Asymmetric Inclusion Process as a Showcase of Complexity

Shlomi Reuveni,[1,2] Iddo Eliazar,[3] and Uri Yechiali[2]

[1]School of Chemistry, Tel-Aviv University, Tel-Aviv 69978, Israel

[2]Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel

[3]Department of Technology Management, Holon Institute of Technology, Holon 58102, Israel

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The asymmetric inclusion process is a lattice-gas model which replaces the "fermionic" exclusion interactions of the asymmetric exclusion process by "bosonic" inclusion interactions. Combining together probabilistic and Monte Carlo analyses, we showcase the model's rich statistical complexity—which ranges from "mild" to "wild" displays of randomness: Gaussian load and draining, Rayleigh outflow with linear aging, inverse-Gaussian coalescence, intrinsic power-law scalings and power-law fluctuations and condensation.

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The asymmetric inclusion process (ASIP), a lattice-gas model in one dimension, was introduced and analyzed in [1]. The ASIP is an exactly solvable "bosonic" counterpart of the "fermionic" asymmetric exclusion process (ASEP)—a fundamental model in nonequilibrium statistical physics [2–4]. In both processes, random events cause particles to propagate unidirectionally along a one-dimensional lattice. In the ASEP particles are subject to exclusion interactions that keep them singled apart, whereas in the ASIP particles are subject to inclusion interactions that coalesce them into inseparable clusters. We note that other coalescence (also known as coagulation) processes, in which particles attract or cluster, have been studied in the past and we refer the interested reader to [5] and to references therein.

The formulation of the ASIP is as follows. Consider a one-dimensional lattice of n sites indexed $k = 1, \ldots, n$. Each site is followed by a gate—labeled by the site's index—which controls the site's outflow. Particles arrive at the first site ($k = 1$) following a Poisson process with rate λ, the openings of gate k are timed according to a Poisson process with rate μ_k ($k = 1, \ldots, n$), and the $n + 1$ Poisson processes are mutually independent. Note that from this definition it follows that the times between particle arrivals, and the times between the openings of gate k, are independent and exponentially distributed—the former with mean $1/\lambda$, and the latter with mean $1/\mu_k$ ($k = 1, \ldots, n$). At an opening of gate k all particles present at site k transit simultaneously, and in one cluster (one "batch"), to site $k + 1$—thus joining particles that may already be present at site $k + 1$ ($k = 1, \ldots, n - 1$). At an opening of the last gate ($k = n$) all particles present at the last site ($k = n$) exit the lattice simultaneously. The ASIP model is illustrated in Fig. 1.

The ASIP links together the ASEP with an apparently unrelated model—the tandem Jackson network (TJN), which is a fundamental service model in queuing theory [6–8]. All three models—ASEP, TJN, and ASIP—share the aforementioned sites-gates lattice structure. To pinpoint the difference between the models consider the two following characteristic capacities: (i) site capacity c_{site}—the number of particles that can simultaneously occupy a given site, and (ii) gate capacity c_{gate}—the number of particles that are simultaneously transferred through a given gate when it opens. In the ASEP $c_{\text{site}} = 1$ and $c_{\text{gate}} = 1$ (or $c_{\text{gate}} = \infty$), in the TJN $c_{\text{site}} = \infty$ and $c_{\text{gate}} = 1$, and in the ASIP $c_{\text{site}} = \infty$ and $c_{\text{gate}} = \infty$. The case $c_{\text{gate}} = \infty$ is also referred as "unlimited batch service" and the ASIP can hence be thought of as a TJN with this additional property. The capacity classification is summarized in Table I—from which it is evident that the ASIP is, in effect, the "missing puzzle piece" connecting together the well established ASEP and TJN models.

The analysis conducted in [1] concludes that the ASIP, despite its simple description, displays highly complex stochastic dynamics. An iterative scheme for the computation of the multidimensional probability generating function (PGF) of the ASIP's site occupancies at steady state was established. Yet, the PGF turns out to be analytically intractable even for small n—a fact that is vivid from the very rapid growth in complexity of the explicit PGF expressions for $n = 1, 2, 3$ [1]. Interestingly, the ASIP's load—defined as the total number of particles present in the lattice—is tractable, and explicit results for its steady state distribution were obtained.

Homogeneous ASIPs are characterized by identical gate opening rates: $\mu_1 = \cdots = \mu_n$. The subclass of

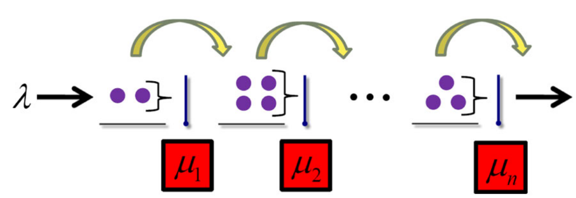

FIG. 1 (color online). An Illustration of the ASIP model.

TABLE I. Capacity classification of the ASEP, TJN, and ASIP models.

	$c_{\text{site}} = 1$	$c_{\text{site}} = \infty$
$c_{\text{gate}} = 1$	ASEP	TJN
$c_{\text{gate}} = \infty$	ASEP	ASIP

homogeneous ASIPs is of special importance. Indeed, it has been shown [1] that amongst the class of general ASIPs the subclass of homogeneous ASIPs is optimal with respect to various measures of efficiency (in what follows $\mu = \mu_1 + \cdots + \mu_n$ denotes the ASIP's total gate opening rate): (i) minimization of the load-mean subject to a given μ; (ii) minimization of the load-variance subject to a given μ; (iii) maximization of the zero-load probability subject to a given μ; (iv) minimization of the load-variance subject to a given load mean.

In this Letter, combining together analytic results and Monte Carlo simulations, we study the homogeneous ASIP with rates: $\lambda = \mu_1 = \cdots = \mu_n = 1$. The analysis, focusing on the ASIPs steady state in large lattices ($n \gg 1$), will reveal a rich assortment of statistical behaviors which manifest the ASIP's intrinsic complexity.

In what follows we denote by $X_k(t)$ the number of particles present in site k at time t, and set $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$. The random vector $\mathbf{X}(t)$ represents the ASIP's occupancy at time t ($t \geq 0$). The stochastic processes $(\mathbf{X}(t))_{t\geq 0}$ is asymptotically stationary, and it converges in law (as $t \to \infty$) to a stochastic limit $\mathbf{X} = (X_1, \ldots, X_n)$ [1]. The random variable X_k represents the number of particles present at site k at steady state, and the random vector $\mathbf{X}$ represents the ASIP's occupancy at steady state. As noted above the PGF of the random vector $\mathbf{X}$ is effectively intractable. Henceforth, given a vector $\mathbf{v} = (v_1, \ldots, v_n)$ we denote by $|\mathbf{v}| = v_1 + \cdots + v_n$ its sum of components, and by $\#(\mathbf{v})$ the number of its nonzero components.

In systems where fluctuations around the average level of the occupancy vector are not too wild, an average based description provides a fair, first order, approximation to the systems' behavior. This holds for the ASEP and TJN, but does not hold for the ASIP. Analysis asserts that the ASIP's mean occupancy at steady state is given by $\langle X_k \rangle = \lambda/\mu_k$ ($k = 1, \ldots, n$) [1]. Thus, for our homogeneous ASIP $\langle X_k \rangle = 1$. On the other hand, Monte Carlo analysis depicted in Fig. 2 asserts that the following power-law asymptotics hold ($k \gg 1$):

$$\begin{aligned} \Pr(X_k > 0) &\approx k^{-1/2}, \\ \langle X_k | X_k > 0 \rangle &\approx k^{1/2}, \\ \sigma(X_k)/\langle X_k \rangle &\approx k^{1/4}, \end{aligned} \tag{1}$$

where $\sigma(X_k)$ denotes the standard deviation of the random variable X_k. Namely, at steady state (i) the probability that site k is occupied decreases like $k^{-1/2}$, (ii) the conditional mean number of particles occupying site k, given that the site is occupied, increases like $k^{1/2}$ [9], and (iii) the standard deviation of the number of particles occupying site k, measured with respect to the mean number of particles occupying site k, increases like $k^{1/4}$. The power-law asymptotics of Eq. (1) imply that "downstream sites" ($k \gg 1$) are rarely occupied, but when they are—they are occupied by a large number of particles (in comparison to the mean occupancy $\langle X_k \rangle = 1$). This "all-or-none" type of steady-state behavior results in large occupancy fluctuations of downstream sites, and hence renders the "averaged" description of the ASIP rather limited.

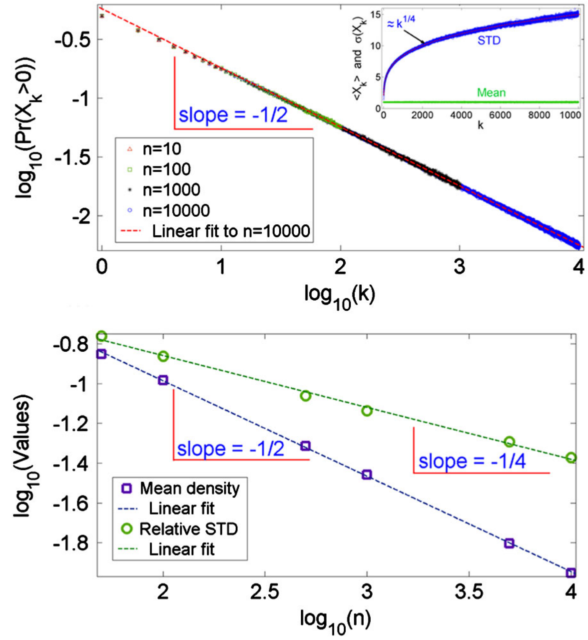

FIG. 2 (color online). Top panel: The probability that site k is occupied, as a function of the index k, on a log-log plot; the linear fit implies a power law decay with exponent $-1/2$. Top panel inset: The mean and the standard deviation (STD) of the occupancy of site k, as a function of the index k. Each site is occupied, on average, by a single particle, yet the fluctuations around the mean grow like $k^{1/4}$, and are hence typically much larger than the mean occupancy itself. Bottom panel: The mean and the relative STD of the density of occupied sites, as a function of the lattice size n, on a log-log plot; the linear fits imply power law decays with respective exponents $-1/2$ and $-1/4$—which, in turn, manifests condensation.

The power-law asymptotics of Eq. (1) are further induced to the ASIP's density of occupied sites: $D_n = \frac{1}{n}\#(\mathbf{X})$. A Monte Carlo analysis depicted in Fig. 2 asserts that the following power-law asymptotics hold ($n \gg 1$):

$$\langle D_n \rangle \approx n^{-1/2}, \qquad \sigma(D_n)/\langle D_n \rangle \approx n^{-1/4}, \tag{2}$$

where $\sigma(D_n)$ denotes the standard deviation of the density D_n. Namely, at steady state the particles occupying the lattice sites condense to a vanishingly small fraction D_n of sites: (i) the mean density of occupied sites decreases to zero like $n^{-1/2}$; (ii) the standard deviation of the density of occupied sites, measured with respect to the mean density of occupied sites, decreases like $n^{-1/4}$. This power-law condensation of particles is yet another manifestation of the 'all-or-none' steady-state behavior noted above.

We now turn to explore four key observables Θ_n of the ASIP at steady state: Load, draining time, interexit time, and coalescence time. A Monte Carlo analysis asserts that these four observables (to be defined momentarily) are random variables admitting asymptotic stochastic approximations of the form ($n \gg 1$):

$$\Theta_n \approx a_n \Theta + b_n, \tag{3}$$

where a_n and b_n are deterministic scaling coefficients, and where Θ is a limiting random variable. For each observable the coefficients a_n, b_n, and the limit Θ will be specified hereinafter.

Load.—As noted above, the ASIP's load is the total number of particles present in the lattice: $\Theta_n = |\mathbf{X}|$. For the ASIP's load the coefficients are $a_n = \sqrt{2n}$ and $b_n = n$, and the limit Θ is Gaussian with zero mean and unit variance. This result is identical, in form, to the standard central limit theorem (CLT) [10]. However, while the standard CLT setting requires the random variables $\{X_k\}_{k=1}^n$ to be independent and identically distributed, in the ASIP the random variables $\{X_k\}_{k=1}^n$ are neither independent nor identically distributed. Rather surprisingly, it was established that, at steady state [1]: The load of an ASIP with rates $(\lambda, \mu_1, \ldots, \mu_n)$ is equal, in law, to the sum of loads of n independent single-site ASIPs with respective rates $(\lambda, \mu_1), \ldots, (\lambda, \mu_n)$. Thus, in the case of homogeneous ASIPs, the CLT can be applied to obtain the load asymptotics.

Draining time.—Consider the ASIP with no inflow ($\lambda = 0$), and assume that at time $t = 0$ the ASIP's occupancy is given by the steady state vector: $\mathbf{X}(0) = \mathbf{X}$. The ASIP's draining time is the time elapsing till the lattice is clear of particles: $\Theta_n = \inf\{t \geq 0 | |\mathbf{X}(t)| = 0\}$. In other words, the ASIP's draining time is the random time required for "draining out" an ASIP at steady state, after having blocked the inflow of new-coming particles. For the ASIP's draining time the coefficients are $a_n = \sqrt{n}$ and $b_n = n$, and the limit Θ is Gaussian with zero mean and unit variance. In effect, Monte Carlo analysis illustrated in Fig. 3 asserts that the draining time Θ_n is approximately gamma with mean n and variance n. In turn, this gamma approximation implies that the draining time Θ_n is equal, in law, to the sum of n independent exponential random variables with unit mean [10]. Thus, the CLT applies to the draining-time asymptotics as well.

Interexit time.—The openings of the last ASIP gate are governed by a Poisson process with unit rate ($\mu_n = 1$), and when the last gate opens all the particles present in the last site exit the lattice. Equation (1) asserts that the steady-state probability that the last site is nonempty is given by $\Pr(X_n > 0) \approx n^{-1/2}$. Consequently, not every opening of

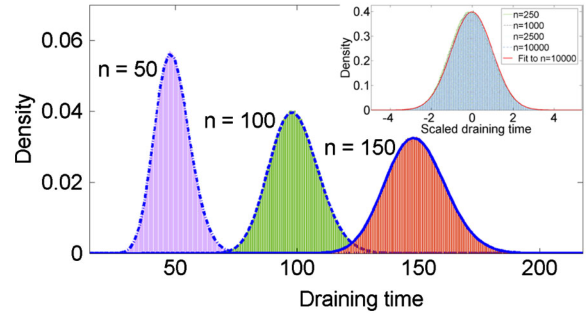

FIG. 3 (color online). Gamma approximation of the draining time; bars represent simulated histograms, and dashed lines represent Gamma density fits. The mean and the variance of the Gamma distribution, for $n = [50, 100, 150]$, are given respectively by [49, 99, 149] and [51.1, 101.3, 151.2]. Inset: The Gaussian limit of the scaled draining time.

the last gate indeed results in an exit of particles from the lattice. The ASIP's interexit time—for an ASIP in steady state—is defined as the time elapsing between two consecutive time epochs at which particles exit the lattice. For the ASIP's interexit time the coefficients are $a_n = \sqrt{\pi n}$ and $b_n = 0$, and the limit Θ is Rayleigh with unit mean and probability tail

$$\Pr(\Theta > t) = \exp(-\pi t^2/4) \tag{4}$$

($t > 0$). In effect, the Monte Carlo analysis illustrated in Fig. 4 shows that the Rayleigh approximation well captures the data even for relatively small n.

The hazard rate $h_T(t)$ ($t > 0$) of a random time T is defined as the limit $h_T(t) = \lim_{\delta\to 0}\frac{1}{\delta}\Pr(T \leq t + \delta | T > t)$ [11]. Namely, given that the random time T did not realize during the time interval $[0, t]$, the realization rate of the random time T immediately after time t is $h_T(t)$. On the one hand, the ASIP's interarrival time is exponential—which is the unique random time characterized by a constant hazard rate. On the other hand, the ASIP's interexit time is approximately Rayleigh—which is the unique random time characterized by a linear hazard rate. Namely, the interarrival time is memory less (due to its constant

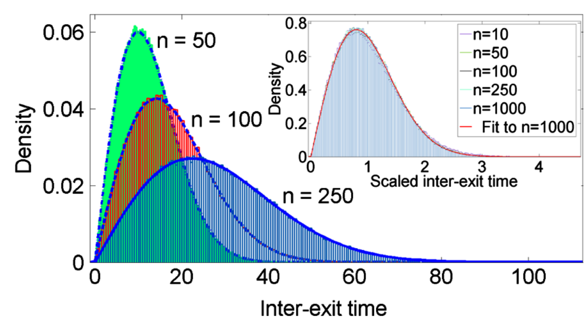

FIG. 4 (color online). Rayleigh approximation of the interexit time; bars represent simulated histograms, and dashed lines represent Rayleigh density fits. The mean and variance of the Rayleigh distribution, for $n = [50, 100, 250]$, are given respectively by [12.6, 17.8, 28.1] and [43.3, 86.2, 215.4]. Inset: The Rayleigh limit of the scaled interexit time.

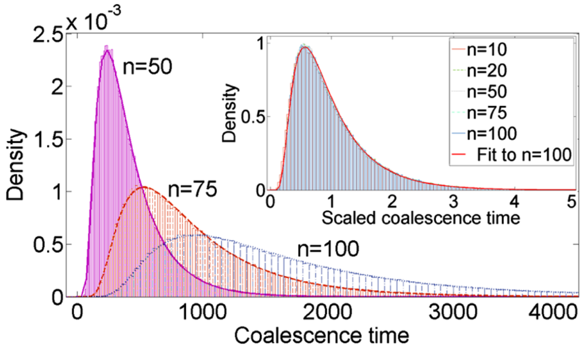

FIG. 5 (color online). Inverse-Gaussian approximation of the coalescence time; bars represent simulated histograms, and dashed lines represent inverse-Gaussian density fits. The mean and the variance, for $n = [50, 75, 100]$, are given respectively by [416, 937, 1669] and [69 984, 354 334, 1 126 670]. Inset: The inverse-Gaussian limit of the scaled coalescence time.

hazard rate), whereas the interexit time is aging linearly: $h_\Theta(t) = (\pi/2) \cdot t$. Thus, in the transition from the ASIP's inflow to the ASIP's outflow an aging effect emerges.

Coalescence time.—Consider a circular ASIP in which the output of the last site is the input of the first site, and assume that at time $t = 0$ all sites are occupied: $\#(\mathbf{X}(0)) = n$. As time progresses, gates open and particles coalesce into larger and larger particle clusters. Eventually, all particles will coalesce to a single 'super cluster'. The ASIP's coalescence time is the time elapsing till all particles coalesce together and form the "super cluster": $\Theta_n = \inf\{t \geq 0 | \#(\mathbf{X}(t)) = 1\}$. For the ASIP's coalescence time the coefficients are $a_n = n^2/6$ and $b_n = 0$, and the limit Θ is inverse Gaussian with unit mean and probability density function

$$\frac{d}{dt}\Pr(\Theta \leq t) = \frac{1}{\sqrt{2\pi\nu}} t^{-3/2} \exp\left(-\frac{(t-1)^2}{2\nu t}\right) \quad (5)$$

($t > 0$; $\nu = 2/5$). In effect, the Monte Carlo analysis illustrated in Fig. 5 shows that the inverse-Gaussian approximation well captures the data even for relatively small n. We note that the inverse-Gaussian distribution characterizes the first passage time of Brownian motion with linear drift [12]: the probability distribution of the random time $\inf\{t \geq 0 | t + \sqrt{\nu} B(t) = 1\}$, where $(B(t))_{t \geq 0}$ is Brownian motion, is quantified by the probability density function of Eq. (5).

The three random times explored above—the draining time, the interexit time, and the coalescence time—are, in effect, first passage times of the ASIP [13]. The four asymptotic stochastic approximation results are summarized in Table II.

The ASEP and the TJN are two fundamental models describing unidirectional flow along one-dimensional lattices, the first applied in nonequilibrium statistical physics, and the latter applied in queueing theory. When classified according to their local capacities, the ASIP model emerges as a missing puzzle piece. From a statistical-physics

TABLE II. Asymptomatic stochastic approximations of the ASIP observables: load, draining time, interexit time, and coalescence time.

	a_n	b_n	Θ	$\langle\Theta\rangle$	$\sigma^2(\Theta)$
Load	$\sqrt{2n}$	n	Gaussian	0	1
Draining time	$\sqrt{n}$	n	Gaussian	0	1
Interexit time	$\sqrt{\pi n}$	0	Rayleigh	1	$\frac{4-\pi}{\pi}$
Coalescence time	$n^2/6$	0	Inv. Gauss.	1	$2/5$

perspective the ASIP replaces the ASEP's fermionic exclusion interactions by Bosonic inclusion interactions. From a queueing-theory perspective the ASIP is a TJN model with unlimited batch service.

In this Letter, we combined together probabilistic and Monte Carlo analyses to explore the ASIP's rich statistical complexity which ranges from various "mild" and "intermediate" forms of randomness (displayed by the load and by the first passage times), to "wild" forms of randomness (displayed by the occupancies and by the density of occupied sites) [14]. Full mathematical proofs for the vast majority of the results presented herein have already been obtained and will be discussed in detail in two forthcoming research papers. We hope that this Letter will stimulate the transfer of knowledge between the statistical-physics and the queueing-theory communities, two scientific communities sharing the common goal of modeling complex stochastic systems.

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