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AN $M/M/1$ QUEUE IN RANDOM ENVIRONMENT WITH DISASTERS

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We study a $M/M/1$ queue in a multi-phase random environment, where the system occasionally suffers a disastrous failure, causing all present jobs to be lost. The system then moves to a repair phase. As soon as the system is repaired, it moves to phase i with probability $q_i \geq 0$. We use two methods of analysis to study the probabilistic behavior of the system in steady state: (i) via probability generating functions, and (ii) via matrix geometric approach. Due to the special structure of the Markov process describing the disaster model, both methods lead to explicit results, which are related to each other. We derive various performance measures such as mean queue sizes, mean waiting times, and fraction of lost customers. Two special cases are further discussed.

Keywords: $M/M/1$ queue; random environment; disasters; lost customers; generating functions; matrix geometric.

1. Introduction

The $M/M/1$ queue in random environment, where the underlying environment is a n-phase continuous-time Markov chain (MC), has been studied intensively by various authors (see e.g., Gupta *et al.*, 2006; Neuts, 1981; Yechiali and Naor, 1971; Yechiali, 1973). In the present work, we further consider such a system, but one that suffers random disastrous failures (catastrophes, see e.g., Artalejo and Gomez-Coral,

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1999; Chakravarthy, 2009; Sengupta, 1990; Sudhesh, 2010; Yechiali, 2006). When occurs, a failure causes all present jobs to be cleared (flushed) out of the system and lost. The system itself then moves to a repair phase (phase 0) that its duration is exponentially distributed. Being repaired, the system moves to an operative phase $i \geq 1$ with probability $q_i \geq 0$, where $\sum_{i=1}^{n} q_i = 1$. Such a model may represent a manufacturing process or a storage process (Kella and Whitt, 1992) operating in randomly changing environment where a severe failure causes all units in process to be discarded (see e.g., Artalejo and Gomez-Coral, 1999; Sudhesh, 2010; Yechiali, 2006). The model can also represent a dynamically changing road traffic evolution when an incident (such as an accident or a broken traffic light) requires that all present vehicles be directed to another roadway. In Sudhesh (2010), the author argues that "queueing models with disasters seem to be appropriate in some computer network applications or telecommunications applications that depend on satellites or in Internet applications".

The paper is constructed as follows: In Sec. 2, we describe the model and define the two-dimensional stochastic process underlying its dynamics. In Sec. 3, we construct the balance equations for the system's steady-state probabilities and calculate the fraction of time the system resides in phase i $(0 \leq i \leq n)$. Probability generating functions (PGFs) are employed in Sec. 4, while the roots of corresponding quadratic functions are explicitly calculated and used to derive explicit solutions for the probabilities that the system is in phase i and no customers are present. In Sec. 5, we use matrix geometric analysis. Due to the special structure of the model, key matrices are upper diagonal which enables explicit and direct calculation of the basic matrix R, without the usual iteration procedure. The connection between the roots related to the PGFs and the entries of the basic matrix R are revealed in Sec. 6, along with other relationships. In Secs. 7 and 8, various performance measures, such as mean queue sizes, mean sojourn times and mean number of customers lost per unit time are calculated. Finally, two special cases are investigated.

2. The Model

Consider a $M/M/1$ type queue operating in a special "random environment" as follows. The underlying environment is an $(n+1)$-dimensional continuous-time MC, with phases $i = 0, 1, 2, \ldots, n$, governed by the matrix $[q]$ of transition probabilities:

$$[q] = \begin{pmatrix} 0 & q_1 & q_2 & \ldots & q_n \\ 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

When in operative phase $i \geq 1$ the system acts as an $M(\lambda_i)/M(\mu_i)/1$ queue, with Poisson arrival rate $\lambda_i \geq 0$ and service rate $\mu_i \geq 0$. The duration of time the system resides in phase i is an exponentially distributed random variable with mean $1/\eta_i$, $i = 1, 2, \ldots, n$.

Furthermore, when in operative phase $i \geq 1$, the system suffers occasionally a disastrous failure (catastrophe), causing it to move to a "failure" phase, denoted by $i = 0$. A disaster causes all present customers (jobs) to be cleared (flushed) out of the system. When in the failure phase $i = 0$, the system undergoes a repair process, having exponentially distributed duration with mean $1/\eta_0$. The arrival process continues with rate $\lambda_0 \geq 0$ but no service is rendered, i.e., $\mu_0 = 0$. When the system is repaired, it jumps from the failure phase to some operative phase $i \geq 1$ with probability $q_i \geq 0$, $\sum_{i=1}^{n} q_i = 1$. That is, there are no direct moves from phase $i \geq 1$ to phase $j \geq 1$: in each "active" phase the system stays until a breakdown occurs, which sends it to phase 0. Only then, after a repair duration, the system can move back to one of the operating phases $i = 1, 2, \ldots, n$.

A stochastic process $\{U(t), X(t)\}$ describes the system's state at time t as follows: $U(t)$ denotes the phase in which the system operates at time t, while $X(t)$ counts the number of customers present in the system at that time. The system is said to be in state (i, m) if it is in phase i, and there are m customers in the system. Accordingly, let p_{im} be the steady-state probability of the system in state (i, m). That is, $p_{im} = \lim_{t\to\infty}\{P(U(t) = i, X(t) = m)\}$ $\forall\, t \geq 0$, $0 \leq i \leq n$, $m = 0, 1, 2, \ldots$. Figure 1 below depicts a transition-rate diagram of the above queueing system (see Note below).

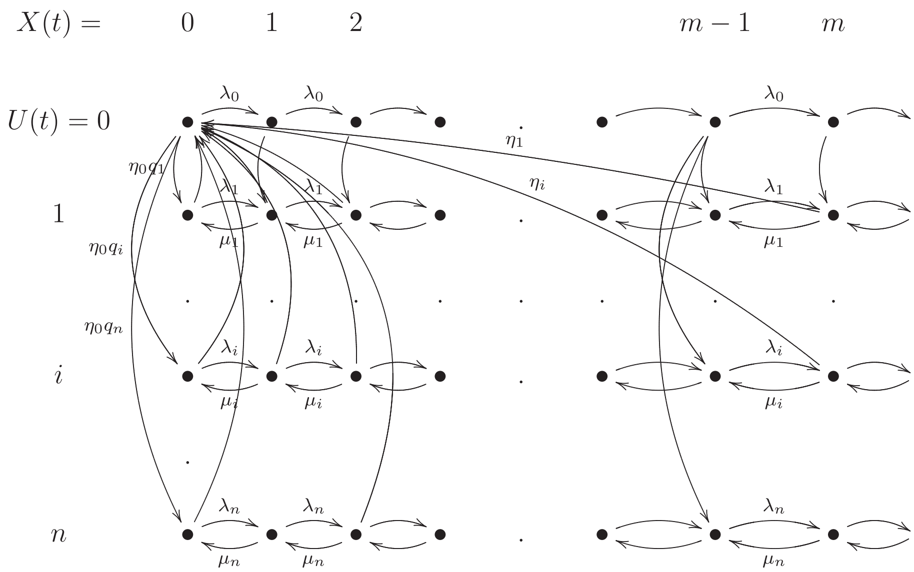

Fig. 1. A transition-rate diagram of a single-server queueing system in random environment and with system failures. In case of a disaster the system moves to state $(0, 0)$. After the system is repaired, it moves to phase i with probability q_i.

Note: For clarity of exposition, not all transitions are shown.

3. Balance Equations

The system's steady-state balance equations are given as follows:

For the failure phase $i = 0$, while $m = 0$,

$$(\lambda_0 + \eta_0)p_{00} = \sum_{i=1}^{n} \eta_i \sum_{m=0}^{\infty} p_{im} = \sum_{i=1}^{n} \eta_i p_{i\bullet}, \tag{1}$$

and while $m \geq 1$,

$$(\lambda_0 + \eta_0)p_{0m} = \lambda_0 p_{0,m-1}. \tag{2}$$

For $i = 1, 2, \ldots, n$ and $m = 0$,

$$(\lambda_i + \eta_i)p_{i0} = \mu_i p_{i1} + \eta_0 q_i p_{00}, \tag{3}$$

and when $m \geq 1$,

$$(\lambda_i + \mu_i + \eta_i)p_{im} = \lambda_i p_{i,m-1} + \mu_i p_{i,m+1} + \eta_0 q_i p_{0m}. \tag{4}$$

From (1) and (2) we get that

$$p_{0m} = \left(\frac{\lambda_0}{\lambda_0 + \eta_0}\right)^m p_{00} \quad m \geq 0, \tag{5}$$

implying that

$$p_{0\bullet} = \frac{\lambda_0 + \eta_0}{\eta_0} p_{00}, \tag{6}$$

where $p_{i\bullet} = \sum_{m=0}^{\infty} p_{im}$, $i = 0, 1, 2, \ldots, n$.

The limit probabilities of the underlying MC Q, $d_j = \lim_{t\to\infty}\{P(U(t) = j)\}$, satisfy $\sum_{j=0}^{n} d_j = 1$, $d_0 = \sum_{j=1}^{n} d_j$, and $d_j = d_0 q_j$ for $j \geq 1$. Therefore, $d_0 = \frac{1}{2}$, and $d_j = \frac{q_j}{2}$ for $j \geq 1$. (Intuitively, $d_0 = \frac{1}{2}$ since the MC constantly alternates between phase $i = 0$ and one of the other phases $j \geq 1$, and thus visits phase 0 half of the times). Hence, the proportion of time the system resides in phase i is given by

$$\begin{aligned} p_{i\bullet} &= \frac{\frac{d_i}{\eta_i}}{\sum_{k=0}^{n} \frac{d_k}{\eta_k}} = \frac{\frac{q_i}{\eta_i}}{\frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k}} \quad 1 \leq i \leq n, \\ p_{0\bullet} &= \frac{\frac{d_0}{\eta_0}}{\sum_{k=0}^{n} \frac{d_k}{\eta_k}} = \frac{\frac{1}{\eta_0}}{\frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k}} = \frac{1}{\alpha\eta_0}, \end{aligned} \tag{7}$$

where

$$\alpha = \frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k}.$$

From (7) it follows that

$$\eta_i p_{i\bullet} = \eta_0 q_i p_{0\bullet} \quad i = 1, 2, \ldots, n. \tag{8}$$

Now, given $p_{0\bullet}$, p_{00} is calculated from (6), namely,

$$p_{00} = \frac{1}{\alpha(\lambda_0 + \eta_0)}, \tag{9}$$

and all p_{0m}, for $m \geq 0$, are explicitly determined by (5).

We note that because of the disaster effect, the system is positive recurrent.

4. Generating Functions

Define a set of (partial) PGFs:

$$G_i(z) = \sum_{m=0}^{\infty} p_{im} z^m \quad i = 0, 1, \ldots, n \;\; (0 \le z \le 1). \tag{10}$$

For $i = 0$, using Eqs. (1) and (2), we get

$$(\lambda_0(1-z) + \eta_0) G_0(z) = \sum_{i=1}^{n} \eta_i p_{i\bullet}. \tag{11}$$

Writing $G_i(1) = p_{i\bullet}$ and setting $z = 1$ in (11) we have

$$\eta_0 p_{0\bullet} = \sum_{i=1}^{n} \eta_i p_{i\bullet}. \tag{12}$$

Equation (12) reflects the fact that, when failure occurs in some phase $i \ge 1$, the system always moves to phase $i = 0$.

Alternatively, using (5),

$$G_0(z) = \sum_{m=0}^{\infty} p_{0m} z^m = p_{00} \sum_{m=0}^{\infty} \left(\frac{\lambda_0 z}{\lambda_0 + \eta_0} \right)^m = p_{00} \frac{\lambda_0 + \eta_0}{\lambda_0(1-z) + \eta_0}. \tag{13}$$

Using (6) and (12) brings us back to Eq. (11).

For $i \ge 1$, using (3) and (4) and arranging terms, we get

$$(\lambda_i(1-z)z + \mu_i(z-1) + \eta_i z) G_i(z) - \eta_0 q_i z G_0(z) = \mu_i(z-1) p_{i0}. \tag{14}$$

Now, given $G_0(z)$ in Eq. (11), each $G_i(z)$ in (14) is fully determined once p_{i0} is known, for $1 \le i \le n$. Define

$$\begin{aligned} f_0(z) &= \lambda_0(1-z) + \eta_0, \\ f_i(z) &= (\lambda_i z - \mu_i)(1-z) + \eta_i z \quad i \ge 1. \end{aligned} \tag{15}$$

The quadratic polynomials $f_i(z)$, $i \ge 1$, each have two real roots. Let z_i denote the (only) root of $f_i(z)$ in the interval $(0, 1)$. This follows since

$$f_i(0) = -\mu_i < 0, \quad f_i(1) = \eta_i > 0, \quad f_i(\pm\infty) < 0.$$

The root $z_i \in (0, 1)$ is given by

$$z_i = \frac{(\lambda_i + \mu_i + \eta_i) - \sqrt{(\lambda_i + \mu_i + \eta_i)^2 - 4\lambda_i \mu_i}}{2\lambda_i}. \tag{16}$$

In fact, z_i represents the Laplace–Stieltjes Transform (LST), evaluated at point η_i, of the busy period in a $M/M/1$ queue with arrival rate λ_i and service rate μ_i. From

Eqs. (11), (12) and (14) it follows that

$$p_{i0} = \frac{\eta_0 q_i z_i G_0(z_i)}{\mu_i(1-z_i)} = \frac{\eta_0^2 p_{0\cdot} q_i z_i}{[\lambda_0(1-z_i)+\eta_0]\mu_i(1-z_i)} \quad 1 \le i \le n. \tag{17}$$

Finally, with p_{i0} given in (17), each PGF $G_i(z)$ is completely determined by Eqs. (11) and (14). Now any probability p_{im} can be calculated, either by differentiation of $G_i(z)$ at $z = 0$, or recurssively from Eqs. (2) to (5).

5. Matrix Geometric

Another approach to analyze our system is by using Neuts' (Neuts, 1981) Matrix Geometric approach. This analysis also reveals the relation between the roots z_i and the entries of Neuts' matrix R (see below). Specifically, the continuous-time MC underlying the queueing process has the following rate matrix Q

$$Q = \begin{pmatrix} B + A_2 + A_1 & A_0 & 0 & 0 & \dots \\ B + A_2 & A_1 & A_0 & 0 & \dots \\ B & A_2 & A_1 & A_0 & \dots \\ B & 0 & A_2 & A_1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \eta_1 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \eta_n & 0 & \dots & 0 \end{pmatrix},$$

$$A_0 = \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_n),$$

$$A_1 = \begin{pmatrix} -\lambda_0 - \eta_0 & q_1\eta_0 & q_2\eta_0 & \dots & q_n\eta_0 \\ 0 & -\lambda_1 - \eta_1 - \mu_1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & \dots & \dots & -\lambda_{n-1} - \eta_{n-1} - \mu_{n-1} & 0 \\ 0 & \dots & \dots & 0 & -\lambda_n - \eta_n - \mu_n \end{pmatrix},$$

$$A_2 = \mathrm{diag}(0, \mu_1, \dots, \mu_n).$$

The steady-state probability vector $P = (P_0, P_1, \dots)$ of Q, with $P_m = (p_{0m}, p_{1m}, \dots, p_{nm})$ being a $1 \times (n+1)$ vector, has a matrix geometric form:

$$\begin{aligned} &P_m = P_0 R^m, \quad m \ge 0 \\ &P_0(B + A_2 + A_1) + P_0 R(B + A_2) + \left(\sum_{m=2}^{\infty} P_m\right) B = 0 \end{aligned} \tag{18}$$

implying that

$$\sum_{m=0}^{\infty} P_m \underline{e} = P_0 \left(\sum_{m=0}^{\infty} R^m \right) \underline{e} = P_0 (I - R)^{-1} \underline{e} = 1, \tag{19}$$

where the $(n+1) \times (n+1)$ matrix R is the smallest non-negative solution (see Neuts, 1981) of

$$A_0 + RA_1 + R^2 A_2 = 0. \tag{20}$$

Due to the (upper) diagonal structures of the matrices A_0, A_1 and A_2, it follows from Eq. (20) that R is also an upper diagonal having the following form

$$R = \begin{pmatrix} r_{00} & r_{01} & r_{02} & \dots & r_{0n} \\ 0 & r_1 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & \dots & 0 & r_{n-1} & 0 \\ 0 & \dots & \dots & 0 & r_n \end{pmatrix}. \tag{21}$$

Thus, from (20) and (21), the nonzero entries of R can be directly calculated:

$$r_i = \frac{(\lambda_i + \mu_i + \eta_i) - \sqrt{(\lambda_i + \mu_i + \eta_i)^2 - 4\lambda_i \mu_i}}{2\mu_i}, \quad i > 0,$$

$$r_{00} = \frac{\lambda_0}{\lambda_0 + \eta_0},$$

$$r_{0i} = \frac{r_{00} q_i \eta_0}{(\lambda_i + \mu_i + \eta_i) - (r_{00} + r_i)\mu_i}, \quad i > 0.$$

6. Connections Between the Two Approaches

It is important to emphasize the relationship between the roots z_i and the entries r_i. Indeed, a simple relationship exists, namely, $r_i = \frac{\lambda_i}{\mu_i} z_i$. The reader is also directed to another case Zhang and Tian (2004) where the matrix R is upper diagonal and its entries are explicitly calculated. A case where an explicit solution for the corresponding matrix R can be determined may be found in Drekic and Grassmann (2002). A connection between the two methods in terms of calculating the vector of probabilities $P_0 = (p_{00}, p_{10}, \dots, p_{n0})$ is the following: In order to compute the vector P_0 we use the matrix G satisfying $RA_2 = A_0 G$ and is the smallest non-negative solution (see Neuts, 1981) to

$$A_2 + A_1 G + A_0 G^2 = 0. \tag{22}$$

The matrix G has the same structure as R and therefore we have, for $i > 0$, $g_i = \frac{\mu_i r_i}{\lambda_i} = z_i$; $g_{0i} = \frac{\mu_i r_{0i}}{\lambda_0}$; and $g_{00} = 0$. g_i represents the probability of starting in state $(i, 1)$ and returning to state $(i, 0)$ with no disaster, while g_{0i} represents the same probability starting in state $(0, 1)$. Also the mean time spent in state $(0, 0)$

between two consecutive disasters is $\frac{1}{(\lambda_0+\eta_0)}$, and in state $(i,0)$

$$\frac{\lambda_0}{\lambda_0+\eta_0} g_{0i} \frac{1}{\eta_i+\lambda_i(1-g_i)} + \frac{\eta_0}{\lambda_0+\eta_0} q_i \frac{1}{\eta_i+\lambda_i(1-g_i)}.$$

Hence, P_0 is proportional to the vector

$$V = \left(1, \frac{\lambda_0 g_{01}+\eta_0 q_1}{\eta_1+\lambda_1(1-g_1)}, \ldots, \frac{\lambda_0 g_{0n}+\eta_0 q_n}{\eta_n+\lambda_n(1-g_n)}\right). \tag{23}$$

That is, $P_0 = cV$. The coefficient c is now calculated by replacing P_0 with cV in Eq. (19). Thus, c is readily calculated, and so is $P_0 = cV$. In view of (23) $P_{00} = c$. All other probability vectors P_m can now be calculated via (18).

7. Mean Queue Sizes and Mean Customers Cleared Per Unit Time

Let $G_i'(1) = E[L_i] = \sum_{m=1}^{\infty} m p_{im}$, $i = 0, 1, \ldots, n$. Then, taking derivatives of (11) and (14) at $z = 1$, we obtain [using (6) and (7)]

$$E[L_0] = \frac{\lambda_0}{(\frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k})\eta_0^2} = \frac{\lambda_0}{\eta_0}\left(1+\frac{\lambda_0}{\eta_0}\right) p_{00} = \frac{\lambda_0}{\eta_0} p_{0\boldsymbol{.}}, \tag{24}$$

and

$$(-\lambda_i+\mu_i+\eta_i)p_{i\boldsymbol{.}} + \eta_i E[L_i] - \eta_0 q_i (p_{0\boldsymbol{.}} + E[L_0]) = \mu_i p_{i0} \quad i \geq 1. \tag{25}$$

This leads to

$$\begin{aligned} E[L_i] &= \frac{1}{\eta_i}\left[\mu_i p_{i0} + \left(\frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i}\right) \frac{q_i}{\frac{1}{\eta_0} + \sum_{k=1}^{n} \frac{q_k}{\eta_k}}\right] \\ &= \frac{1}{\eta_i}\left[\mu_i p_{i0} + \left(\frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i}\right) p_{i\boldsymbol{.}}\eta_i\right] \quad i \geq 1. \end{aligned} \tag{26}$$

From (17) and (26), the total number of customers in the system is given by

$$\begin{aligned} E[L] &= \sum_{i=0}^{n} E[L_i] \\ &= \frac{\lambda_0}{\eta_0}\left(1+\frac{\lambda_0}{\eta_0}\right) p_{00} + \sum_{i=1}^{n} \frac{1}{\eta_i}\left[\frac{\eta_0^2 p_{0\boldsymbol{.}} q_i z_i}{(\lambda_0(1-z_i)+\eta_0)(1-z_i)}\right. \\ &\quad \left. + \left(\frac{\lambda_0}{\eta_0} + \frac{\lambda_i}{\eta_i} - \frac{\mu_i}{\eta_i}\right) p_{i\boldsymbol{.}}\eta_i\right]. \end{aligned} \tag{27}$$

Finally, let C be the number of customers cleared from the system per unit time. Then

$$E[C] = \sum_{i=1}^{n} \eta_i \sum_{m=1}^{\infty} m p_{im} = \sum_{i=1}^{n} \eta_i E[L_i].$$

The fraction of customers receiving full service is therefore

$$\frac{\lambda - E[C]}{\lambda} = 1 - \frac{E[C]}{\lambda}.$$

8. Sojourn Times

Let W_{im} be the sojourn time of a customer that arrives to the system when it is in state (i, m). We claim

Proposition 1.

$$E[W_{im}] = \frac{1}{\eta_i}\left[1 - \left(\frac{\mu_i}{\mu_i + \eta_i}\right)^{m+1}\right]. \tag{28}$$

Proof. When in state (i, m), the time until departure, whether as a result of a failure, or as a result of service completion, is the minimum of two independent variables: (a) time to failure, distributed exponentially with parameter η_i, and (b) sum of $(m+1)$ service completions, which has Erlang (Gamma) distribution with $(m+1)$ stages and parameter μ_i, denoted as Erlang $(\mu_i, m+1)$.

Thus, let $X_i \sim$ Erlang $(\mu_i, m+1)$ and $Y_i \sim \exp(\eta_i)$, so that

$$W_{im} = \min(X_i, Y_i).$$

Then

$$P(W_{im} \geq w) = P(X_i \geq w)P(Y_i \geq w) = \mathrm{e}^{-\eta_i w} \sum_{k=0}^{m} \mathrm{e}^{-\mu_i w} \frac{(\mu_i w)^k}{k!}.$$

Therefore,

$$\begin{aligned} E[W_{im}] &= \int_{w=0}^{\infty} P(W_{im} \geq w) dw \\ &= \int_{w=0}^{\infty} \sum_{k=0}^{m} \mathrm{e}^{-\mu_i w} \frac{(\mu_i w)^k}{k!} \mathrm{e}^{-\eta_i w} dw \\ &= \sum_{k=0}^{m} \frac{\mu_i^k}{(\mu_i + \eta_i)^{k+1}} \int_{w=0}^{\infty} \mathrm{e}^{-(\mu_i + \eta_i)w} \frac{(\mu_i + \eta_i)^{k+1} w^k}{k!} dw \\ &= \frac{1}{\eta_i}\left[1 - \left(\frac{\mu_i}{\mu_i + \eta_i}\right)^{m+1}\right]. \end{aligned}$$

□

Now, define $E[W]$ as the mean sojourn time of an arbitrary customer. Then,

$$E[W] = \sum_{i=0}^{n} \sum_{m=0}^{\infty} p_{im} E[W_{im}]. \tag{29}$$

In fact, using Little's law, $E[W] = \frac{1}{\hat{\lambda}} E[L]$ where $\hat{\lambda} = \sum_{i=0}^{n} \lambda_i p_{i\bullet}$.

9. Two Special Cases

9.1. *One operative phase*

A special case of the general model introduced in Sec. 2 is when $n = 1$. That is, there is only one active phase and one failure phase. We consider the homogeneous case where the arrival rates are $\lambda_0 = \lambda_1 = \lambda$. (Clearly, since $n = 1$, $q_1 = 1$, and this case becomes a special case of Yechiali (Yechiali, 1973), where an $M/M/1$ system with breakdowns and customers' impatience is studied. Assuming in Yechiali (1973) that $\xi = 0$, the two models coincide). It thus follows that

$$p_{0\boldsymbol{\cdot}} = \frac{\eta_1}{\eta_0 + \eta_1}, \quad p_{1\boldsymbol{\cdot}} = \frac{\eta_0}{\eta_0 + \eta_1}, \tag{30}$$

and

$$p_{10} = \frac{p_{0\boldsymbol{\cdot}}\eta_0^2 z_1}{(\lambda(1 - z_1) + \eta_0)\mu_1(1 - z_1)}, \tag{31}$$

where z_1 is the root of $f_1(z)$, as discussed in Sec. 4.

Furthermore $E[L_0] = \lambda p_{0\boldsymbol{\cdot}}/\eta_0$, which equates the mean rate of arrivals to phase 0, $\lambda p_{0\boldsymbol{\cdot}}$, to the mean rate of departures from this phase, $\eta_0 E[L_0]$. In addition,

$$\eta_1 E[L_1] = \lambda p_{1\boldsymbol{\cdot}} + \eta_0 E[L_0] - \mu_1(p_{1\boldsymbol{\cdot}} - p_{10}). \tag{32}$$

Again, Eq. (32) equates the rates of customers inflow and outflow at phase 1.

9.2. *Arrival stops when the system is down*

Another special case of the general n-phase model of Sec. 2 is when the arrival process *stops* whenever the system is down. That is, $\lambda_0 = 0$, implying that $p_{0m} = 0$, $\forall\, m > 0$. The set of probabilities $p_{i\boldsymbol{\cdot}}$ is unchanged and is given by (7).

The equivalent of (11) and (14) is

$$G_0(z) = p_{00}, \quad G_i(z) = \frac{1}{f_i(z)}[\mu_i(z-1)p_{i0} + \eta_0 q_i p_{00} z] \quad i \geq 1, \tag{33}$$

where $f_i(z)$ are given by (15). Finally,

$$E[L_0] = 0, \quad \eta_i E[L_i] + \mu_i(p_{i\boldsymbol{\cdot}} - p_{i0}) = \lambda_i p_{i\boldsymbol{\cdot}}, \quad i \geq 1, \tag{34}$$

which, again, equates the inflow and outflow rates at phase i.

10. Conclusion

We have considered an extended version of the $M/M/1$ queue in a n-phase Markovian random environment with disasters that, when occurs, destroy all customers present in the system. Two methods of analysis have been used: (i) PGFs and (ii) matrix geometric approach, while some relationships between them have been revealed. Mean queue sizes, mean (conditional and unconditional) waiting times, and fraction of customers lost were calculated, and two special cases were further investigated.

References

Artalejo, JR and A Gomez-Coral (1999). Performance analysis of a single-server queue with repeated attempts. *Mathematical and Computer Modeling*, 30, 79–88.

Chakravarthy, SR (2009). A disaster queue with markovian arrivals and impatient customers. *Applied Mathematics and Computation*, 214, 48–59.

Drekic, S and WK Grassmann (2002). An eigenvalue approach to analyzing a finite source priority queueing model. *Annals of Operations Research*, 112, 139–152.

Gupta, V, AS Wolf, M Harchol-Balter and U Yechiali (2006). Fundamental characteristics of queues with flactuating load. In *Proc. ACM SIGMETRICS 2006 Conf. Management and Modeling of Computer Systems*, Saint Malo, France, pp. 203–215.

Kella, O and W Whitt (1992). A storage model with two-state random environment. *Operations Research*, 40, 257–262.

Neuts, MF (1981). *Matrix-Geometric Solutions in Stochastic Models*: *An Algorithmic Approach*. Baltimore, USA: John Hopkins University Press.

Sengupta, B (1990). A queue with service interruptions in an alternating random environment. *Operations Research*, 38, 308–318.

Sudhesh, R (2010). Transient analysis of a queue with system disasters and customer impatience. *Queueing Systems*, 66, 95–105.

Yechiali, U and P Naor (1971). Queueing problems with heterogeneous arrivals and service. *Operations Research*, 19, 722–734.

Yechiali, U (1973). A queueing-type birth-and-death process defined on a continuous-time markov chain. *Operations Research*, 21, 604–609.

Yechiali, U (2006). Queues with system disasters and impatient customers when system is down. *Queueing Systems*, 56, 195–202.

Zhang, ZG and N Tian (2004). An analysis of queueing systems with multi-task servers. *European Journal of Operational Research*, 156, 375–389.

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