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CUSTOMERS' OPTIMAL JOINING RULES FOR THE *GI/M/s* QUEUE*

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A $GI/M/s$ queue with a stationary balking sequence is considered. For the infinite horizon average reward criterion, it is shown that among all stationary joining policies the optimal ones are nonrandomized control limit rules of the form: join if and only if the queue size is smaller than some specific number. It is shown that, in general, exercising self-optimization by individual customers does not optimize public good. The $M/M/s$ queue is then treated as an example, and a "direct" proof for the optimality of the control limit rule is given.

1. Introduction

We are concerned with a $GI/M/s$ queueing process where customers arrive at instants $\tau_0, \tau_1, \tau_2, \cdots, \tau_m, \cdots$, and the interarrival times $\tau_{m+1} - \tau_m$, $m = 0, 1, 2, \cdots$, are independent, identically distributed positive random variables with common distribution function $H(\cdot)$ and finite mean $1/\lambda$. There are $s \geqq 1$ identical servers in the system and the distribution of service times is exponential with parameter μ. The queue discipline (for customers who join the system) is 'first come first served.' Let $\eta(t)$ be the queue size (customers waiting or being served) at instant t. Let $\eta_m = \eta(\tau_m - 0)$; that is, η_m denotes the queue size just before the mth arrival. The system is said to be in state i at the mth step if $\eta_m = i$. Suppose that upon arrival each customer can take one of two actions: either join the queue or balk. Let $\{\Delta_m\}$, $m = 0, 1, 2, \cdots$, denote the sequence of decisions (i.e., actions) made by the arriving customers. Let $\Delta_m = 1$ stand for the decision to join and $\Delta_m = 0$ for the decision to balk. Thus, regarding the system as making its transitions at instants of arrival, we obtain a Markovian Decision Process [5], [6], $\{\eta_m, \Delta_m\}$, $m = 0, 1, 2, \cdots$, whose stationary transition probabilities, $q_{ij}(k) = P\{\eta_{m+1} = j \mid \eta_m = i, \Delta_m = k\}$, $i, j = 0, 1, 2, \cdots$; $k = 0, 1$ are calculated as follows (see [12] or [15]):

(1)

(i) If $j > i + 1$, $q_{ij}(1) = 0$ for all $i = 0, 1, 2, \cdots$

(ii) If $j \leqq i + 1 \leqq s$, $q_{ij}(1) = \binom{i+1}{j} \int_0^\infty e^{-j\mu x}(1 - e^{-\mu x})^{i+1-j}\, dH(x)$.

(iii) If $i + 1 \geqq j \geqq s$ and $i \geqq s$, $q_{ij}(1) = \int_0^\infty e^{-s\mu x}((s\mu x)^{i+1-j}/(i + 1 - j)!)\, dH(x)$.

(iv) If $i + 1 > s > j$,

$$q_{ij}(1) = \int_0^\infty \left[\int_0^x \frac{(s\mu)^{i+1-s}}{(i-s)!} t^{i-s} e^{-s\mu t} \binom{s}{j} e^{-\mu(x-t)j}(1 - e^{-\mu(x-t)})^{s-j}\, dt\right] dH(x)$$

$$= \binom{s}{j} \int_0^\infty e^{-j\mu x} \left[\int_0^x \frac{(s\mu t)^{i-s}}{(i-s)!} (e^{-\mu t} - e^{-\mu x})^{s-j} s\mu\, dt\right] dH(x).$$

It is also easy to see that:

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$$q_{ij}(0) = q_{i-1,j}(1) \quad \text{for} \quad i = 1, 2, \cdots; j = 0, 1, 2, \cdots, \tag{2}$$

and $q_{00}(0) = 1$.

Our objective is to find optimal joining and balking rules for arriving customers under the long-run average reward criterion. However, we assume that the only information available to a newly arriving customer is the current state of the system. This assumption together with the Markovian property of the service times amounts to considering only the so-called stationary Markovian policies [6]. Let D_{ik} denote the stationary conditional probability of taking action k when state i is observed. That is, $D_{ik} = P\{\Delta_m = k \mid \eta_m = i\}$, $m = 0, 1, 2, \cdots$. Since $k = 0$ or 1 let $D_{i1} = D_i$ and let $D_{i0} = 1 - D_i$. The sequence $\{D_i\}$ will be called a *balking sequence.*

For any *given* balking sequence, $\{D_i\}$, the sequence of random variables $\{\eta_m\}$ forms a homogeneous Markov chain (imbedded at instants of arrival) with transition probabilities

$$p_{ij} = q_{ij}(1)D_i + q_{ij}(0)(1 - D_i), \qquad i, j = 0, 1, 2, \cdots. \tag{3}$$

Let $p_{ij}^{(t)}$ denote the t-step transition probability from state i to state j. Then it is well known [7] that the limiting probabilities $\pi_j = \lim_{t\to\infty} p_{ij}^{(t)}$, $i, j = 0, 1, 2, \cdots$, always exist, they are nonnegative, and for an ergodic chain they form a distribution which uniquely satisfies the set of linear equations

$$\pi_j = \textstyle\sum_i \pi_i p_{ij}, \qquad \text{all } j, \qquad \textstyle\sum_j \pi_j = 1. \tag{4}$$

2. The $GI/M/s/n$ Queue

Suppose now that the service facility has a limited waiting room of size $n \geqq 0$. That is, there could be at most $n + s$ customers in the system. An arrival balks with probability one if he finds that all the s servers and all the n waiting positions are occupied. Suppose also that customers who find the system in state $i < s + n$ join the the queue with positive probability. From probabilistic point of view the physical limitation on the number of customers in the system is equivalent to a balking sequence for which $0 < D_i \leqq 1$ for $i < s + n$ and $D_i = 0$ for all $i \geqq s + n$. A $GI/M/s$ queueing process with such a balking sequence will be denoted as a $GI/M/s/n$ queue.

As before, the $GI/M/s/n$ process may be imbedded at instants of arrival to yield a homogeneous Markov chain whose transition probabilities, $\{p_{ij}(n)\}$, are identical with those of the $GI/M/s$ queue except for the $s + n$th (= last) row of the transition matrix. We have:

$$\begin{aligned} p_{ij}(n) &= q_{ij}(1)D_i + q_{ij}(0)(1 - D_i), \qquad i < s + n; j \leqq s + n, \\ p_{s+n,j}(n) &= q_{s+n,j}(0), \qquad j \leqq s + n. \end{aligned} \tag{5}$$

Since the finite Markov chain thus obtained is aperiodic and irreducible, the limiting probabilities of the $GI/M/s/n$ process, $\pi_j(n) = \lim_{t\to\infty} p_{ij}^{(t)}(n)$, $j = 1, 2, \cdots, s + n$, are all positive and clearly satisfy (4) with $i, j = 0, 1, 2, \cdots, s + n$.

Related Models

The special balking sequence, $\{D_i \mid D_i = 1, i < s + n; D_i = 0, i \geqq s + n\}$, i.e., where balking occurs if and only if the system is full, was treated in [15] by Takács who obtained an explicit form for the limiting probabilities $\{\pi_j(n)\}$. In [8] Finch derived an explicit solution for the $GI/M/1/n$ process with a general balking sequence

$\{D_i \mid 0 < D_i \leqq 1, i < n + 1; D_i = 0, i \geqq n + 1\}$. Haight [9] studied the $M/M/1$ and $M/M/1/n$ queues with a general stationary balking sequence, and Homma [10] analyzed the $GI/M/s$ process with a nondecreasing balking sequence $\{D_0 \geqq D_1 \geqq D_2 \geqq \cdots\}$ and gave conditions for recurrence and transiency. Yechiali [18] found optimal balking rules for the $GI/M/1$ queue with a general balking sequence.

Waiting Times

Consider the stationary process of the $GI/M/s/n$ queue with its limiting probabilities $\{\pi_j(n)\}$. We then have:

THEOREM 1. *The conditional distribution function $F_W(\cdot)$ of the waiting time W (time from arrival until the start of service) of an arbitrary customer, given that he joins the queue, is*:

$$(6)\qquad F_W(x) = 1 - [1/\textstyle\sum_{i=0}^{s+n} D_i\pi_i(n)]\left[\sum_{j=s}^{s+n} D_j\pi_j(n)\left(\sum_{k=0}^{j-s} e^{-s\mu x}\frac{(s\mu x)^k}{k!}\right)\right],$$

and the conditional expected waiting time is given by:

$$(7)\qquad EW = \left(s\mu \textstyle\sum_{i=0}^{s+n} D_i\pi_i(n)\right)^{-1} \sum_{j=s}^{s+n} (j - s + 1)D_j\pi_j(n).$$

PROOF. The proof is similar to the ones given in [15] and [18] for related models and therefore will be omitted.

REMARK. It is readily seen that results (6) and (7) may be extended to the $GI/M/s$ process as well.

3. Two Imbedded Markov Chains

A slightly different avenue of approach for the analysis of the $GI/M/s$ process with balking is to imbed it at instants of *joining* rather than at *instants of arrival.* This is the approach that was taken by Homma in [10]. Consider the limiting probabilities $\{\pi_j\}$ (or $\{\pi_j(n)\}$) of the process imbedded at instants of arrival and let $\{P_j\}$ ($\{P_j(n)\}$) denote the corresponding stationary limiting probabilities of the process imbedded at instants of *joining*; i.e., P_j is the probability that the system is in state j when joining occurs. We then have:

THEOREM 2. $P_j = D_j\pi_j/\sum_i D_i\pi_i$ *for all* j.

PROOF. Let A_j be the event, "An arrival finds the system in state j," and let C be the event, "An arrival joins the queue." Clearly, $P(A_j) = \pi_j$ and $P(C \mid A_j) = D_j$. Since the $\{A_j\}$'s form a partition of the sample space, and $P_j = P(A_j \mid C)$, the result follows immediately by using Bayes' theorem.

Note. Since $\sum_i D_i\pi_i \leqq 1$, $P_j \geqq D_j\pi_j$ for every j, where equality holds if and only if $D_i = 1$ for all i. In particular, if $D_k < 1$ for some k then $P_j > D_j\pi_j$ for *all* relevant j. This is clearly the case for the $GI/M/s/n$ process.

4. Customers' Optimal Joining Rules

We associate with our queueing process a linear cost-reward structure as follows:

(i) A reward $G \geqq 0$ is obtained by each customer upon successful completion of his service.

(ii) A service charge (= toll) θ is paid by the customer to the service agency for the service rendered.

(iii) The waiting costs for each arrival who joins the queue are incurred at the rate of $c \geqq 0$ per unit time.

(iv) The decision to balk is accompanied with a fixed penalty l. We let $g = G - \theta$

be the *net reward* of a customer who has been served and we let $b = g + l$ be the *benefit* of a customer who joins the queue. In order to eliminate trivialities we assume that G, θ, C and l are finite and that $b \geqq c/\mu$. (This very last assumption will be clarified later.)

Our objective now is to find rules that maximize the *net* benefit of the customers. We distinguish between two cases. (a) Self-optimization, i.e. every customer finds his own optimal "balking or joining" rule. (However, since G, θ, c and l are equal for all customers, the same optimal rule applies for all.) (b) Social (or public) optimization, i.e., we consider the long-run average expected net benefit of *all* customers and obtain the balking rule that maximizes this objective function.

More specific, a *rule* or *policy*, R, for controlling the system is a set of functions $\{D_k^R(H_{m-1}, \eta_m)\}$, $m = 0, 1, \cdots$, where $H_m = \{\eta_0, \Delta_0, \cdots, \eta_m, \Delta_m\}$, $D_k^R(\cdot) \geqq 0$ and $\sum_{k=0}^1 D_k^R(\cdot) = 1$; and where $D_k^R(H_{m-1}, \eta_m)$ is to be interpreted as the probability of implementing decision k $(k = 0, 1)$ at time m given the "history" H_{m-1} and the present state η_m. (However, in all the preceding sections we have assumed that

$$D_k^R(H_{m-1}, \eta_m \mid \eta_m = i) = D_k^R(H_{m-1}, i) = D_{ik}$$

for every $m = 0, 1, 2 \cdots$ and thus we have obtained the Markov chains represented by (3) and (5).)

Let B_m, $m = 0, 1, \cdots$, denote the net benefit obtained at time τ_m, and define its stationary conditional expectation:

$$E\{B_m \mid \eta_m = i, \Delta_m = k\} = b_{ik}.$$

Since (for any m) a customer who joins the queue when the system is in state i spends there an expected total time of $1/\mu$ if $i < s$, and an expected total time of

$$(i - s + 1)/s\mu + 1/\mu = (i + 1)/s\mu$$

if $i \geqq s$, we immediately have:

$$\begin{aligned} b_{i0} &= -l, \quad \text{for all} \quad i, \\ b_{i1} &= g - \frac{c}{\mu}, \quad \text{if} \quad i < s, \qquad (8) \\ &= g - \frac{c}{s\mu}(i+1), \quad \text{if} \quad i \geqq s. \end{aligned}$$

Before proceeding to find optimal joining rules we note that because of the Markovian property of the service times reneging is never optimal; i.e., once having decided to join the queue it never pays to leave the system before service is completed.

Self-Optimization

We wish to find a rule $R = \{D_k^R(\cdot)\}$ consisting of joining probabilities $\{D_1^R(H_{m-1}, i)\}$ (recall that $D_0^R(\cdot) = 1 - D_1^R(\cdot)$) such that for every history H_{m-1} and every i

$$E(B_m \mid \eta_m = i) = D_1^R(H_{m-1}, i)b_{i1} + [1 - D_1^R(H_{m-1}, i)]b_{i0}$$

is maximized.

By (8) we have

$$\begin{aligned} E(B_m \mid \eta_m = i) &= D_1^R(H_{m-1}, i)\left(g - \frac{c}{\mu}\right) - [1 - D_1^R(H_{m-1}, i)]l, \quad \text{if} \quad i < s, \\ &= D_1^R(H_{m-1}, i)\left[g - \frac{c}{s\mu}(i+1)\right] - [1 - D_1^R(H_{m-1}, i)]l, \text{ if } i \geqq s. \end{aligned}$$

Since $b_{i1} = g - c(i + 1)/\mu$ is a decreasing function of i then, for $i > s$, there exists a *smallest integer* $n(s) \geqq 0$, defined by $i = s + n(s)$, for which balking is better than joining. That is, $b_{s+n(s),1} < -l$ whereas $b_{s+n(s)-1,1} \geqq -l$. It follows then that $n(s)$ satisfies

$$g - \frac{c}{s\mu}(s + n(s) + 1) < -l \leqq g - \frac{c}{s\mu}(s + n(s)). \tag{9}$$

It is therefore clear that the rule

$$R = \{D_1^R(\cdot) \mid D_1^R(H_{m-1}, i) = 1, i \leqq s + n(s) - 1; D_1^R(H_{m-1}, i) = 0, \\ i \geqq s + n(s)\} \tag{10}$$

is the one which maximized $E(B_m \mid \eta_m = i)$ for every history H_{m-1} and every i. Note that if strict inequality holds in (9) then (10) is the unique optimal policy.

From (9) we immediately obtain

$$\frac{s\mu}{c}\left(b - \frac{c}{\mu}\right) - 1 < n(s) \leqq \frac{s\mu}{c}\left(b - \frac{c}{\mu}\right) \tag{11}$$

and since rule (10) is applied by all customers the process reduces to the $GI/M/s/n(s)$ process with balking sequence $\{D_i \mid D_i = 1, i < s + n(s); D_i = 0, i \geqq s + n(s)\}$. Such a sequence will be called a *deterministic control limit rule* [18] with *control limit* being equal to $s + n(s)$. The interpretation is obvious; the rule says: Join if and only if the number of customers in the system is less than the control limit, $s + n(s)$.

As noted in [18] it is readily seen that the control limit rule (10) is independent of the arrival distribution. This is clearly understandable since each individual customer observes—upon arrival—only a "local" situation, i.e., the current state of the system, and is not interested in the forthcoming customers. However, if the arrival pattern is taken into consideration one might suspect that a better optimum—a "global" one—may be achieved. This is indeed the case, as will be shown in the following section.

Public Optimization

Our objective now is to find a rule R which maximizes the long-run average expected net benefit of the entire customers' population. For any given policy R and an initial state $\eta_0 = j$ the expected net benefit at instant τ_m is:

$$E_R(B_m \mid \eta_0 = j) = \sum_i \sum_{k=0}^{1} b_{ik} P_R(\eta_m = i, \Delta_m = k \mid \eta_0 = j), \tag{12}$$

where E_R and P_R denote the expectation and probability under the policy R. Let

$$\Phi_R(j) = \lim_{T\to\infty} \sup \left[\frac{1}{T+1} \sum_{m=0}^{T} E_R(B_m \mid \eta_0 = j)\right] \tag{13}$$

be the long-run average expected net benefit.

For public optimization we seek the rule $R \in C_s$ that maximizes $\Phi_R(j)$ for all j, where C_s is the class of all Markovian stationary policies $R = \{D_i \mid 0 \leqq D_i \leqq 1, \text{all } i\}$. (Recall that we have restricted ourselves to this class by assuming that the only information obtainable by a newly arriving customer is the current state of the system.)

Consider now the class C_{DCL} of all deterministic control limit rules R_k^D, where we say that $R_k^D \in C_{DCL}$ if $R_k^D = \{D_i \mid D_i = 1, i < k; D_i = 0, i \geqq k\}$ for some $k = 1, 2, 3, \cdots$ and we let $R_\infty^D = \{D_i \mid D_i = 1, \text{for all } i = 0, 1, 2, \cdots\}$. Clearly, $C_{DCL} \subset C_s$. In [18] it has been shown for the $GI/M/1$ process that for the infinite horizon average expected net benefit criterion there exists an optimal policy $R \in C_{DCL}$ which maximizes $\Phi_R(j)$ for all j. Moreover, it has been shown that $R_\infty{}^D$ may be considered only if the system under this rule is ergodic, i.e. only if $\lambda < \mu$. However, the same arguments may be carried out for the $GI/M/s$ case and therefore, in searching for an optimal policy for social optimization in C_s, it is sufficient to consider only rules $R_k{}^D \in C_{DCL}$ where $R_\infty{}^D$ may be considered only if $\lambda < s\mu$. Moreover, since $b \geqq c/\mu$ (i.e., $g - c/\mu \geqq -l$) implies that the expected net benefit for an individual customer is always nonnegative if he joins the system when it is in state $i < s$, we may consider only rules $R_k{}^D$ with $k \geqq s$.

Now, for any $k \geqq s$ and rule $R_k = \{D_i \mid 0 < D_i, i < k; D_i = 0, i \geqq k\}$ applying Chung's well-known theorem [2, pp. 92–94] to expressions (12) and (13) yields, independently of the initial state $\eta_0 = j$,

$$\Phi_{R_k} = \Phi_{R_k}(j) = \textstyle\sum_{i=0}^{k} \pi_i(R_k)[D_i b_{i1} + (1 - D_i)b_{i0}] \tag{14}$$

where the $\{\pi_i(R_k)\}$'s are the corresponding limiting probabilities of the process under R_k and they satisfy (4).

Alternatively, since the expected total time spent in the system by a customer who joins is $EW + 1/\mu$, the expected net benefit of an arbitrary arrival is:

$$\Phi_{R_k} = \textstyle\sum_{i=0}^{k} D_i\pi_i(R_k)\left[g - c\left(EW + \frac{1}{\mu}\right)\right] - [1 - \sum_{i=0}^{k} D_i\pi_i(R_k)]l \tag{15}$$

where $\sum_{i=0}^{k} D_i\pi_i(R_k)$ is the (unconditional) probability of joining. Using (8) for (14) or result (7) of Theorem 1 for (15) yields:

$$\Phi_{R_k} = \textstyle\sum_{i=0}^{s-1} D_i\pi_i(R_k)\left[b - \frac{c}{\mu}\right] + \sum_{i=s}^{k} D_i\pi_i(R_k)\left[b - \frac{c}{s\mu}(i + 1)\right] - l. \tag{16}$$

In particular, if we consider $R_k{}^D$ instead of R_k we have ($D_k = 0$):

$$\Phi_{R_k{}^D} = \textstyle\sum_{i=0}^{s-1} \pi_i(R_k{}^D)\left[b - \frac{c}{\mu}\right] + \sum_{i=s}^{k-1} \pi_i(R_k{}^D)\left[b - \frac{c}{s\mu}(i + 1)\right] - l. \tag{17}$$

We now show that the optimal rule for public optimization is a control limit rule with finite control limit, $s + n(p)$, and that self-optimization need not bring upon social optimization. That is, we show that $\Phi_{R^D_{s+n(p)}} = \sup_{R \in C_s} \Phi_R$ which implies that $\Phi_{R^D_{s+n(p)}} \geqq \Phi_{R^D_{s+n(s)}}$. For this purpose we use Howard's algorithm [11] and start the "Value Determination Operation" with a policy

$$R^D_{s+n(s)} = \{D_i \mid D_i = 1, i < s + n(s); D_i = 0, s + n(s) \leqq i \leqq s + n\},$$

where, by the remarks above, we may assume that the system is physically limited to a waiting room of size $n > n(s)$. We thus have to find $\Phi, v_0, v_1, \cdots, v_{s+n}$ that satisfy:

$$\begin{aligned}
\Phi + v_i &= g - \frac{c}{\mu} + \textstyle\sum_{j=0}^{i+1} q_{ij}(1)v_j, \quad 0 \leqq i \leqq s - 1,\\
\Phi + v_i &= g - \frac{c}{s\mu}(i + 1) + \textstyle\sum_{j=0}^{i+1} q_{ij}(1)v_j, \quad s \leqq i \leqq s + n(s) - 1,\\
\Phi + v_i &= -l + \textstyle\sum_{j=0}^{i} q_{ij}(0)v_j, \quad s + n(s) \leqq i \leqq s + n.
\end{aligned} \tag{18}$$

We need the following

LEMMA 1. *For any solution of* (18) *we have*:

$$v_0 \geqq v_1 \geqq \cdots \geqq v_{s+n(s)-1} \geqq v_{s+n(s)} \geqq \cdots \geqq v_{s+n} . \tag{19}$$

PROOF. The details of the proof will be omitted since it is similar to the one given in [18] for the $GI/M/1$ queue. In principle, it is an inductive proof in three steps. We first subtract the $(s + n(s) + 1)$st row of (18) from the $(s + n(s))$th row to obtain: $v_{s+n(s)-1} \geqq v_{s+n(s)}$. Then we use backward induction to show that $v_0 \geqq v_1$, and finally, a forward induction is employed to show that $v_{s+n-1} \geqq v_{s+n}$.

The interpretation of the result of this lemma is probably better expressed by quoting Howard who remarks that "the difference in the relative values $v_i - v_{i+1}$ is equal to the amount that a rational man would be just willing to pay in order to start his transitions from state i rather than state $i + 1$ if he is going to operate the system for many, many transitions." This interpretation makes it clear why, for the $R^D_{s+n(s)}$ policy, (19) always holds.

We can now prove the following:

THEOREM 3. $\exists n(p) \leqq n(s) \ni \Phi_{R^D_{s+n(p)}} = \sup_{R \in C_s} \Phi_R$.

PROOF. Since only deterministic control limit rules need to be considered, it suffices to show that a rule R^D_{s+n} for some $n > n(s)$ is never an improvement on $R^D_{s+n(s)}$. To show this we define for every i,

$$g_i(k) = b_{ik} + \textstyle\sum_j q_{ij}(k) v_j , \qquad k = 0, 1,$$

and therefore, it suffices to show (using Howard's "Policy Improvement Routine" after starting with $R^D_{s+n(s)}$ and obtaining the corresponding Φ, $v_0, v_1, \cdots, v_{s+n}$) that $g_{s+i}(0) > g_{s+i}(1)$ for all $n(s) \leqq i < n$. Using (18) and the fact that $q_{s+i,j}(1) = q_{s+i+1,j}(0)$, we obtain:

$$\begin{aligned} g_{s+i}(1) - g_{s+i}(0) &= g - \frac{c}{s\mu}(s + i + 1) + \textstyle\sum_{j=0}^{s+i+1} q_{s+i+1,j}(0) v_j \\ &\qquad - [-l + \textstyle\sum_{j=0}^{s+i} q_{s+i,j}(0) v_j] \\ &= g - \frac{c}{s\mu}(s + i + 1) + (\Phi + v_{s+i+1} + l) - [-l + (\Phi + v_{s+i})] \\ &= g - \frac{c}{s\mu}(s + i + 1) + l + v_{s+i+1} - v_{s+i} . \end{aligned}$$

From (9) it follows that for $i \geqq n(s)$, $g - c(s + i + 1)/s\mu + l < 0$. From Lemma 1 it follows that $v_{s+i+1} - v_{s+i} \leqq 0$. Hence, $g_{s+i}(1) < g_{s+i}(0)$ for $n(s) \leqq i < n$ and therefore R^D_{s+n} is worse than $R^D_{s+n(s)}$. That is, $\Phi_{R^D_{s+n(s)}} > \Phi_{R^D_{s+n}}$ for all $n > n(s)$. This implies immediately that $\Phi_{R^D_{s+n(s)}} \geqq \Phi_{R_\infty^D}$. Hence, $\exists n(p) \leqq n(s)$ such that

$$\Phi_{R^D_{s+n(p)}} = \sup_{R \in C_{DCL}} \Phi_R = \sup_{R \in C_s} \Phi_R . \qquad \text{Q.E.D.}$$

As was pointed out in [18], in most cases we will have $n(p) < n(s)$ which implies that acting individually seldom optimizes public good.

Formulation as a Linear Program

$n(p)$ could now be calculated by solving a linear program. The relations between Markovian decision processes and linear programming are well known and following [13], [3], [17] and [18] the problem of maximizing Φ as given by (16) can be written in the following way:

$$\text{Maximize}\left\{\sum_{i=0}^{s-1}\chi_{i1}\left(b-\frac{c}{\mu}\right)+\sum_{i=s}^{n(s)}\chi_{i1}\left(b-\frac{c}{s\mu}(i+1)\right)\right\}$$

$$\text{Subject to } \sum_{k=0}^{1}\chi_{jk}-\sum_{i=0}^{n(s)}\sum_{k=0}^{1}\chi_{ik}q_{ij}(k)=0, \qquad j=0,1,\cdots,n(s),$$

$$\sum_{j=0}^{n(s)}\sum_{k=0}^{1}\chi_{ik}=1, \qquad \chi_{ik}\geqq 0, \qquad i=0,1,\cdots,n(s);\ k=0,1,$$

where

$$\chi_{ik}=\pi_i(R^D_{s+n(s)})D_{ik}; \qquad i=0,1,\cdots,n(s);\ k=0,1,$$

i.e.

$$\chi_{i1}=\pi_i(R^D_{s+n(s)})D_i, \quad \chi_{i0}=\pi_i(R^D_{s+n(s)})(1-D_i).$$

Since, as was shown in [17], at most $n(s)+1$ variables χ_{jk} will assume positive values in the optimal solution, and since $\chi_{i1}=0 \Rightarrow \pi_i=0$, $n(p)$ is readily obtained from:

$$n(p)=\max\{i \mid \chi_{i1}>0\}. \tag{20}$$

5. Example with a "Direct" Proof: The $M/M/s$ Queue

For the $M/M/s$ queue with stationary balking sequence $\{D_i \mid 0<D_i\leqq 1, i<s+n;\ D_i=0, i\geqq s+n; n>n(s)\}$ a "direct" proof of Theorem 3 may be given. It is well known [12] that from the set of Kolmogorov's backward differential equations written for this process, the set of limiting probabilities $\{\pi_i(n)\}$ may be expressed as a function of $\pi_0(n)$ in the following way:

$$\begin{aligned}\pi_k(n)&=\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k\left(\prod_{i=0}^{k-1}D_i\right)\pi_0(n), && k=1,2,\cdots,s,\\ \pi_{s+k}(n)&=\left(\frac{\lambda}{s\mu}\right)^k\left(\prod_{i=s}^{s+k-1}D_i\right)\pi_s(n), && k=1,2,\cdots,n.\end{aligned} \tag{21}$$

Letting $\sum_{i=1}^{s+n}\pi_i(n)=1$ we obtain

$$\pi_0(n)=\left[1+\sum_{k=1}^{s}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k\left(\prod_{i=0}^{k-1}D_i\right)+\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\sum_{k=1}^{n}\left(\frac{\lambda}{s\mu}\right)^k\left(\prod_{i=0}^{s+k-1}D_i\right)\right]^{-1}. \tag{22}$$

If we consider now a sequence of probabilities $\{d_i \mid 0<d_i\leqq 1\}$, $i=0,1,2,\cdots$, and generate a sequence of control limit rules $R_k=\{D_i \mid D_i=d_i, i<k; D_i=0, i\geqq k\}$, $k=1,2,\cdots$, then for any rules R_{s+n} and R_{s+n+1} belonging to this sequence we have:

LEMMA 2. *$\pi_i(n)>\pi_i(n+1)$ for $i=0,1,2,\cdots,n$ and all $n=1,2,\cdots$.*

PROOF. From (21) it is clear that $\pi_i(n)>\pi_i(n+1)$ if and only if $\pi_0(n)>\pi_0(n+1)$. From (22) it follows that $\pi_0(n)>\pi_0(n+1)$ if and only if

$$\sum_{k=1}^{n}\left(\frac{\lambda}{s\mu}\right)^k\left(\prod_{i=0}^{s+k-1}d_i\right)<\sum_{k=1}^{n+1}\left(\frac{\lambda}{s\mu}\right)^k\left(\prod_{i=0}^{s+k-1}d_i\right).$$

Since $(\lambda/s\mu)^{n+1}\left(\prod_{i=0}^{s+n}d_i\right)>0$ the proof is complete.

With the aid of Lemma 2, Theorem 3 can now be proved directly. We show that for any $n>n(s)$, $\Phi_{R_{s+n(s)}}>\Phi_{R_{s+n}}$ for all control limit rules $\{R_{s+n}\}$ (which obviously contain all deterministic control limit rules).

Using (16) we have:

$$\Phi_{R_{s+n(s)}} - \Phi_{R_{s+n}} = \sum_{i=0}^{s-1} d_i \left(b - \frac{c}{\mu}\right)[\pi_i(n(s)) - \pi_i(n)]$$

$$(23) \qquad + \sum_{i=s}^{s+n(s)-1} d_i \left[b - \frac{c}{s\mu}(i+1)\right][\pi_i(n(s)) - \pi_i(n)]$$

$$- \sum_{i=s+n(s)}^{s+n-1} d_i \left[b - \frac{c}{s\mu}(i+1)\right]\pi_i(n).$$

Since $b - c(i+1)/s\mu \geqq 0$ for $i \leqq s + n(s) - 1$, and is < 0 for $i \geqq s + n(s)$, the right-hand side of (23) is comprised of three nonnegative sums of which at least one is positive. This shows that $\Phi_{R_{s+n(s)}} > \Phi_{R_{s+n}}$ for all $n > n(s)$ and only rules R_{s+k} with $k \leqq n(s)$ need be considered. Clearly, as it is well known for finite state Markovian decision processes, an optimal rule may be found among the nonrandomized rules and hence it suffices to search through the $n(s) + 1$ deterministic control limit rules $\{R_{s+k}^D, k = 0, 1, \cdots n(s)\}$, as was pointed out earlier for the general case.

6. Remarks on Optimal Toll Charges

In [14] and [18] it was shown for the $M/M/1$ and $GI/M/1$ queues, respectively, that—permitting every customer to act individually—an overall optimality still may be achieved by levying an extra toll, θ_0, such that the new $n(s)$—calculated from (11) with $(\theta + \theta_0)$ replacing θ—will be equal to $n(p)$ as given by (20). Following the same arguments it is readily obtained that θ_0 is given by

$$b - \frac{c}{s\mu}[n(p) + 2] < \theta_0 \leqq b - \frac{c}{s\mu}[n(p) + 1].$$

The new toll at the level of $\theta + \theta_0$ will cause the selfish customers to act according to the overall optimal criteria, that is, their new control limit will be smaller and the (average) queue size will be reduced to the desired magnitude.

Station optimization may also be studied in the framework of a "competition" model between the service agency—which is then considered as a profit-making organization—and the customers, whether acting individually or collectively. It is then assumed that the service agency's objective is to maximize its long-run average revenue, i.e., to find θ_r so as to maximize $[1 - \pi_{s+n}(R_{s+n}^D)]\theta$ over all possible θ, where n is a function of θ and is either $n(s)$ or $n(p)$. As in [18] it can be shown that the situation is analogous to the monopoly model of Price Theory where the "demand" function—that is, $n(s)$ or $n(p)$—is completely known to the service station—the monopolist—for any given θ.

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