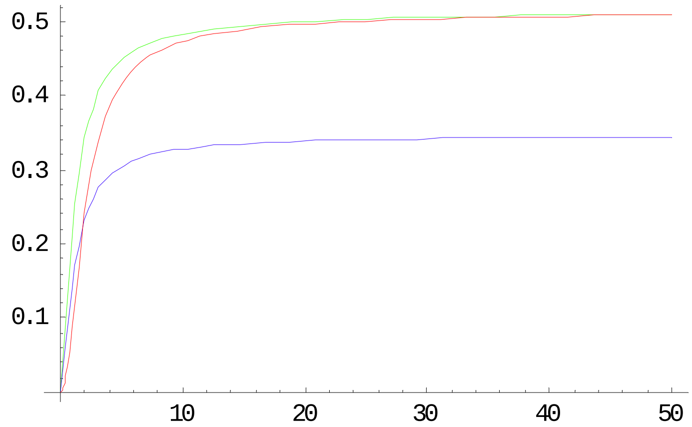

Figure 5.1: $P_{\bullet 0} = P(L_2 = 0)$ as a function of λ_1

Figure 5.1 depicts 3 graphs, each one for each model, showing the change in $P_{\bullet 0} = P(L_2 = 0)$ as a function of λ_1, while the other parameters are those of Table 5.1. When $\lambda_1 \to \infty$ there are always $N = 2$ customers in Q_1. Thus, in Models 1 and 3, Q_2 is effectively an $M(1)/M(2)/1/4$ queue, for which the probabilities $P_{\bullet m} = P(L_2 = m)$, $0 \le m \le 4$ are given by $P_{\bullet m} = \left(\frac{1}{2}\right)^{m+1} \Big/ \left(1 - \left(\frac{1}{2}\right)^5\right)$. In particular, for $m = 0$, $P_{\bullet 0} = 0.516$, as seen in the corresponding graphs.

For Model 2, when $\lambda_1 \to \infty$, Q_2 effectively becomes an $M(1)/M(1)/2/4$ queue.

The graph also demonstrates that $P(L_2 = 0 \mid \text{Model } 1) > P(L_2 = 0 \mid \text{Model } j)$, $j = 2, 3$.

6 Summary

In this paper we extend the scope of analytic investigation of 2-queue models where customers of each queue act as servers for the other queue. In contrast to the case where only one queue is finite, we study the case where both queues are finite. Three models are considered, distinguish by the way in which customers of one queue serve the customers of the other queue. For each model we derive the steady state probability vectors and calculate the mean total number of customers in each queue. The derivation of the probability vectors uses the special 3-diagonal structure of the generator of the QBD process to reduce the required computational effort. We further show in Model 1 that the carried load at Q_1 is always larger than the carried load at Q_2, while in Model 2, the carried loads are equal. In Model 3 the effective arrival rate is smaller than the realized service rate for both queues.

7 References

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