Limit laws for the asymmetric inclusion process

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The Asymmetric Inclusion Process (ASIP) is a unidirectional lattice-gas flow model which was recently introduced as an exactly solvable 'Bosonic' counterpart of the 'Fermionic' asymmetric exclusion process. An iterative algorithm that allows the computation of the probability generating function (PGF) of the ASIP's steady state exists but practical considerations limit its applicability to small ASIP lattices. Large lattices, on the other hand, have been studied primarily via Monte Carlo simulations and were shown to display a wide spectrum of intriguing statistical phenomena. In this paper we bypass the need for direct computation of the PGF and explore the ASIP's asymptotic statistical behavior. We consider three different limiting regimes: heavy-traffic regime, large-system regime, and balanced-system regime. In each of these regimes we obtain—analytically and in closed form—stochastic limit laws for five key ASIP observables: traversal time, overall load, busy period, first occupied site, and draining time. The results obtained yield a detailed limit-laws perspective of the ASIP, numerical simulations demonstrate the applicability of these laws as useful approximations.

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I. INTRODUCTION

The Asymmetric Inclusion Process (ASIP), a lattice-gas flow model in one dimension, was introduced and analyzed in [1,2]. The ASIP is an exactly solvable 'Bosonic' counterpart of the 'Fermionic' Asymmetric Exclusion Process (ASEP)—a fundamental model in non-equilibrium statistical physics [3–5]. In both processes, random events cause particles to propagate unidirectionally along a one-dimensional lattice. In the ASEP particles are subject to *exclusion* interactions that keep them singled apart, whereas in the ASIP particles are subject to *inclusion* interactions that coalesce them into inseparable clusters.

The ASIP, schematically illustrated in Fig. 1, is described as follows. Consider a one-dimensional lattice of n sites indexed $k = 1,\dots,n$. Each site is followed by a gate—labeled by the site's index—which controls the site's outflow. Particles arrive at the first site ($k = 1$) following a Poisson process Π_0 with rate λ, the openings of gate k are timed according to a Poisson process Π_k with rate μ_k ($k = 1,\dots,n$), and the $n + 1$ Poisson processes are mutually independent. Note that from this definition it follows that the times between particle arrivals are independent and exponentially distributed with mean $1/\lambda$, and that the times between the openings of gate k are independent and exponentially distributed with mean $1/\mu_k$ ($k = 1,\dots,n$). At an opening of gate k all particles present at site k transit simultaneously, in one cluster (one 'batch'), to site $k + 1$—thus joining particles that may already be present at site $k + 1$ ($k = 1,\dots,n - 1$). At an opening of the last gate ($k = n$) all particles present at site n exit the lattice simultaneously.

The statistical physics perspective on the ASIP is complemented by a queueing theory perspective. Queueing theory is the mathematical field concerned with the stochastic analysis of queues [6]. The theory has found its traditional applications in telecommunication [7–9], traffic engineering [10] and performance modeling and evaluation [11–13]. More recently there has been an ongoing wave of interest in the applications of queueing theory to "non-traditional" problems and it has been applied in the study of human dynamics [14–17], gene expression [18–21], intracellular transport [22], and non-equilibrium statistical physics [23–29]. The ASIP links together the ASEP with the tandem Jackson network—a fundamental service model in queuing theory [6,30,31]. From a queueing perspective, the ASIP is a tandem Jackson network with *unlimited batch service* [32,33]. Unlimited batch service in queueing systems has also been considered in the context of polling systems where a single server attends n queues according to a predetermined, or dynamic, visit-order rule [34–36].

The analysis conducted in [1] concluded that the ASIP, despite its simple description, displays highly complex stochastic dynamics. An iterative scheme for the computation of the multidimensional probability generating function (PGF) of the ASIP's site occupancies at steady state was established. Yet, this PGF turns out to be analytically intractable even for small n—a fact that is vivid from the very rapid growth in complexity of the explicit PGF expressions for $n = 1,2,3$ [1]. Understanding the behavior of large ASIPs (i.e., ASIPs with large lattice size n) is therefore a challenge.

Homogeneous ASIPs are characterized by identical gate opening rates: $\mu_1 = \cdots = \mu_n$, and are of special importance—as amongst the class of general ASIPs the subclass of homogeneous ASIPs is optimal with respect to various measures of efficiency [1]. Interestingly, it was recently demonstrated that large homogeneous ASIPs–despite their relative simplicity—are a showcase of complexity [2]. Indeed, a series of Monte Carlo simulations showed that the statistical behavior of homogeneous ASIPs is rich and ranges from 'mild' to 'wild' displays of randomness. In particular it was shown that as the lattice size n tends to infinity the ASIPs statistical behavior is well captured by a set of limiting probability distributions.

Motivated by these findings, in this paper we analytically study the stochastic limit laws of five key ASIP observables: (i) Traversal Time—the time it takes a particle to traverse the lattice; (ii) Overall Load—the total number of particles

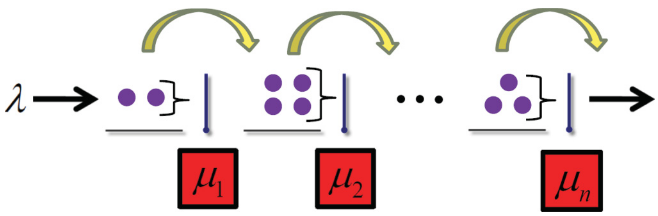

FIG. 1. (Color online) An illustration of the ASIP model.

present in the lattice in steady state; (iii) Busy Period—the time elapsing from the instant a particle arrives at an empty lattice, till the first instant the lattice is empty once again; (iv) First Occupied Site—the index of the first non-empty site; (v) Draining Time—the time elapsing from the instant the arrival flow is blocked, in steady state, till the first instant the lattice is empty.

The stochastic limit laws of the key observables are established in three different limiting regimes: (i) Heavy-Traffic regime—in which the particles' arrival rate λ tends to infinity; (ii) Large-System regime—in which the lattice size n tends to infinity; (iii) Balanced-System regime—in which the lattice size n tends to infinity, the gates' opening rates μ_k ($k = 1,\ldots,n$) tend to infinity, and these limits are kept in balance. Our results hold for homogeneous and general (inhomogeneous) ASIPs alike.

We emphasize that despite the inherent complexity of the ASIP, all the results established herein are obtained analytically and in closed form. In particular, the stochastic limit laws obtained for the Overall Load and Draining Time, in the large-system limiting regime, analytically validate and considerably generalize the numerical results reported in [2].

The reminder of the paper is organized as follows. We begin with a preliminary analysis of the ASIP's key observables (Sec. II), analyze the asymptotic statistical behavior of homogeneous ASIPs (Sec. III), and then analyze the asymptotic statistical behavior of general ASIPs (Sec. IV).

A note about notation: Throughout the paper $E[\xi]$ and Var $[\xi]$ denote, respectively, the mathematical expectation and variance of a real-valued random variable ξ. Also, IID is the acronym for "Independent and Identically Distributed".

II. KEY OBSERVABLES

In this section we analyze five key observables of the ASIP: Traversal Time, Overall Load, Busy Period, First Occupied Site, and Draining Time.

A. Traversal time

The ASIP's *traversal time* T is the random time it takes a particle to traverse the lattice. Namely, T is the time elapsing from the instant a particle arrives at the first site ($k = 1$), till the instant it leaves the last site ($k = n$).

Due to the memory-less property of the exponential distribution [37], the time elapsing from the arrival of a particle to site k (at an arbitrary time epoch), till the first opening of gate k thereafter, is exponentially distributed with mean $1/\mu_k$—independently of the particle's arrival epoch to site k. A particle arriving to the lattice would thus wait an exponentially distributed random time (with mean $1/\mu_1$) till

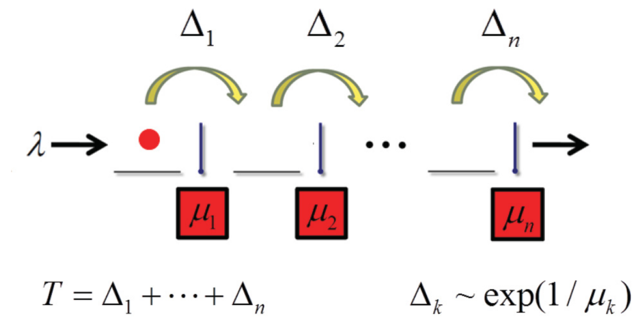

FIG. 2. (Color online) An illustration of the traversal time.

moving from the first site to the second site, then wait an exponentially distributed random time (with mean $1/\mu_2$) till moving from the second site to the third site, and so forth. Since the gate-openings are governed by independent Poisson processes we conclude that the traversal time T admits the stochastic representation

$$T = \sum_{k=1}^{n} \Delta_k, \tag{1}$$

where $\{\Delta_1,\ldots,\Delta_n\}$ is a sequence of independent and exponentially distributed random times with corresponding means $\{1/\mu_1,\ldots,1/\mu_n\}$.

Consequently, Eq. (1) straightforwardly implies that the mean and the Laplace transform of the traversal time T are given, respectively, by

$$E[T] = \sum_{k=1}^{n} \frac{1}{\mu_k} \tag{2}$$

and

$$E[\exp(-\theta T)] = \prod_{k=1}^{n} \frac{\mu_k}{\mu_k + \theta} \tag{3}$$

($\theta \geqslant 0$). We note that the traversal time is, in effect, the first passage time of a particle to leave the system [38], and that other ASIP first passage times were discussed in [2]. The traversal time is illustrated schematically in Fig. 2.

B. Overall load

Consider the ASIP in *steady state*, and let X_k denote the number of particles present in site k ($k = 1,\ldots,n$). The ASIP's *overall load* L is the *total number of particles* present in the lattice in steady state:

$$L = \sum_{k=1}^{n} X_k. \tag{4}$$

An analysis presented in [1] shows that the mean overall load is given by

$$E[L] = \sum_{k=1}^{n} \frac{\lambda}{\mu_k}. \tag{5}$$

Equation (5), in conjunction with Eq. (2), implies that $E[L] = \lambda E[T]$. Namely, the mean overall load $E[L]$ equals the product of the inflow rate λ and the mean traversal time $E[T]$—the

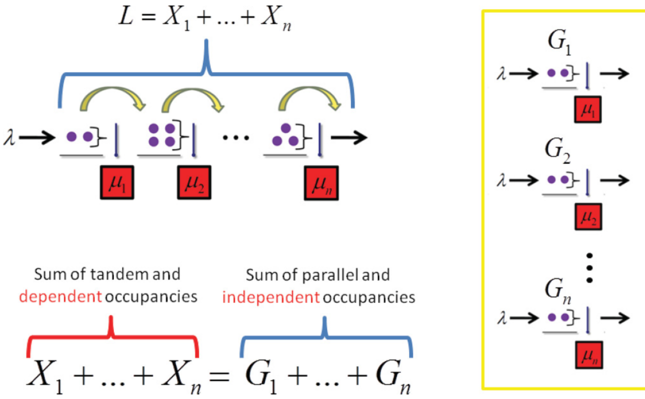

FIG. 3. (Color online) An illustration of the overall load.

mean sojourn time of an arbitrary particle in the lattice. Equation (5) is the ASIP's version of the well-known *Little's law* in Queueing Theory [39].

An analysis presented in [1] further establishes that the probability generating function of the overall load L is given by

$$E[z^L] = \prod_{k=1}^{n} \frac{\mu_k}{\mu_k + \lambda(1-z)} \tag{6}$$

($|z| \leqslant 1$). Equation (6) has several important implications. First, it implies that the overall load L of a *single-site* ASIP ($n = 1$) follows a *geometric distribution*: For $n = 1$ the probability generating function of Eq. (6) yields the probability distribution $\Pr(L = j) = (1 - p_1)^j p_1$ ($j = 0,1,2,\ldots$), where $p_1 = \mu_1/(\mu_1 + \lambda)$. Second, the *product-form* structure of Eq. (6) implies that the overall load L admits the stochastic representation

$$L = \sum_{k=1}^{n} G_k, \tag{7}$$

where $\{G_1,\ldots,G_n\}$ is a sequence of independent geometrically distributed random variables: $\Pr(G_k = j) = (1 - p_k)^j p_k$ ($j = 0,1,2,\ldots$), where $p_k = \mu_k/(\mu_k + \lambda)$. The overall load L is hence equal, in law, to the sum of the overall loads of n *independent single-site* ASIPs with respective parameters $(\lambda,\mu_1),\ldots,(\lambda,\mu_n)$. Third, Eq. (6) implies the following distributional form of the aforementioned *ASIP Little's law*: $L = \Pi_0(T)$, the equality being in law. Namely, the number of particles arriving to the lattice, $\Pi_0(T)$, during a traversal time T is equal, in law, to the ASIP's overall load L. The proof of the distributional Little's law is given in the Appendix. Fourth, setting $z = 0$ in Eq. (6) implies that the probability that the lattice is empty is given by

$$\Pr(L = 0) = \prod_{k=1}^{n} \frac{\mu_k}{\mu_k + \lambda}. \tag{8}$$

The overall load is illustrated schematically in Fig. 3.

C. Busy period

The ASIP's *busy period* B is the random duration of time in which the lattice is continuously non-empty. Namely, B is the time elapsing from the instant a particle arrives at an empty lattice, till the first instant thereafter the lattice is empty once again. The busy period is a key variable in queueing theory, as every queueing system continuously alternates between random busy and idle periods [40–42].

Consider the two following scenarios: (i) a particle arrives at an empty lattice and traverses it before a second particle arrives; (ii) a particle arrives at an empty lattice and a second particle arrives before the first particle traversed the lattice. Let T denote the traversal time of the first particle, and let $\triangle_0$ denote the time elapsing between the arrival epochs of the two particles. Clearly, the random variables T and $\triangle_0$ are independent.

The first scenario is the event $\{T < \triangle_0\}$, and in this scenario the busy period equals the traversal time: $B = T$. The second scenario is the event $\{\triangle_0 \leqslant T\}$, and in this scenario the busy period equals the interarrival time $\triangle_0$ plus an additional and independent random time whose distribution is equal in law to that of a busy period: $B = \triangle_0 + B'$, where B' is an IID copy of B which is independent of T and $\triangle_0$. Thus, we obtain that the busy period B satisfies the following stochastic regeneration formula:

$$B = \begin{cases} T & \text{if} \quad T < \triangle_0, \\ \triangle_0 + B' & \text{if} \quad \triangle_0 \leqslant T. \end{cases} \tag{9}$$

Consequently, Eq. (9) implies that the mean and the Laplace transform of the busy period B are given, respectively, by

$$E[B] = \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left[1 + \frac{\lambda}{\mu_k}\right] - 1\right) \tag{10}$$

and

$$E[\exp(-\theta B)] = \frac{\lambda + \theta}{\lambda + \theta \prod_{k=1}^{n}\left[1 + \frac{\lambda+\theta}{\mu_k}\right]} \tag{11}$$

($\theta \geqslant 0$).

The derivations of Eqs. (10) and (11) are given in the Appendix. Equation (10) can also be obtained via a renewal argument which we now describe.

Note that the lattice alternates between empty and non-empty periods. The empty periods are IID copies of the generic interarrival period $\triangle_0$, the non-empty periods are IID copies of the generic busy period B, and these alternating periods are mutually independent. Renewal theory implies that—over infinitely large time windows—the fraction of time the lattice is empty is given by the ratio [43]: $E[\triangle_0]/(E[\triangle_0] + E[B])$. On the other hand, the fraction of time the lattice is empty equals the probability, in steady state, of a zero overall load: $\Pr(L = 0)$. Thus, we obtain that $\Pr(L = 0) = E[\triangle_0]/(E[\triangle_0] + E[B])$. Now, since $E[\triangle_0] = 1/\lambda$, and since $\Pr(L = 0)$ is given by Eq. (8), we can extract the mean busy period $E[B]$. Doing so indeed yields Eq. (10). The renewal argument is illustrated schematically in Fig. 4.

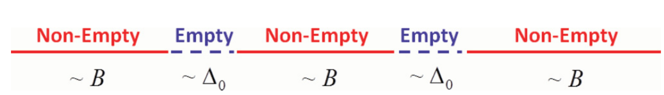

FIG. 4. (Color online) An illustration of the renewal argument (in the context of the mean busy period).

D. The first occupied site

Consider the ASIP in *steady state*, and let I denote the index of the first occupied site: $I = \min\{k|X_k > 0\}$. If all the sites are empty then we set $I = \infty$ by convention. Clearly, $\Pr(I = 1) = \Pr(X_1 > 0)$. In addition,

$$\begin{aligned}\Pr(I = k) &= \Pr(X_1 = 0,\ldots,X_{k-1} = 0, X_k > 0)\\ &= \Pr(X_1 = 0,\ldots,X_{k-1} = 0)\\ &\quad - \Pr(X_1 = 0,\ldots,X_k = 0)\end{aligned} \tag{12}$$

$(1 < k \leqslant n)$, and

$$\Pr(I = \infty) = \Pr(X_1 = 0,\ldots,X_n = 0). \tag{13}$$

Since the event $\{X_1 = 0,\ldots,X_n = 0\}$ is equivalent to the event $\{L = 0\}$, the probability appearing on the right-hand side of Eq. (13) is given by Eq. (8). In order to compute the probabilities appearing on the right-hand side of Eq. (12) we apply the embedding property of the ASIP, which we now describe.

Consider two ASIPs: ASIP A with n' gates and parameters $\{\lambda,\mu_1,\ldots,\mu_{n'}\}$, and ASIP B with n gates and parameters $\{\lambda,\mu_1,\ldots,\mu_n\}$, where $n' \leqslant n$. In [1] we have shown that the steady state distribution of ASIP A coincides with the steady state distribution of the first n' sites of ASIP B. An intuitive understanding of the embedding phenomenon follows from the fact that in an ASIP model with n gates the operation of the first n' gates is independent of whatever happens in the following gates $\{n'+1,\ldots,n\}$. In other words, an observation of the first n' gates in an ASIP with n gates is indistinguishable from an observation of an ASIP with n' gates (and the same parameters).

Due to the embedding phenomenon Eq. (8) implies that $\Pr(X_1 = 0,\ldots,X_{n'} = 0) = \prod_{k=1}^{n'} \frac{\mu_k}{\mu_k+\lambda}$ $(1 \leqslant n' \leqslant n)$. Substituting these probabilities into Eqs. (12) and (13) we obtain that the distribution of the first occupied site I is given by

$$\begin{aligned}\Pr(I = k) &= \frac{\lambda}{\mu_k+\lambda}\prod_{j=1}^{k-1}\frac{\mu_j}{\mu_j+\lambda}(1 \leqslant k \leqslant n),\\ \Pr(I = \infty) &= \prod_{j=1}^{n}\frac{\mu_j}{\mu_j+\lambda}.\end{aligned} \tag{14}$$

E. Draining time

Consider the ASIP in *steady state*, and assume that—starting at an arbitrary time epoch—we *block* the inflow of newcoming particles to the lattice. The ASIP's *draining time* D is the duration of time it takes the lattice to clear. Namely, D is the random time elapsing from the blocking epoch till the first instant the lattice is empty.

Consider the index I of the first occupied site at the blocking epoch. If $I = k$ $(k = 1,\ldots,n)$ then the draining time equals the traversal time of the gates $\{k,k+1,\ldots,n\}$. Consequently—analogous to the derivation of Eq. (1)—we obtain that if $I = k$ then $D = \sum_{j=k}^{n}\triangle_j$, where $\{\triangle_k,\ldots,\triangle_n\}$ is a sequence of independent and exponentially distributed random times with corresponding means $\{1/\mu_k,\ldots,1/\mu_n\}$. On the other hand, if $I = \infty$ then the lattice is empty and hence $D = 0$. Thus, we obtain that the draining time admits

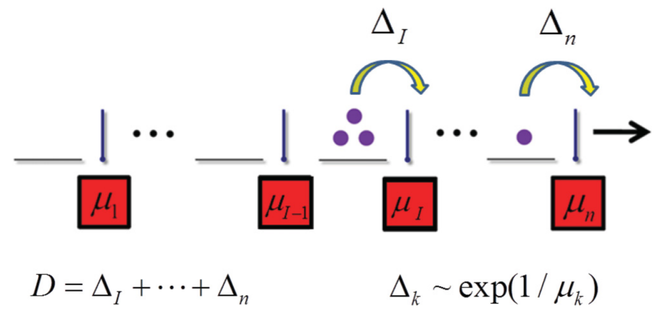

FIG. 5. (Color online) An illustration of the draining time.

the stochastic representation

$$D = \begin{cases}\sum_{j=k}^{n}\triangle_j & \text{if} \quad I = k \quad (1 \leqslant k \leqslant n),\\ 0 & \text{if} \quad I = \infty,\end{cases} \tag{15}$$

where the exponentially distributed random variables $\{\triangle_1,\ldots,\triangle_n\}$ are independent of the first occupied site I.

Since we blocked the inflow to an ASIP in *steady state* the distribution of the first occupied site I is given by Eq. (14). Consequently, combining together Eqs. (14) and (15) we obtain that the mean and the Laplace transform of the draining time D are given, respectively, by

$$E[D] = \sum_{k=1}^{n}\frac{\lambda}{\mu_k+\lambda}\left(\prod_{j=1}^{k-1}\frac{\mu_j}{\mu_j+\lambda}\right)\left(\sum_{j=k}^{n}\frac{1}{\mu_j}\right) \tag{16}$$

and

$$\begin{aligned}E[\exp(-\theta D)] &= \prod_{j=1}^{n}\frac{\mu_j}{\mu_j+\lambda} + \sum_{k=1}^{n}\frac{\lambda}{\mu_k+\lambda}\left(\prod_{j=1}^{k-1}\frac{\mu_j}{\mu_j+\lambda}\right)\\ &\quad\times\left(\prod_{j=k}^{n}\frac{\mu_j}{\mu_j+\theta}\right),\end{aligned} \tag{17}$$

$(\theta \geqslant 0)$. The derivations of Eqs. (16) and (17) are given in the Appendix. The draining time is illustrated schematically in Fig. 5.

III. ASYMPTOTIC ANALYSIS: THE HOMOGENEOUS CASE

Homogeneous ASIPs are characterized by identical gate-opening rates: $\mu_1 = \cdots = \mu_n$. Amongst the class of general ASIPs the subclass of homogeneous ASIPs turns out to be optimal with respect to various measures of efficiency. Indeed, referring to $\mu_1 + \cdots + \mu_n$ as the ASIP's "aggregate rate", it was shown that homogeneous ASIPs [1]: (i) minimize the mean overall load subject to a given aggregate rate; (ii) minimize the variance of the overall load subject to a given aggregate rate; (iii) maximize the probability of a zero overall load, i.e., the probability of an empty lattice in steady state, subject to a given aggregate rate; and (iv) minimize the variance of the overall load subject to a given mean overall load (this is the ASIP analog of the "Markowitz optimization" of financial portfolios [44]).

Due to the optimality of the aforementioned properties, in this section we consider homogeneous ASIPs, and set $\mu_1 = \cdots = \mu_n = 1/m$. Namely, m is the mean sojourn time of an arbitrary particle in a single site. In what follows we shall establish stochastic limit laws for the ASIP's key observables presented in the previous section: Traversal Time, Overall Load, Busy Period, First Occupied Site, and Draining Time. The stochastic limit laws shall be established for the three following limiting regimes:

(i) The *Heavy-Traffic regime*—in which the inflow rate tends to infinity: $\lambda \to \infty$.

(ii) The *Large-System regime*—in which the lattice size tends to infinity: $n \to \infty$.

(iii) The *Balanced-System regime*—in which the lattice size tends to infinity, the particles' mean sojourn time at a site tends to zero, and the product of these parameters tends to a positive limit: $n \to \infty$, $m \to 0$, and $nm \to \tau \in (0,\infty)$.

Throughout this section we denote by Z a Gauss-distributed random variable with zero mean and unit variance ("Standard Normal"), by $\mathcal{E}$ an exponentially distributed random variable with unit mean ("Standard Exponential"), and by Γ_n an Erlang distributed random variable with n degrees of freedom. Namely, Γ_n is the sum of n IID copies of the random variable $\mathcal{E}$. In what follows the sign "$\approx$" will denote asymptotic equivalence in law.

A. Heavy traffic

The heavy-traffic regime considers ASIPs in which the inflow rate is increased to infinity: $\lambda \to \infty$. The ASIP stochastic limit laws—under the heavy-traffic regime—are as follows:

Traversal Time. As is clear from Eq. (3), the inflow rate does not affect the traversal time T. On the other hand, in the homogeneous ASIP the traversal time T is the sum of n IID exponential random variables each with mean m. Hence, the traversal time T admits the stochastic representation

$$T = m\Gamma_n \tag{18}$$

(recall that a random variable $\mathcal{E}$ is exponentially distributed with unit mean if and only if the random variable $m\mathcal{E}$ is exponentially distributed with mean m).

Overall Load. Increasing the inflow rate λ is expected to result in an increase of the overall load L. And indeed, Eq. (5) implies that the mean of the overall load L scales linearly with λ. Consequently, we normalize the overall load L by the dimensionless term λm and analyze the stochastic limit of the normalized overall load $L/(\lambda m)$ (as $\lambda \to \infty$). Setting $z = \exp(-\theta/(\lambda m))$ in Eq. (6) we obtain the limit

$$\lim_{\lambda\to\infty} E\left[\exp\left(-\theta\frac{L}{\lambda m}\right)\right] = \left(\frac{1}{1+\theta}\right)^n, \tag{19}$$

($\theta \geqslant 0$). Since the right-hand side of Eq. (19) is the Laplace transform of the Erlang distribution with n degrees of freedom, we obtain that, as $\lambda \to \infty$, the overall load L admits the stochastic approximation

$$L \approx \lambda m \Gamma_n, \tag{20}$$

($\lambda \to \infty$).

Busy Period. As in the case of the overall load, increasing the inflow rate λ is expected to result in an increase of the duration of the busy period B. And indeed, Eq. (10) implies that the mean of the busy period B scales like λ^{n-1}. Consequently, we normalize the busy period B by the dimensionless term $(m\lambda)^{n-1}$ and analyze the stochastic limit of the normalized busy period $B/(m\lambda)^{n-1}$ (as $\lambda \to \infty$). Using Eq. (11) we obtain the limit

$$\lim_{\lambda\to\infty} E\left[\exp\left(-\theta\frac{B}{(m\lambda)^{n-1}}\right)\right] = \frac{1}{1+m\theta}, \tag{21}$$

($\theta \geqslant 0$). Since the right-hand side of Eq. (21) is the Laplace transform of the exponential distribution with mean m, we obtain that the busy period B admits the stochastic approximation

$$B \approx \lambda^{n-1} m^n \mathcal{E} \tag{22}$$

($\lambda \to \infty$).

First Occupied Site. Increasing the inflow rate λ is expected to increase to one the probability of finding the first site occupied. And indeed, Eq. (14) yields the limit

$$\lim_{\lambda\to\infty} \Pr(I = 1) = 1. \tag{23}$$

Draining Time. Equation (23) implies that for large λ the first occupied site is effectively the first site. Consequently, for large λ the draining time D should coincide with the traversal time T. And indeed, taking the limit $\lambda \to \infty$ in Eq. (17) confirms this conjecture and yields the stochastic approximation

$$D \approx m\Gamma_n, \tag{24}$$

($\lambda \to \infty$).

B. Large systems

The large-system regime considers ASIPs in which the lattice size increases to infinity: $n \to \infty$. The ASIP stochastic limit laws—under the large-system regime—are as follows:

Traversal Time. In the homogeneous ASIP the traversal time T is a sum of n IID exponential random variables—each with mean m and variance m^2. Consequently, the Central Limit Theorem [37] implies that the traversal time T admits the Gaussian stochastic approximation

$$T \approx nm + \sqrt{n} m Z, \tag{25}$$

($n \to \infty$).

Overall Load. Equation (6) asserts that in the homogeneous ASIP the overall load L is a sum of n IID geometric random variables—each with mean λm and variance $\lambda m + (\lambda m)^2$. Consequently, the Central Limit Theorem [37] implies that the overall load L admits the Gaussian stochastic approximation

$$L \approx n\lambda m + \sqrt{n}\sqrt{\lambda m + (\lambda m)^2} Z, \tag{26}$$

($n \to \infty$).

Busy Period. Increasing the lattice size n is expected to result in an increase of the busy period. Indeed, Eq. (10) implies that for large n the mean of the busy period scales like $(1+\lambda m)^n$. Consequently, we normalize the busy period by the dimensionless term $(1+\lambda m)^n$ and analyze the stochastic limit of the normalized busy period $B/(1+\lambda m)^n$ (as $n \to \infty$). Using Eq. (11) we obtain the limit

$$\lim_{n\to\infty} E\left[\exp\left(-\theta \frac{B}{(1+\lambda m)^n}\right)\right] = \frac{\lambda}{\lambda+\theta}, \tag{27}$$

($\theta \geqslant 0$). Since the right-hand side of Eq. (27) is the Laplace transform of the exponential distribution with mean $1/\lambda$, we obtain that the busy period B admits the stochastic approximation

$$B \approx (1+\lambda m)^n \frac{1}{\lambda}\mathcal{E}, \tag{28}$$

($n \to \infty$).

First Occupied Site. Taking the limit $n \to \infty$ in Eq. (14) yields the geometric distribution

$$\lim_{n\to\infty} \Pr(I = k) = q(1-q)^{k-1}, \tag{29}$$

($k = 1,2,3,\cdots$), where $q = \lambda m/(1+\lambda m)$.

Draining Time. Equation (29) implies that—in the limit $n \to \infty$—the First Occupied Site is geometrically distributed with mean $1/q$ and that $E[\sum_{k=1}^{I-1} \triangle_k]$ is a finite constant that does not depend on n. Consequently, since the number of sites tends to infinity ($n \to \infty$), the draining time D effectively equals the traversal time T. Combining this observation together with Eq. (25) implies that the draining time D admits the Gaussian stochastic approximation

$$D \approx nm + \sqrt{n} m Z \tag{30}$$

($n \to \infty$). A rigorous proof of this result is given in the Appendix.

C. Balanced systems

The balanced-system regime considers ASIPs in which the number of sites increases to infinity ($n \to \infty$) and the mean sojourn time at a site decrease to zero ($m \to 0$), while their product tends to a positive limit: $nm \to \tau \in (0,\infty)$. Namely, in this regime the large number of sites is balanced by rapid gate-opening rates. With no loss of generality we henceforth set $m = \tau/n$ and consider the limit $n \to \infty$. The ASIP stochastic limit laws, under the balanced-system regime, are as follows:

Traversal Time. Setting $m = \tau/n$ into Eq. (3) we obtain the limit

$$\lim_{n\to\infty} E[\exp(-\theta T)] = \exp(-\theta\tau), \tag{31}$$

($\theta \geqslant 0$). The right-hand side of Eq. (31) is the Laplace transform of a degenerate random variable which admits the value τ with probability one. Thus, the traversal time converges in law to the *deterministic* value τ.

Overall Load. Setting $m = \tau/n$ into Eq. (6) we obtain the limit

$$\lim_{n\to\infty} E[z^L] = \exp(-\tau\lambda(1-z)), \tag{32}$$

($|z| \leqslant 1$). The right-hand side of Eq. (32) is the probability generating function of the Poisson distribution with mean $\tau\lambda$. Thus, the overall load converges in law to a Poisson random variable with mean $\tau\lambda$.

Busy Period. Setting $m = \tau/n$ into Eq. (11) we obtain the limit

$$\lim_{n\to\infty} E[\exp(-\theta B)] = \frac{\lambda+\theta}{\lambda+\theta\exp(\tau(\lambda+\theta))}, \tag{33}$$

($\theta \geqslant 0$). The right-hand side of Eq. (33) can be derived from the stochastic regeneration formula of Eq. (9) by setting $T = \tau$ (see Appendix for details). We note that (in the limit $n \to \infty$) the busy period B is equal to τ with probability $\exp(-\lambda\tau)$, and is larger than τ otherwise. Indeed, $B = \tau$ if and only if there are no particle arrivals during the traversal time τ—an event which takes place with probability $\exp(-\lambda\tau)$. Also, Eq. (33) implies that (in the limit $n \to \infty$) the mean of the busy period is given by $(\exp(\lambda\tau) - 1)/\lambda$.

First Occupied Site. Consider the scaled first occupied site $\hat{I} = I/n$. Setting $m = \tau/n$ into Eq. (14) we obtain the limits

$$\begin{aligned} \lim_{n\to\infty} \Pr(\hat{I} \leqslant x) &= 1 - \exp(-\tau\lambda x) \\ \lim_{n\to\infty} \Pr(\hat{I} = \infty) &= \exp(-\lambda\tau) \end{aligned} \tag{34}$$

($0 \leqslant x \leqslant 1$); recall that the event $\{\hat{I} = \infty\}$ represents the (steady-state) scenario in which all sites are empty. Equation (34) implies that the scaled first occupied site $\hat{I}$ converges, in law, to a limit which has the density of an exponential random variable, with mean $\lambda\tau$, on the unit interval and an atom with probability $\exp(-\lambda\tau)$ at infinity. The derivation of Eq. (34) is given in the Appendix. Hence the scaled first occupied site $\hat{I}$ admits the asymptotic stochastic approximation

$$\hat{I} \approx \begin{cases} \mathcal{E}/(\lambda\tau) & \text{if } \mathcal{E} \leqslant \lambda\tau \\ \infty & \text{if } \mathcal{E} > \lambda\tau \end{cases}, \tag{35}$$

($n \to \infty$).

Draining Time. Setting $m = \tau/n$ into Eq. (17) we obtain the limit

$$\lim_{n\to\infty} E[\exp(-\theta D)] = \frac{\theta\exp(-\lambda\tau) - \lambda\exp(-\theta\tau)}{\theta - \lambda}, \tag{36}$$

($\theta \geqslant 0$). The derivation of Eq. (36) is given in the Appendix. Equation (36) implies that an asymptotic stochastic approximation for the draining time is given by an amalgamation of probability density function

$$f_D(x) = \begin{cases} \exp(-\lambda(\tau - x)) & 0 < x \leqslant \tau \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

and a probability mass, $\exp(-\lambda\tau)$, at zero. The validity of this approximation is easily verified by taking Laplace transform and recovering the right-hand side of Eq. (36).

Mapping the ASIP's lattice to the unit interval—positioning site k at the interval $[(k-1)/n, k/n]$—the balanced-system limiting regime is illustrated schematically in Fig. 6.

1. The $M/D/\infty$ queue

The balanced-system limiting regime may be better understood in light of the mapping between homogeneous ASIPs in this regime and the $M/D/\infty$ queue [45], which we describe in this subsection.

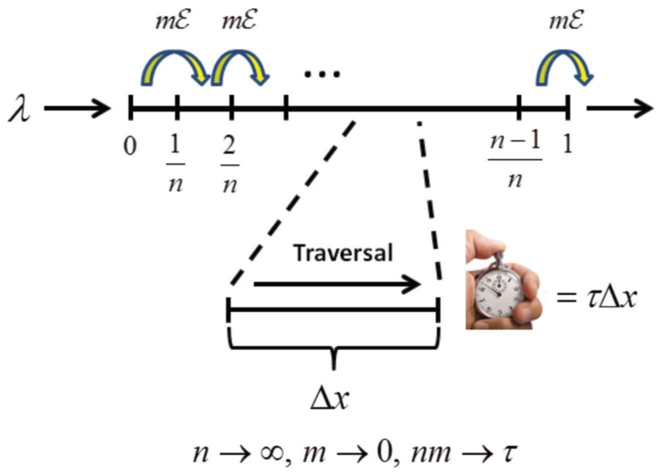

FIG. 6. (Color online) The homogeneous ASIP in the balanced-system regime mapped onto the unit interval. The traversal time of an interval of length Δx is deterministic and equals to $\tau \Delta x$. Particles do not cluster, remain separated at all times and leave the lattice exactly τ units of time after their respective arrival epochs.

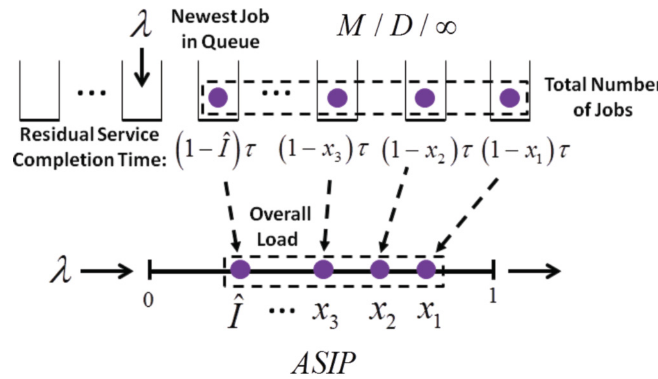

FIG. 7. (Color online) The $M/D/\infty$ queue mapped onto the ASIP in the balanced-system limiting regime. Each job in the $M/D/\infty$ queue is mapped to a particle on the unit interval. The position of the particle is chosen such that the residual service completion time in the $M/D/\infty$ queue is equal to the residual traversal time in the ASIP. Since particles move at a constant speed $1/\tau$, the latter is given by τ times the distance to the right edge of the unit interval. The residual service completion time of the newest job in the $M/D/\infty$ queue determines the normalized position of the first occupied site, $\hat{I}$, in the ASIP.

In queueing theory, the $M/D/\infty$ queue represents a system consisting of an infinite number of servers (or infinite "broadband"), to which particles ("jobs" in the queueing jargon) arrive following a Poisson process with rate λ. Each particle, upon its arrival, is immediately attended by one of the available servers; upon service completion a served particle leaves the system. Service times are deterministic and of common length τ, and the particles are served independently. Note that the common deterministic service times assure that particles will leave the system *exactly* τ units of time after their respective arrival epochs, and will do so in a *First In First Out* (FIFO) manner.

From the particles' perspective the ASIP—in the balanced-system limiting regime—is identical to the $M/D/\infty$ queue. Indeed, particles arrive to the lattice following a Poisson process with rate λ. Each particle, upon its arrival to the lattice, starts traversing it. In the balanced-system limiting regime the particles' traversal times are deterministic and of common length τ; upon "traversal completion" the particles leave the lattice. Here again, the common deterministic traversal times assure that particles will leave the lattice *exactly* τ units of time after their respective arrival epochs, and will do so separately, i.e., one by one, and in a FIFO manner. We emphasize that in the balanced-system limiting regime the particles remain separated and thus do *not* coalesce into clusters.

Thus, the ASIP in the balanced-system regime—with deterministic traversal time τ—is identical to a $M/D/\infty$ queue with deterministic service time τ. This observation gives rise to additional analogies between ASIPs in the limiting balanced-system regime and the $M/D/\infty$ queue. (i) The ASIP's overall load is equal in law to the $M/D/\infty$ queue size—the total number of jobs present (i.e., being served) in the $M/D/\infty$ queue in steady state, and is distributed according to the Poisson distribution with mean $\tau\lambda$ [45]. (ii) The ASIP's busy period is equal in law to the busy period of the $M/D/\infty$ queue. Indeed, the busy period in the $M/D/\infty$ queue follows the stochastic regeneration formula:

$$B = \begin{cases} \tau & \text{if} \quad \tau < \triangle_0, \\ \triangle_0 + B' & \text{if} \quad \triangle_0 \leqslant \tau, \end{cases} \tag{38}$$

in which τ is the service time, $\triangle_0$ is the exponential time elapsing between the arrival epochs of two jobs and B' is an IID copy of the busy period B. Note that Eq. (38) is a special case of Eq. (9), in which the general traversal time T is replaced by the deterministic traversal time τ. (iii) The ASIP's draining time is equal in law to the residual service-completion time of the newest job in the $M/D/\infty$ queue size. Indeed, the cumulative distribution function of the residual service-completion time, T_{res}, in the $M/D/\infty$ queue is equal to the probability that the interarrival time between jobs will exceed the value $\tau - t$ and is hence given by

$$\Pr(T_{\text{res}} \leqslant t) = \exp(-\lambda(\tau - t)), \tag{39}$$

($0 \leqslant t \leqslant \tau$). One can easily verify that the Laplace transform of T_{res} coincides with the right hand side of Eq. (36). The mapping between the balanced-system limit of the ASIP and the $M/D/\infty$ queue is illustrated schematically in Fig. 7.

D. Comparison with simulations

In this section we compare the limit laws obtained above with numerical simulations. Our main aim is to visually demonstrate convergence, and to illustrate how general thumb rules regarding the applicability of the limiting distributions as useful approximations can be attained. Throughout the section we use the following set of parameters:

(i) *Heavy-Traffic regime:* $m = 1$ and $n = 10$.
(ii) *Large-System regime:* $m = 1$ and $\lambda = 1$.
(iii) *Balanced-System regime:* $\lambda = 1$ and $nm = 1$.

We note that without loss of generality either λ or m can always be set to unity.

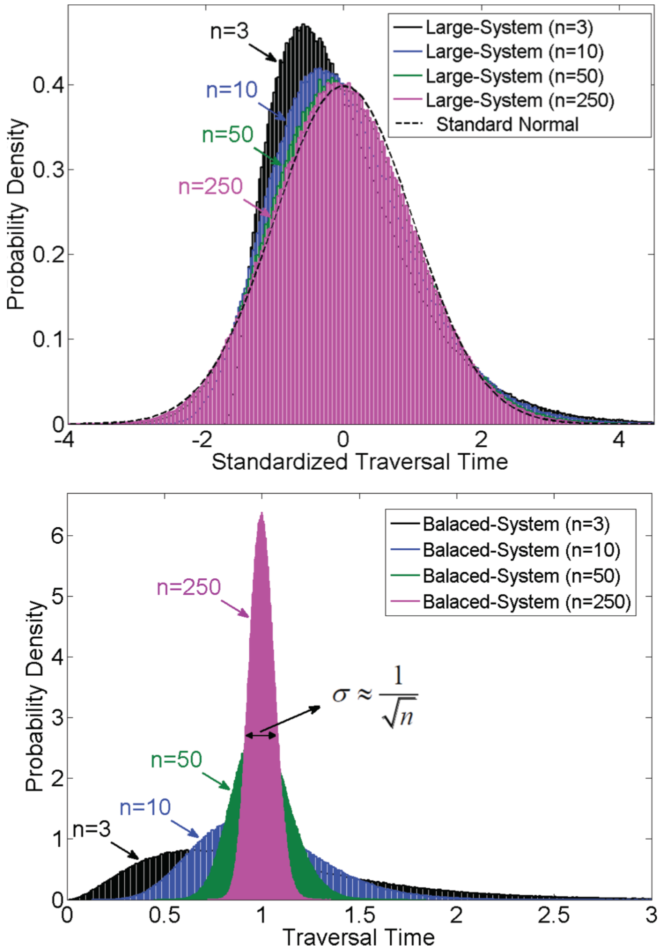

FIG. 8. (Color online) Asymptotic behavior of the traversal time in the Large-System (top panel) and Balanced-System (bottom panel) regimes.

Traversal Time. Recalling that the traversal time does not depend on the particles' arrival rate λ, we examine the asymptotic behavior of this observable in the Large-System and Balanced-System regimes only.

In the top panel of Fig. 8 we plot histograms of the standardized traversal time, $(T-n)/\sqrt{n}$, in the Large-System regime. As predicted by Eq. (25), with $m=1$, convergence to the standard Gaussian distribution is visible as the bell shape curve gradually takes the place of the positively skewed, Erlang like, distribution that characterizes the standardized traversal time for small n.

In the bottom panel of Fig. 8 we plot histograms of the traversal time, T, in the Balanced-System regime. As n increases, histograms become sharply peaked around unity reflecting the convergence to a deterministic random variable, as predicted by Eq. (31). To that end we note that in the Balanced-System regime the standard deviation of the traversal time is of the order $1/\sqrt{n}$.

Overall Load. In the top panel of Fig. 9 we plot cumulative distribution functions of the load in the Heavy-Traffic regime. As predicted by Eq. (19), with $n=10$, convergence to the Erlang distribution with ten degrees of freedom is clearly visible as simulated curves virtually collapse onto the Erlang curve for $\lambda \geqslant 25$. Convergence is also visible when plotting histograms as we do in the inset. Doing so, one should keep in mind that the load is a discrete random variable for which there is no proper probability density. This fact creates a somewhat deceiving impression regarding convergence, as the onset of "density like" histograms strongly depends on the preselected bin widths which in turn affect bar heights in the histogram.

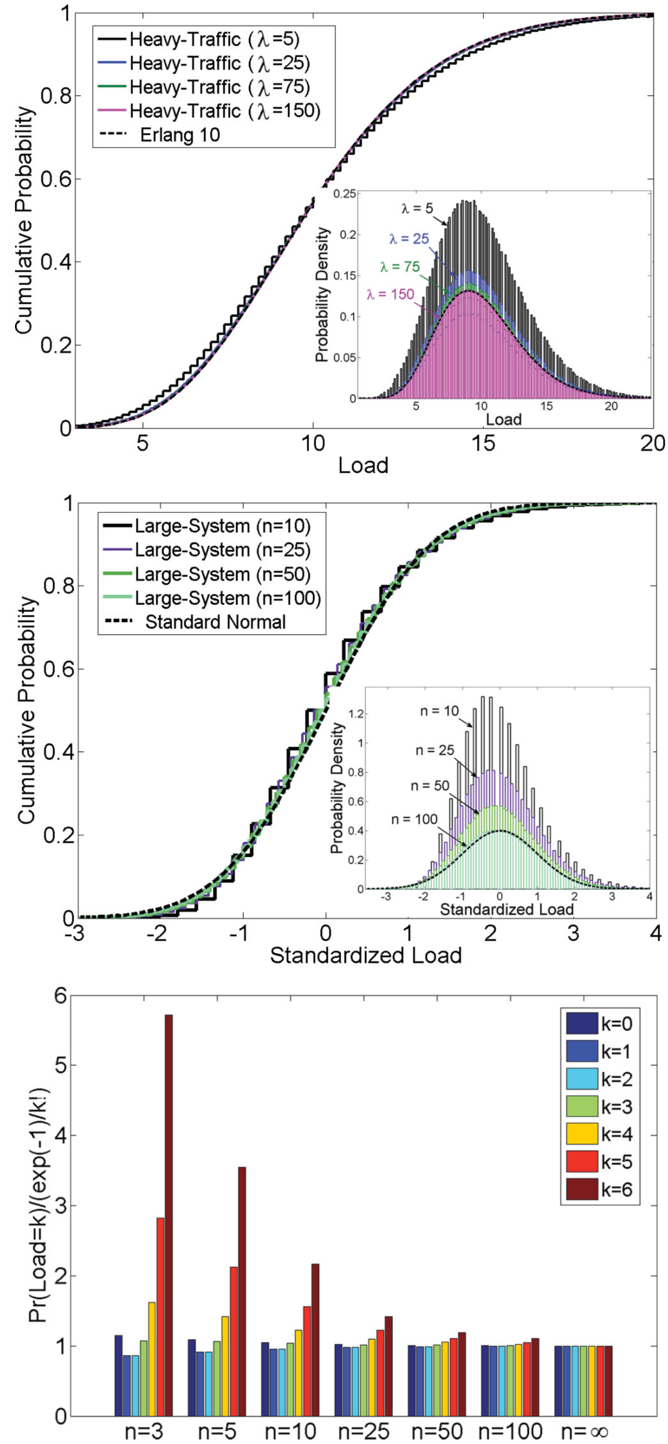

FIG. 9. (Color online) Asymptotic behavior of the overall load in the Heavy-Traffic (top panel), Large-System (middle panel) and Balanced-System (bottom panel) regimes.

In the middle panel of Fig. 9 we plot cumulative probability functions of the standardized load, $(L-n)/\sqrt{2n}$, in the

Large-System regime. As predicted by Eq. (26), with $\lambda = m = 1$, convergence to the standard Gaussian distribution is clearly visible as the simulated curves closely follow the standard Gaussian curve even for moderate values of n. Convergence is also visible when plotting histograms as we do in the inset.

In the bottom panel of Fig. 9 we plot the ratio between the probability that the overall load in the system is k ($k = 0,1,2,\ldots,6$) and the limiting probability of this event in the Balanced-System regime, for several different values of n. In this type of plot, every deviation of the ratio from unity can be interpreted as a deviation from the Poissonian limit law given by Eq. (32). Values that are smaller/larger than unity mean that the observed probability will be over/under estimated by the Poissonian approximation. As n increases convergence to the Poissonian limit (bars of unit height) is clearly visible and it can be considered a fair approximation even for moderate values of n. We note that under the chosen set of parameters, the total error made by neglecting $k > 6$ terms is given by the probability tail $Pr(L > 6) \cong 8.32 \times 10^{-5}$.

Busy Period. In the top panel of Fig. 10 we plot probability density curves of the normalized busy period, B/λ^9, in the Heavy-traffic regime. As predicted by Eq. (21), with $m = 1$ and $n = 10$, convergence to the exponential distribution with unit mean ("Standard Exponential") is clearly visible. Under this choice of parameters the exponential approximation can be considered very good for $\lambda \geqslant 250$.

In the middle panel of Fig. 10 we plot probability density curves of the normalized busy period, $B/2^n$, in the Large-System regime. As predicted by Eq. (27), with $\lambda = m = 1$, rapid convergence to the exponential distribution with unit mean ("Standard Exponential") is clearly visible. Under this choice of parameters the exponential approximation can be considered very good even for relatively small ($n \geqslant 10$) values of n.

In the bottom panel of Fig. 10 we plot cumulative distribution functions of the busy period in the Balanced-System regime. Convergence to the asymptotic cumulative distribution function predicted by numerical inversion of the Laplace transform given in Eq. (33), with $\lambda = \tau = 1$, is clearly visible. However, convergence seems slower than in the Heavy-Traffic and Large-System regimes and the asymptotic distribution can be considered a fair approximation only for relatively large values of n ($n \geqslant 1250$). This slow convergence is due to the discontinuity (at $B = 1$) of the cumulative distribution function of the limiting busy-period.

First Occupied Site and Draining Time. In the top panel of Fig. 11 we plot histograms of the draining time in the Heavy-Traffic regime. As predicted by Eq. (24), with $n = 10$, convergence to the Erlang distribution with ten degrees of freedom is clearly visible and the Erlang approximation seems excellent even for relatively small particles' arrival rate ($\lambda = 3$). In the inset we plot the probability of finding the first site occupied as a function of λ. Under the chosen set of parameters this probability is given by $\Pr(I = 1) = \lambda/(1 + \lambda)$. As delineated by Eq. (23) this probability rapidly approaches unity as λ increases.

In the middle panel of Fig. 11 we plot histograms of the standardized draining time, $(D - n)/\sqrt{n}$, in the Large-System regime. As predicted by Eq. (30), with $m = 1$, convergence to the standard Gaussian distribution is clearly visible and the Gaussian approximation seems fair for $n \geqslant 250$. Equation (14) asserts that for homogeneous ASIPs the first occupied site follows a *truncated* geometric distribution for which $\Pr(I = k) = q(1 - q)^{k-1}$ ($k = 1,2,3,\ldots,n$), where $q = \lambda m/(1 + \lambda m)$ and $\Pr(I = \infty) = (1 + \lambda m)^n$. Hence, the probability that $I = \infty$, i.e., all sites are empty, can also be understood as the total error made in approximating $\Pr(I = k)$ ($k = 1,2,3,\ldots$) by the geometric limit, given in Eq. (29). Under the chosen set

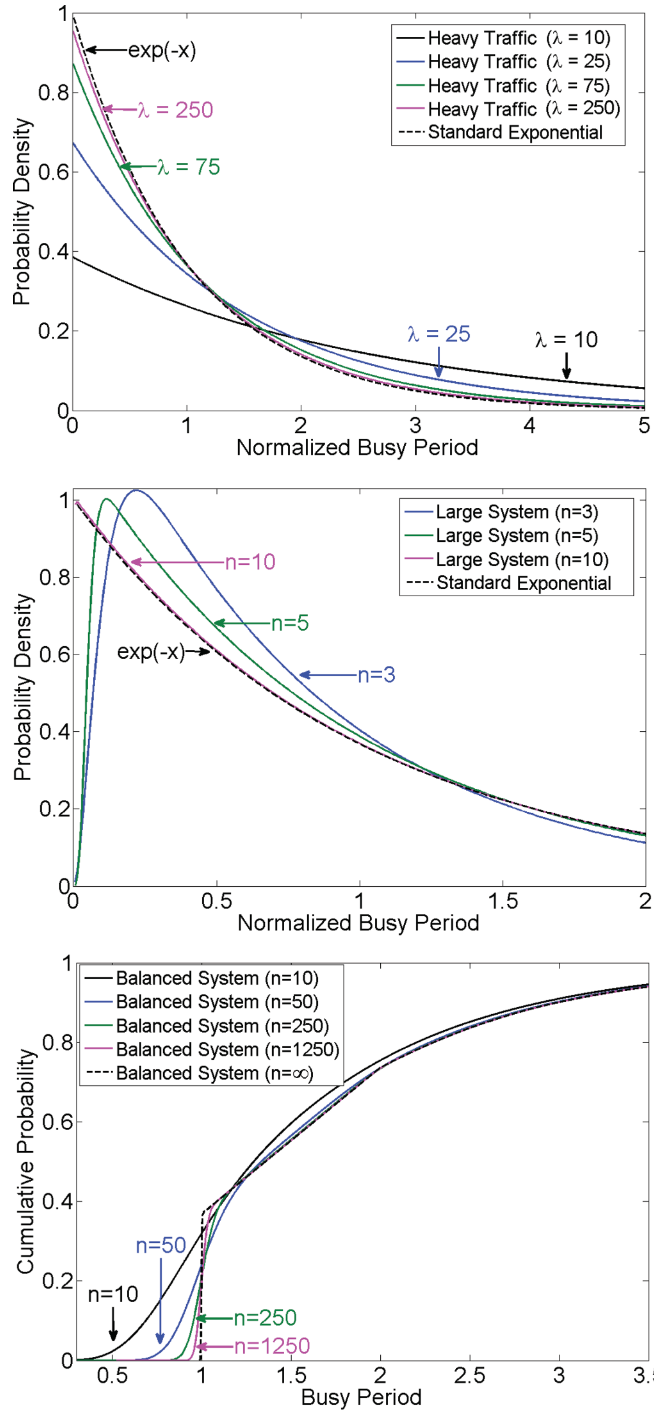

FIG. 10. (Color online) Asymptotic behavior of the busy period the Heavy-Traffic (top panel), Large-System (middle panel), and Balanced-System (bottom panel) regimes; "Standard exponential" is a shorthand for the exponential distribution with unit mean.

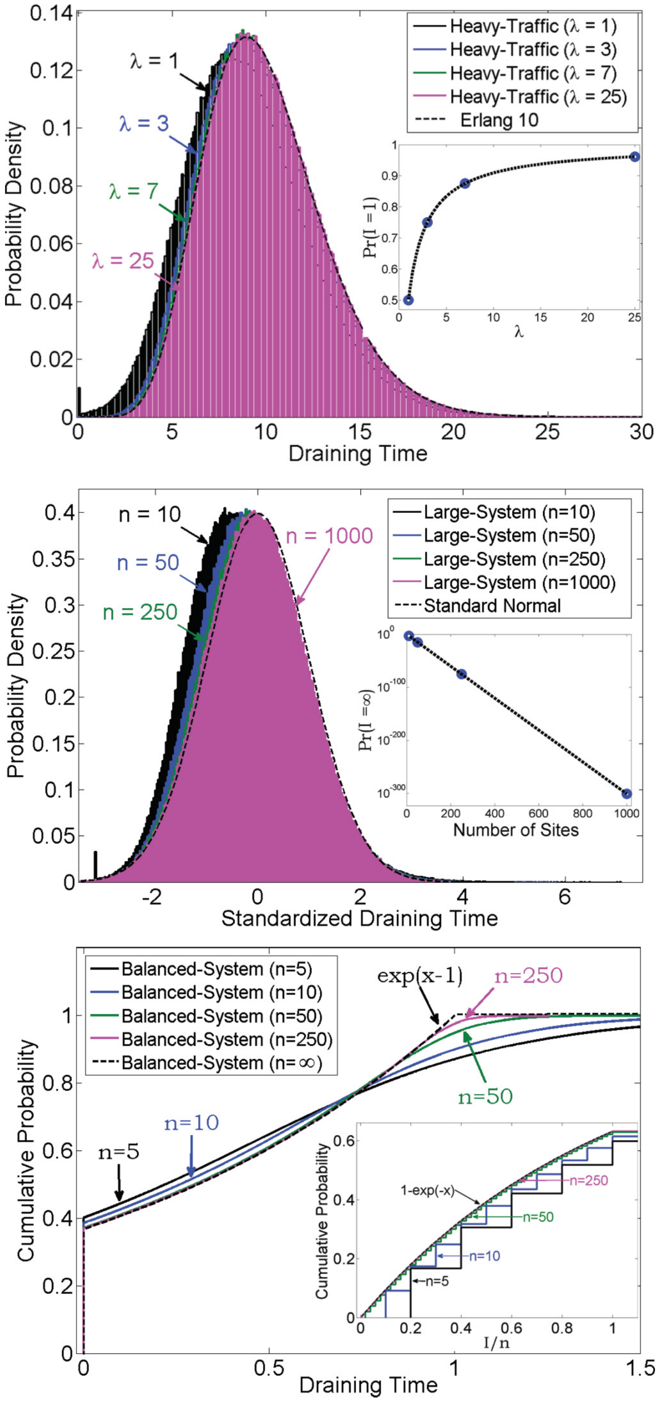

FIG. 11. (Color online) Asymptotic behavior of the first occupied siteand draining time in the Heavy-Traffic (top panel), Large-System (middle panel), and Balanced-System (bottom panel) regimes.

of parameters $\Pr(I = \infty) = 2^{-n}$, and the total error made by making use of the geometric approximation rapidly decays to zero as is clearly illustrated in the inset.

In the bottom panel of Fig. 11 we plot cumulative distribution functions of the draining time in the Balanced-System regime. Convergence to the asymptotic cumulative distribution function predicted by integrating over the density in Eq. (37) with $\lambda = \tau = 1$, while taking into account the atom at zero (i.e., the probability that $D = 0$) is clearly visible. Similarly, cumulative distribution function of the first occupied site are plotted in the inset and are shown to converge to the asymptotic cumulative distribution function predicted by Eq. (34).

IV. ASYMPTOTIC ANALYSIS: THE GENERAL CASE

In this section we shift from homogeneous ASIPs to general (inhomogeneous) ASIPs, and extend the stochastic limit laws established in Sec. III to the general case. Throughout this section we denote by $m_k = 1/\mu_k$ the mean sojourn time of particles in site k, by $\mathcal{E}$ an exponentially distributed random variable with unit mean, and by Z a Gauss-distributed random variable with zero mean and unit variance.

A. Heavy traffic

We remind the reader that the Heavy-Traffic regime considers ASIP lattices in which the inflow rate tends to infinity: $\lambda \to \infty$. Throughout this subsection we set

$$\langle m \rangle = \frac{1}{n}\sum_{k=1}^{n} m_k \tag{40}$$

and

$$\langle m^2 \rangle = \frac{1}{n}\sum_{k=1}^{n} m_k^2. \tag{41}$$

The ASIP stochastic limit laws, under the heavy-traffic regime, are as follows:

Traversal Time. As is clear from Eq. (3) the inflow rate does not affect the traversal time T. The traversal time is a sum of n independent exponential random variables with corresponding means $\{m_1, \ldots, m_n\}$. Consequently, the Laplace transform of the traversal time is given by

$$E[\exp(-\theta T)] = \prod_{k=1}^{n} \frac{1}{1 + m_k\theta}, \tag{42}$$

$(\theta \geqslant 0)$.

Overall Load. Increasing the inflow rate λ is expected to result in an increase of the overall load L. And indeed, Eq. (5) implies that the mean of the overall load L scales linearly with λ. Consequently, we normalize the overall load L by the dimensionless term $\langle m \rangle\lambda$ and analyze the stochastic limit of the normalized overall load $L/(\langle m \rangle\lambda)$ (as $\lambda \to \infty$). Setting $z = \exp(-\theta/(\langle m \rangle\lambda))$ in Eq. (6) we obtain the limit

$$\lim_{\lambda\to\infty} E\left[\exp\left(-\theta \frac{L}{\langle m \rangle\lambda}\right)\right] = \prod_{k=1}^{n} \frac{1}{1 + \frac{m_k}{\langle m \rangle}\theta}, \tag{43}$$

$(\theta \geqslant 0)$. Equation (43) implies that the limiting overall load is equal, in law, to the sum of n independent exponential random variables with corresponding means $\{m_1/\langle m \rangle, \ldots, m_n/\langle m \rangle\}$.

Busy Period. As in the case of the overall load, increasing the inflow rate λ is expected to result in an increase of the duration of the busy period B. And indeed, Eq. (10) implies that the mean of the busy period B scales like λ^{n-1}. Consequently, we normalize the busy period B by the dimensionless term $(\langle m \rangle\lambda)^{n-1}$ and analyze the stochastic limit of the normalized busy period $B/(\langle m \rangle\lambda)^{n-1}$ (as $\lambda \to \infty$). Using

Eq. (11) we obtain the limit

$$\lim_{\lambda\to\infty} E\left[\exp\left(-\theta\frac{B}{(\langle m\rangle\lambda)^{n-1}}\right)\right] = \frac{1}{1+(\langle m\rangle\prod_{k=1}^{n}\frac{m_k}{\langle m\rangle})\theta}, \tag{44}$$

($\theta \geqslant 0$). Equation (44) implies that the limiting busy period is equal, in law, to an exponential random variable with mean $\langle m\rangle\prod_{k=1}^{n}(m_k/\langle m\rangle)$. Note that $\prod_{k=1}^{n}\frac{m_k}{\langle m\rangle} \leqslant 1$ due to the inequality of arithmetic and geometric means.

First Occupied Site. Increasing the inflow rate λ is expected to increase to one the probability of finding the first site occupied. And, indeed, Eq. (14) yields the limit

$$\lim_{\lambda\to\infty}\Pr(I=1)=1. \tag{45}$$

Draining Time. Equation (45) implies that for large λ the first occupied site is effectively the first site. Consequently, for large λ the draining time D should coincide with the traversal time T. And indeed, taking the limit $\lambda\to\infty$ in Eq. (17) confirms this conjecture.

B. Large systems

We remind the reader that the large-system regime considers ASIPs in which the number of sites increases to infinity: $n\to\infty$. In Sec. III B we analyzed the large-system limit of homogeneous ASIP lattices. Throughout our analysis we have encountered sums of IID random variables and, in turn, applied the classic Central Limit Theorem. In this subsection we will make use of Lyapunov's Central Limit Theorem, a variant of the classical Central Limit Theorem in which the random summands $\{\xi_k\}$ are independent, but not necessarily identically distributed [46]. Lyapunov's theorem requires that there exists some $\delta>0$ for which the moments of order $(2+\delta)$ of the random variables $\{|\xi_k|\}$ exist and that the rate of growth of these moments is limited by the condition

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}E[|\xi_k-E[\xi_k]|^{2+\delta}]}{\left(\sum_{k=1}^{n}\mathrm{Var}[\xi_k]\right)^{\frac{2+\delta}{2}}}=0. \tag{46}$$

The theorem then asserts that the sum

$$\sum_{k=1}^{n}\frac{\xi_k-E[\xi_k]}{\sqrt{\sum_{k=1}^{n}\mathrm{Var}[\xi_k]}} \tag{47}$$

converges in distribution to a standard normal random variable Z, as n tends to infinity. A note regarding Lyapunov's condition appears in the Appendix.

Throughout this sections we will assume that the random variables $\{\triangle_1,\triangle_2,\triangle_3,\ldots\}$ and $\{G_1,G_2,G_3,\ldots\}$, that were defined, respectively, in subsections II A and II B obey Lyapunov's condition. In addition, we will assume that

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n-1}(1+\lambda m_k)^{-1}\sum_{k=1}^{n}m_k\right]<\infty. \tag{48}$$

A note regarding the condition in Eq. (48) appears in the Appendix. Provided that the conditions in Eqs. (46) and (48) jointly hold, the ASIP stochastic limit laws, under the large-system regime, are as follows:

Traversal Time. Equation (3) asserts that the traversal time T is a sum of n independent exponential random variables, $\{\triangle_1,\ldots,\triangle_n\}$, with the corresponding means $\{m_1,\ldots,m_n\}$ and variances $\{m_1^2,\ldots,m_n^2\}$. Applying Lyapunov's Central Limit Theorem we obtain that the traversal time T admits the Gaussian stochastic approximation

$$T\approx\sum_{k=1}^{n}m_k+\sqrt{\sum_{k=1}^{n}m_k^2}Z, \tag{49}$$

(as $n\to\infty$).

Overall Load. Equation (6) asserts that the overall load L is a sum of n independent geometric random variables, $\{G_1,\ldots,G_n\}$, with the corresponding means $\{\lambda m_1,\ldots,\lambda m_n\}$ and variances $\{\lambda m_1+(\lambda m_1)^2,\ldots,\lambda m_n+(\lambda m_n)^2\}$. Applying Lyapunov's Central Limit Theorem we obtain that the overall load L admits the Gaussian stochastic approximation

$$L\approx\sum_{k=1}^{n}\lambda m_k+\sqrt{\sum_{k=1}^{n}(\lambda m_k+(\lambda m_k)^2)}Z, \tag{50}$$

(as $n\to\infty$).

Busy Period. Increasing the lattice size n is expected to result in an increase in the length of the busy period. Indeed, Eq. (10) implies that for large n the mean of the busy period scales like $\prod_{k=1}^{n}(1+\lambda m_k)$. Consequently, we analyze the stochastic limit of the normalized busy period $B/\prod_{k=1}^{n}(1+\lambda m_k)$ (as $n\to\infty$). Using Eq. (11) we obtain the limit

$$\lim_{n\to\infty}E\left[\exp\left(-\theta\frac{B}{\prod_{k=1}^{n}(1+\lambda m_k)}\right)\right]=\frac{\lambda}{\lambda+\theta}, \tag{51}$$

($\theta\geqslant 0$). Since the right-hand side of Eq. (51) is the Laplace transform of an exponential distribution with mean $1/\lambda$, we obtain that the busy period B admits the stochastic approximation

$$B\approx\prod_{k=1}^{n}(1+\lambda m_k)\frac{1}{\lambda}\mathcal{E}, \tag{52}$$

(as $n\to\infty$). The derivation of Eq. (51) is given in the Appendix.

First Occupied Site. Taking the limit $n\to\infty$ in Eq. (14) yields

$$\lim_{n\to\infty}\Pr(I=k)=\frac{\lambda m_k}{1+\lambda m_k}\prod_{j=1}^{k-1}\frac{1}{1+\lambda m_j}, \tag{53}$$

($k=1,2,3,\ldots$). This result can be interpreted as an inhomogeneous geometric law. The derivation of Eq. (53) is given in the Appendix.

Draining Time. In the Appendix we show that the regularity condition given in Eq. (48) asserts that $E[\sum_{k=1}^{I-1}\triangle_k]$ is a finite constant that does not depend on n. Consequently, since the number of sites tends to infinity ($n\to\infty$), the draining time D effectively equals the traversal time T. Combining this observation together with Eq. (49) implies that the draining time D admits the Gaussian stochastic approximation

$$D \approx \sum_{k=1}^{n} m_k + \sqrt{\sum_{k=1}^{n} m_k^2} Z, \tag{54}$$

(as $n \to \infty$). The derivation of Eq. (54) is given in the Appendix.

C. Balanced systems

We remind the reader that the balanced-system regime considers ASIPs in which the number of sites tends to infinity ($n \to \infty$), and the mean sojourn time at each site tends to zero ($m_k \to 0$ for all k). In the case of homogeneous ASIPs the balance between the large number of sites and the rapid gate-opening rates was attained by setting $m_k = \tau/n$ where τ is an arbitrary positive parameter. In the case of general ASIPs the balance is attained by setting

$$m_k = \phi\left(\frac{k}{n}\right)\frac{1}{n}, \tag{55}$$

($k = 1, \ldots, n$), where $\phi(u)$ is an arbitrary positive-valued function defined on the unit interval ($0 \leqslant u \leqslant 1$). The integrability conditions that the function $\phi(u)$ needs to meet are $\int_0^1 \phi(u)du < \infty$ and $\int_0^1 \phi(u)^2 du < \infty$. In what follows, and without loss of generality, we further set $\int_0^1 \phi(u)du = \tau$.

Applying this balanced-system setting, and taking the limit $n \to \infty$, the following results are obtained: (i) the traversal time T admits the limit of Eq. (31); (ii) the overall load L admits the limit given by Eq. (32); (iii) the busy period B admits the limit given by Eq. (33); (iv) the draining time D admits the limit given by Eq. (36). Namely, in the balanced-system regime, the aforementioned observables—traversal time, overall load, busy period, and draining time—admit the same stochastic limit laws both in the case of homogeneous ASIPs and in the case of general ASIPs. A difference between homogeneous and general ASIPs is displayed by the first occupied site I. Indeed, setting $\hat{I} = I/n$ to be the scaled first occupied site, we obtain the limits

$$\lim_{n\to\infty} \Pr(\hat{I} > x) = \exp\left(-\lambda \int_0^x \phi(u)du\right),$$
$$\lim_{n\to\infty} \Pr(\hat{I} = \infty) = \exp(-\lambda\tau), \tag{56}$$

($0 \leqslant x \leqslant 1$). We note that the above mentioned results can also be obtained under milder assumptions and we refer the reader to the Appendix for details and proofs.

As in the homogeneous setting, the general balanced-system limiting regime can be understood as an $M/D/\infty$ queue. Indeed, particles arrive to the lattice following a Poisson process with rate λ. Each particle, upon its arrival to the lattice, starts traversing it. Particles' traversal times are deterministic and of common length τ, and upon "traversal completion" the particles leave the lattice. As in the homogeneous setting, the common deterministic traversal times assure that particles will leave the lattice *exactly* τ units of time after their respective arrival epochs, and will do so in a FIFO manner. One should however note the following difference between the homogeneous and inhomogeneous settings. While in the homogeneous setting particles traverse the lattice at a "constant velocity", in the inhomogeneous setting particles do so with a "local velocity" that depends on their position along the lattice. Specifically, in the homogeneous ASIP the traversal velocity is position-independent and equals $1/\tau$ (in units length per unit time), whereas in the inhomogeneous ASIP the traversal velocity is position-dependent and is given by the function $1/\phi(u)$.

V. CONCLUSIONS

In this paper we established stochastic limit laws for five key observables of the ASIP: Traversal Time, Overall Load, Busy Period, First Occupied Site, and Draining Time. Prior to this study, existing knowledge on the ASIP asymptotic statistical behavior was limited to homogeneous ASIPs in the large-system limit. Moreover this knowledge was primarily based on numerical simulations [2]. Here we considered three different asymptotic limiting regimes: the Heavy-Traffic regime, the Large-System regime, and the Balanced-System regime. We showed that each of these limiting regimes yields a set of stochastic limit laws for the ASIP's five key observables. Each set of limit laws established is, in effect, a characteristic "finger print" of the asymptotic limiting regime applied. The results were obtained analytically and in closed form, and cover both homogeneous and inhomogeneous ASIPs. This paper is the first out of two papers in which we analytically validate and considerably generalize the numerical Monte Carlo results reported in [2]. Our work joins a gallery of recent studies bridging statistical physics and queueing theory.

APPENDIX

1. Proof of the distributional Little's law

Let $A(t)$ denote the number of Poisson arrivals during a time interval of length t. Then

$$E[z^{A(T)}] = \mathbf{E_T}[E[z^{A(T)}|T]] = \mathbf{E_T}[e^{-\lambda(1-z)T}], \tag{A1}$$

where in the second equality we have used fact that $A(t)$ follows the Poisson distribution with mean λt. The right-hand side of Eq. (A1) is the Laplace transform of the traversal time T evaluated at the point $\theta = \lambda(1 - z)$ and by use Eq. (3) we therefore have

$$E[z^{A(T)}] = \prod_{k=1}^{n} \frac{\mu_k}{\mu_k + \lambda(1 - z)}. \tag{A2}$$

Comparing this result with Eq. (6) it readily follows that $E[z^L] = E[z^{A(T)}]$.

2. Derivation of Eq. (11)

Considering Eq. (9) and utilizing the law of total expectation we write the Laplace transform of B as

$$\begin{aligned} &E[\exp(-\theta B)] \\ &= Pr(T < \Delta_0)E[\exp(-\theta T)|T < \Delta_0] \\ &\quad + \Pr(\Delta_0 \leqslant T)E[\exp(-\theta(\Delta_0 + B'))|\Delta_0 \leqslant T]. \end{aligned} \tag{A3}$$

The first term in Eq. (A3) is treated by noting that the independence of the random variables Δ_0 and T implies

$$E[\exp(-\theta T)|T < \Delta_0] = \frac{\int_0^\infty f(t)e^{-\theta t}\Pr(t < \Delta_0)dt}{\Pr(T < \Delta_0)}, \tag{A4}$$

where $f(t)$ is the probability density function of T. Since $\triangle_0$ is exponentially distributed with rate λ, $\Pr(\triangle_0 > t) = e^{-\lambda t}$, and we have

$$E[\exp(-\theta T)|T < \triangle_0] = \frac{E[\exp(-(\theta+\lambda)T)]}{\Pr(T < \triangle_0)}. \tag{A5}$$

We now note that the Laplace transform of the random variable T is given by Eq. (3) and we therefore have

$$\Pr(T < \triangle_0)E[\exp(-\theta T)|T < \triangle_0] = \prod_{k=1}^{n}\frac{\mu_k}{\mu_k+\theta+\lambda}. \tag{A6}$$

The second term in Eq. (A3) is treated by noting that the random variables $\{\triangle_0, T, B'\}$ are independent, and that B' is an IID copy of B. Therefore,

$$\begin{aligned} &E[\exp(-\theta(\triangle_0+B'))|\triangle_0 \leqslant T] \\ &\quad = \frac{E[\exp(-\theta B)]\int_0^\infty f(t)\left[\int_0^t g(z)e^{-\theta z}dz\right]dt}{\Pr(\triangle_0 \leqslant T)}, \end{aligned} \tag{A7}$$

where $g(z) = \lambda e^{-\lambda z}$ is the probability density function of $\triangle_0$. The double integral gives

$$\begin{aligned} &\frac{\lambda}{\lambda+\theta}\int_0^\infty f(t)[1-e^{-(\lambda+\theta)t}]dt \\ &\quad = \frac{\lambda}{\lambda+\theta}\left[1-\prod_{k=1}^{n}\frac{\mu_k}{\mu_k+\theta+\lambda}\right], \end{aligned} \tag{A8}$$

and we conclude that

$$\begin{aligned} &E[\exp(-\theta(\triangle_0+B'))|\triangle_0 \leqslant T] \\ &\quad = \frac{\lambda E[\exp(-\theta B)]\left[1-\prod_{k=1}^{n}\frac{\mu_k}{\mu_k+\theta+\lambda}\right]}{\Pr(\triangle_0 \leqslant T)(\lambda+\theta)}. \end{aligned} \tag{A9}$$

Substituting Eqs. (A6) and (A9) into Eq. (A3) and rearranging terms we obtain Eq. (11). Equation (10) can be obtained directly by using $E[B] = -\frac{dE[\exp(-\theta B)]}{d\theta}|_{\theta=0}$.

3. Derivation of Eqs. (16) and (17)

Considering Eq. (15) and conditioning on the value of the the first non-empty site I we obtain the following expressions:

$$\begin{aligned} E[D] &= E[E[D|I]] = \sum_{k=1}^{n}\Pr(I=k)\sum_{j=k}^{n}\frac{1}{\mu_j} \\ E[e^{-\theta D}] &= E[E[e^{-\theta D}|I]] = \Pr(I=\infty) \\ &\quad + \sum_{k=1}^{n}\Pr(I=k)\prod_{j=k}^{n}\frac{\mu_j}{\theta+\mu_j}. \end{aligned} \tag{A10}$$

Equations (16) and (17) follow by substituting Eq. (14) into Eq. (A10).

4. Derivation of Eq. (30)

Intuitively, Eq. (30) is most easily understood by noting that in the large-system limit, the draining and traversal times are both given by infinite sums of independent exponential random variables, $D = \sum_{k=I}^{\infty}\triangle_k$ and $T = \sum_{k=1}^{\infty}\triangle_k$, correspondingly. Moreover, the only difference between the two infinite sums is a sum of $I-1$, independent, exponential random variables whose expected value is

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{I-1}\triangle_k\right] = \mathbf{E_I}\left[E\left[\sum_{k=1}^{I-1}\triangle_k|I\right]\right] = \frac{1}{\lambda}, \tag{A11}$$

a finite constant that does not depend on n. Equation (A11) readily follows from the geometric distribution of the first occupied site in the large system limit, see Eq. (29). The difference between the traversal time and the draining time is hence negligible in the large-system limit.

More precisely, Eq. (30) is derived by substituting $-i\theta$ for θ in (see in the sequel) Eq. (A21) to obtain the characteristic function of the draining time

$$\begin{aligned} E[\exp(i\theta D)] &= \frac{-i\theta}{-i\theta-\lambda}\left(\frac{1}{1+\lambda m}\right)^n \\ &\quad + \frac{\lambda}{\lambda+i\theta}\left(\frac{1}{1-i\theta m}\right)^n. \end{aligned} \tag{A12}$$

The characteristic function of the standardized draining time, $\frac{D-nm}{m\sqrt{n}}$, follows:

$$\begin{aligned} &E\left[\exp\left(i\theta\left[\frac{D-nm}{m\sqrt{n}}\right]\right)\right] \\ &\quad = \left[\frac{-i\theta/(m\sqrt{n})}{-i\theta/(m\sqrt{n})-\lambda}\left(\frac{1}{1+\lambda m}\right)^n \right.\\ &\qquad \left. + \frac{\lambda}{\lambda+i\theta/(m\sqrt{n})}\left(\frac{1}{1-i\theta/\sqrt{n}}\right)^n\right]\exp(-i\theta\sqrt{n}). \end{aligned} \tag{A13}$$

Recalling the Taylor expansion

$$nln\left[\frac{1}{1-i\theta/\sqrt{n}}\right] = i\theta\sqrt{n} - \theta^2/2 + O(1/\sqrt{n}) \tag{A14}$$

and taking the large-system limit of Eq. (A13) we obtain

$$\lim_{n\to\infty} E\left[\exp\left(i\theta\left[\frac{D-nm}{m\sqrt{n}}\right]\right)\right] = \exp(-\theta^2/2), \tag{A15}$$

which is the characteristic function of a normal random variable with zero mean and unit variance.

5. Derivation of Eq. (34)

We now note that

$$\Pr(\hat{I} > x) = \sum_{i/n>x}\frac{\lambda}{n/\tau+\lambda}\frac{1}{(1+\lambda\tau/n)^{i-1}} + \Pr(\hat{I}=\infty) \tag{A16}$$

taking the limit $n\to\infty$ we find that the first term is a Riemann sum that converges to the integral

$$\lim_{n\to\infty}\Pr(\hat{I} > x) = \lambda\tau\int_x^1 \exp(-\lambda\tau u)du, \tag{A17}$$

and that the second term is given by

$$\lim_{n\to\infty}\Pr(\hat{I}=\infty) = \lim_{n\to\infty}\left(\frac{1}{1+\lambda\tau/n}\right)^n = \exp(-\lambda\tau). \tag{A18}$$

Equation (34) readily follows.

6. Derivation of Eq. (36)

In order to derive Eq. (36) we first note that, in the case of a homogeneous ASIP lattice, Eq. (17) reads

$$E[\exp(-\theta D)] = \left(\frac{1}{1+\lambda m}\right)^n + \frac{\lambda m}{1+\lambda m}\sum_{k=1}^{n}\left(\frac{1}{1+\lambda m}\right)^{k-1} \times \left(\frac{1}{1+\theta m}\right)^{n-k+1}. \tag{A19}$$

We sum the series by noting that

$$\sum_{k=1}^{n} a^{k-1}b^{n-k+1} = \frac{b(a^n-b^n)}{a-b}, \tag{A20}$$

a formula that is easily proved by use of either geometric series summation or mathematical induction. We obtain

$$E[\exp(-\theta D)] = \frac{\theta}{\theta-\lambda}\left(\frac{1}{1+\lambda m}\right)^n + \frac{\lambda}{\lambda-\theta}\left(\frac{1}{1+\theta m}\right)^n. \tag{A21}$$

Equation (36) follows from substituting $m=\tau/n$ into Eq. (A21) and taking the limit $n\to\infty$.

7. Derivation of the large system limiting regime—General case

a. *Notes on regularity conditions*

(1) In practice it is usually easiest to check the Lyapunov's condition for $\delta=1$ and it is easily verified that the condition holds for the *special case* in which the following two limits exist:

$$\langle\sigma^2\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{Var}[\xi_k], \qquad \langle\kappa^3\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E[|\xi_k-E[\xi_k]|^3]. \tag{A22}$$

(2) The fact that Lyapunov's condition holds for the random variables $\{\triangle_1,\triangle_2,\triangle_3\cdots\}$ and $\{G_1,G_2,G_3\cdots\}$ implies that

$$\sum_{k=1}^{\infty} m_k = \infty, \qquad \prod_{k=1}^{\infty}(1+\lambda m_k) = \infty. \tag{A23}$$

Indeed, since $\sum_{k=1}^{n}E[|\xi_k-E[\xi_k]|^{2+\delta}]$ is monotonically increasing with n, Eq. (46) implies that $\sum_{k=1}^{\infty}\mathrm{Var}[\xi_k]$ diverges. In the case of the random variables, $\{\triangle_1,\triangle_2,\triangle_3\cdots\}$, this means that $\sum_{k=1}^{\infty}m_k^2$ diverges and in the case of the random variables, $\{G_1,G_2,G_3\cdots\}$, this means that $\sum_{k=1}^{\infty}(\lambda m_k+\lambda^2 m_k^2)$ diverges. In any case, since $\sum_{k=1}^{\infty}m_k<\infty\to\sum_{k=1}^{\infty}m_k^2<\infty$ it follows that $\sum_{k=1}^{\infty}m_k$ must diverge as well. In addition, since

$$l1+\lambda\sum_{k=1}^{n}m_k \leqslant \prod_{k=1}^{n}(1+\lambda m_k) \leqslant e^{\lambda\sum_{k=1}^{n}m_k}, \tag{A24}$$

it follows that $\sum_{k=1}^{\infty}m_k$ and $\prod_{k=1}^{\infty}(1+\lambda m_k)$ converge or diverge together.

(3) The regularity condition in Eq. (48) implies that

$$\lim_{n\to\infty}\left[\frac{\sum_{k=1}^{n}m_k}{\prod_{k=1}^{n-1}(1+\lambda m_k)}\right]=0. \tag{A25}$$

Since

$$\frac{\frac{m_j}{(1+\lambda m_j)}}{\prod_{k=1}^{n}(1+\lambda m_k)} \leqslant \frac{\sum_{k=1}^{n}\frac{m_k}{(1+\lambda m_k)}}{\prod_{k=1}^{n}(1+\lambda m_k)} \leqslant \frac{\sum_{k=1}^{n}m_k}{\prod_{k=1}^{n-1}(1+\lambda m_k)} \tag{A26}$$

it follows that

$$\lim_{n\to\infty}\left[\frac{\frac{m_j}{(1+\lambda m_j)}}{\prod_{k=1}^{n}(1+\lambda m_k)}\right]=0, \qquad \lim_{n\to\infty}\left[\frac{\sum_{k=1}^{n}\frac{m_k}{(1+\lambda m_k)}}{\prod_{k=1}^{n}(1+\lambda m_k)}\right]=0. \tag{A27}$$

b. *Busy period*

From Eq. (11), the Laplace transform of $B/\prod_{k=1}^{n}(1+\lambda m_k)$ is given by

$$E\left[\exp\left(-\theta\frac{B}{\prod_{k=1}^{n}(1+\lambda m_k)}\right)\right] = \frac{\lambda+\theta/\prod_{k=1}^{n}(1+\lambda m_k)}{\lambda+\theta\prod_{k=1}^{n}\left[1+\theta m_k/\left((1+\lambda m_k)\prod_{j=1}^{n}(1+\lambda m_j)\right)\right]}. \tag{A28}$$

Equation (A23) asserts that the second term in the nominator of the right-hand side of Eq. (A28) is negligible in the large n limit. Taking the logarithm of the second term in the right hand side of the denominator of Eq. (A28) and using Eq. (A27) we have

$$\log\left[\prod_{k=1}^{n}\left[1+\theta m_k\Big/\left((1+\lambda m_k)\prod_{j=1}^{n}(1+\lambda m_j)\right)\right]\right] \cong \theta\left[\frac{\sum_{k=1}^{n}\frac{m_k}{(1+\lambda m_k)}}{\prod_{k=1}^{n}(1+\lambda m_k)}\right]\longrightarrow 0, \tag{A29}$$

(as $n\to\infty$). The result in Eq. (51) follows from the continuity of the exponential function.

c. *First occupied site*

In order to obtain Eq. (53) it is enough to take the limit $n\to\infty$ in Eq. (14) and use Eq. (A23).

d. *Draining time*

Provided that Lyapunov's condition holds for the random variables, $\{\triangle_1,\triangle_2,\triangle_3,\ldots\}$, Eq. (49) asserts that

$$\frac{\sum_{k=1}^{n}\triangle_k-\sum_{k=1}^{n}m_k}{\sqrt{\sum_{k=1}^{n}m_k^2}}\xrightarrow{d} Z, \tag{A30}$$

(as $n \to \infty$). In order to show that

$$\frac{D - \sum_{k=1}^{n} m_k}{\sqrt{\sum_{k=1}^{n} m_k^2}} \xrightarrow{d} Z, \tag{A31}$$

(as $n \to \infty$), we note that $D = \sum_{k=I}^{n} \triangle_k$ and recall that if ξ_n is a random variable that converges in distribution to ξ and the difference between the random variables ξ_n and ζ_n converges in probability to zero, then ζ_n also converges in distribution to ξ [47]. It is therefore sufficient to show that

$$\lim_{n\to\infty} \Pr\left(\left|\frac{\sum_{k=1}^{I-1} \triangle_k}{\sqrt{\sum_{k=1}^{n} m_k^2}} > \varepsilon\right|\right) \to 0\ . \tag{A32}$$

By use of Markov's inequality we have

$$\Pr\left(\left|\frac{\sum_{k=1}^{I-1} \triangle_k}{\sqrt{\sum_{k=1}^{n} m_k^2}} > \varepsilon\right|\right) \leqslant \frac{E\left[\sum_{k=1}^{I-1} \triangle_k\right]}{\varepsilon\sqrt{\sum_{k=1}^{n} m_k^2}}\ . \tag{A33}$$

By use of Eq. (53) the nominator in the right-hand side of Eq. (A33) gives

$$E\left[\sum_{k=1}^{I-1} \triangle_k\right] = \mathbf{E_I}\left[E\left[\sum_{k=1}^{I-1} \triangle_k | I\right]\right]$$
$$\cong \sum_{n=1}^{\infty} \frac{\lambda m_n}{1+\lambda m_n} \prod_{k=1}^{n-1} (1+\lambda m_j)^{-1} \sum_{k=1}^{n-1} m_k\ . \tag{A34}$$

Since

$$\frac{\frac{\lambda m_n}{1+\lambda m_n} \sum_{k=1}^{n-1} m_k}{\prod_{k=1}^{n-1}(1+\lambda m_k)} \leqslant \frac{\sum_{k=1}^{n} m_k}{\prod_{k=1}^{n-1}(1+\lambda m_k)}\ , \tag{A35}$$

the regularity condition in Eq. (48) asserts that $E[\sum_{k=1}^{I-1} \triangle_k]$ is a finite constant that does not depend on n. Recalling that Lyapunov's condition for the random variables, $\{\triangle_1, \triangle_2, \triangle_3, \ldots\}$ implies that $\sum_{k=1}^{\infty} m_k^2 = \infty$ (see Sec. A 7a), we conclude that

$$\lim_{n\to\infty} \frac{E\left[\sum_{k=1}^{I-1} \triangle_k\right]}{\varepsilon\sqrt{\sum_{k=1}^{n} m_k^2}} = 0 \tag{A36}$$

and the desired result, Eq. (54), follows.

8. Derivation of the balanced system limiting regime—General case

a. Regularity conditions

In proving the results presented in Sec. IV C we first note that the setup depicted there is a special case of a more general setup. Here we will assume that the set $\{m_k(n)\}$ is chosen such that there exists a positive-valued function $\phi(u)$ that obeys

$$\lim_{n\to\infty} \sum_{k/n<x} m_k(n) = \int_0^x \phi(u)du < \infty,$$
$$\lim_{n\to\infty} \sum_{k/n<x} m_k(n)^2 = 0, \tag{A37}$$

$(0 \leqslant x \leqslant 1)$. In particular, and without loss of generality, we denote

$$F(x) = \int_0^x \phi(u)du, \tag{A38}$$

$(0 \leqslant x \leqslant 1)$, and set $F(1) = \tau$. One can now easily verify that Eq. (A37) holds for the special case in which $m_k(n) = \phi(\frac{k}{n})\frac{1}{n}, \int_0^1 \phi(u)du = \tau$, and $\int_0^1 \phi(u)^2 du < \infty$.

b. Traversal time

Taking the logarithm of Eq. (3) we obtain

$$\log[E[\exp(-\theta T)]] = -\sum_{k=1}^{n} \log[1 + m_k(n)\theta], \tag{A39}$$

$(\theta \geqslant 0)$. We now note that

$$-\sum_{k=1}^{n} \log[1 + m_k(n)\theta] \cong -\theta \sum_{k=1}^{n} m_k(n) - \frac{\theta}{2} \sum_{k=1}^{n} m_k(n)^2 \tag{A40}$$

and after taking the balanced-system limit of this equation we have

$$\lim_{n\to\infty} -\sum_{k=1}^{n} \log[1 + m_k(n)\theta] = -\theta\tau\ . \tag{A41}$$

The desired result, Eq. (31), follows from the continuity of the exponential function.

c. Overall load

Taking the logarithm of Eq. (6) we obtain

$$\log[E[z^L]] = -\sum_{k=1}^{n} \log[1 + m_k(n)\lambda(1-z)], \tag{A42}$$

$(|z| \leqslant 1)$. We now note that

$$-\sum_{k=1}^{n} \log[1 + m_k(n)\lambda(1-z)]$$
$$\cong -\lambda(1-z) \sum_{k=1}^{n} m_k(n) - \frac{\lambda(1-z)}{2} \sum_{k=1}^{n} m_k(n)^2\ , \tag{A43}$$

and after taking the balanced-system limit of this equation we have

$$\lim_{n\to\infty} -\sum_{k=1}^{n} \log\left[1 + m_k(n)\lambda(1-z)\right] = -\lambda(1-z)\tau\ . \tag{A44}$$

The desired result, Eq. (32), follows from the continuity of the exponential function.

d. Busy period

Taking the balanced-system limit of Eq. (11) we have (by use of continuity)

$$\lim_{n\to\infty} E[\exp(-\theta B)]$$
$$= \frac{\lambda + \theta}{\lambda + \theta \lim_{n\to\infty} \prod_{k=1}^{n}[1 + (\lambda+\theta)m_k(n)]}\ , \tag{A45}$$

($\theta \geqslant 0$). We now note that

$$\log\left[\prod_{k=1}^{n}[1+(\lambda+\theta)m_k(n)]\right] = \sum_{k=1}^{n}\log[1+(\lambda+\theta)m_k(n)] \simeq (\lambda+\theta)\sum_{k=1}^{n}m_k(n)+\frac{\lambda+\theta}{2}\sum_{k=1}^{n}m_k(n)^2, \quad \text{(A46)}$$

and after taking the balanced-system limit of this equation we have

$$\lim_{n\to\infty}\log\left[\prod_{k=1}^{n}[1+(\lambda+\theta)m_k(n)]\right] = (\lambda+\theta)\tau\,. \quad \text{(A47)}$$

The desired result, Eq. (33), follows from the continuity of the exponential function.

e. First occupied site

We note that

$$\Pr(\hat{I} > x) = \Pr(\hat{I}=\infty) + \sum_{k/n>x}\frac{\lambda}{1/m_k(n)+\lambda}\frac{1}{\prod_{j=1}^{k-1}(1+\lambda m_j(n))}\,. \quad \text{(A48)}$$

Taking the balanced-system limit of both sides we find that the first term is given by

$$\lim_{n\to\infty}\Pr(\hat{I}=\infty) = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{1}{1+\lambda m_k(n)} = \exp(-\lambda\tau)\,. \quad \text{(A49)}$$

The second term converges to the integral

$$\Pr(x < \hat{I} \leqslant 1) = \lambda\int_x^1 \phi(u)\exp\left(-\lambda F(u)\right)du \quad \text{(A50)}$$

which in turn gives

$$\Pr(x < \hat{I} \leqslant 1) = \exp\left(-\lambda F(x)\right) - \exp(-\lambda\tau)\,. \quad \text{(A51)}$$

Equation (56) readily follows.

f. Draining time

We first note that

$$E[\exp(-\theta D)] = \mathbf{E}_{\hat{\mathbf{I}}}[E[\exp(-\theta D)|\hat{I}]]\,. \quad \text{(A52)}$$

Taking the balanced-system limit of both sides we have

$$\lim_{n\to\infty}E[\exp(-\theta D)] = \exp(-\lambda\tau)+\lambda\int_0^1\phi(u)\exp\left(-\lambda F(u)\right) \times \exp\left(-\theta(\tau-F(u))\right)du\,, \quad \text{(A53)}$$

and the desired result, Eq. (36), follows.

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